LNF–89/091

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Estratto da: Nuovo Cimento "Note Brevi" 104B, n. 4 481 (1989)

IL NUOVO CIMENTO VOL. 104 B, N. 4 Ottobre 1989

NOTE BREVI

Dynamics of a Charged Particle in an Accelerating Field.

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(ricevuto il 25 Luglio 1989)

Summary. — In this paper we study the dynamics of a charged particle in an accelerating field. We make use of a simple perturbation method to derive a general analytic expression of the transfer matrix for the transverse motion. The results have been compared with numerical calculations. In the asymptotic regime, *i.e.* when the energy gain per cell is small compared to the initial energy, the agreement is very good.

PACS 41.70 – Particles in electromagnetic fields (including syncrotron radiation).

1. – Introduction.

The study of transverse particle dynamics in either travelling wave or standing wave linear accelerating structures is particularly important because of its effects on monitoring and controlling the transverse beam size and the various beam instabilities. In general, the spatial distribution of the fields is such that even for the simplest accelerating modes the equations of motion can be solved only by numerical integration. However, considerable physical insight is gained by making simplifying approximations which permit analytic solutions.

In this paper we describe a perturbation method which can be used in the case of a relativistic particle traversing a multicell accelerating structure where the energy gain per cell is small compared with the initial energy (asymptotic case). We also derive the transfer matrices for two significant examples of standing wave cavities.

2. – The equations of motion.

Particle motion is described by Lorentz' force:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}), \tag{1}$$

where $\boldsymbol{v}$, $\boldsymbol{E}$ and $\boldsymbol{B}$ are functions of time and position. We are interested in the particle motion near the axis of the accelerating structure (z-axis) where, in case of cylindrical symmetry, the accelerating field $E_z(z, t)$ is independent of the radius r, while the transverse fields $E_r(r, z, t)$ and $B_\phi(r, z, t)$ depend on r linearly.

If the particle is relativistic, *i.e.* $\mathrm{d}z/\mathrm{d}t \sim c$, we can neglect the effect of the radial velocity on the longitudinal motion and write eq. (1) as a function of z and r only:

$$\frac{\mathrm{d}p_z(z)}{\mathrm{d}t} = eE_z(z), \tag{2}$$

$$\frac{\mathrm{d}p_r(z)}{\mathrm{d}t} = f(z)\,r, \tag{3}$$

where

$$f(z) = e\frac{\mathrm{d}}{\mathrm{d}r}[E_r - cB_\phi]. \tag{4}$$

In general, the analytic solution of eqs. (2) and (3) will be hard to find out. In our method we consider a small increment Δz of the longitudinal position z of the particle and the corresponding changes in the r, p_r variables:

$$r(z + \Delta z) = r(z) + r'(z)\,\Delta z, \qquad p_r(z + \Delta z) = p_r(z) + p_r'(z)\,\Delta z. \tag{5}$$

It is customary to put eqs. (5) in a matrix form, using eq. (3) and remembering that $r'(z) = p_r(z)/p(z)$, where $p(z)$ is the total momentum:

$$\begin{pmatrix} r(z+\Delta z) \\ p_r(z+\Delta z) \end{pmatrix} = \begin{bmatrix} 1 & \dfrac{\Delta z}{p(z)} \\ \dfrac{f(z)}{c}\Delta z & 1 \end{bmatrix} \begin{pmatrix} r(z) \\ p_r(z) \end{pmatrix}. \tag{6}$$

The above matrix has determinant 1 neglecting terms of the 2nd order in Δz. So symplecticity is assured and the matrix can be propagated through the accelerating structure, in order to determine the particle trajectory in phase space (r, p_r) as a function of z.

In the asymptotic case, however, the particle momentum $p(z)$ does not change appreciably along its path through a cell, so it can be taken constant and equal to its average value. With this assumption it is possible to obtain an approximate analytic solution with the wished degree of accuracy by means of a simple perturbation method (1).

3. – Analytical method.

The total transfer matrix $\boldsymbol{M}$ from $z_0 = 0$ to z is just

$$M(z) = \prod_i M(z_i,\ \Delta z_i), \tag{7}$$

where the infinitesimal matrices $\boldsymbol{M}(z,\ \mathrm{d}z)$ are given by eq. (6). It is convenient to introduce a new variable s, defined by $\Delta s = \Delta z/p(z)$.

Also, let us put $\mathscr{f}(s) = p(z)f(z)/c$. Equation (6) becomes

$$\boldsymbol{M}(s,\ \Delta s) = \begin{pmatrix} 1 & \Delta s \\ \mathscr{f}(s)\,\Delta s & 1 \end{pmatrix} = \boldsymbol{I} + \Delta s\,\boldsymbol{T}(s)\,, \tag{8}$$

where $\boldsymbol{I}$, the unit matrix, and $\boldsymbol{T}(s)$ are

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \boldsymbol{T}(s) = \begin{pmatrix} 0 & 1 \\ \mathscr{f}(s) & 0 \end{pmatrix}. \tag{9}$$

Substituting eq. (8) in eq. (7) and expanding we get

$$\boldsymbol{M}(s) = \boldsymbol{I} + \int_0^s \boldsymbol{T}(s_1)\,\mathrm{d}s_1 + \int_0^s \boldsymbol{T}(s_1)\,\mathrm{d}s_1 + \int_0^{s_1} \boldsymbol{T}(s_2)\,\mathrm{d}s_2 + \dots\,. \tag{10}$$

Each element can now be expressed as

$$M_{ij}(s) = \sum m_{ij}^n(s)\,. \tag{11}$$

For $n = 0$ we have the unit matrix. For $n = 1$

$$\begin{cases} m_{11}^1(s) = 0\,, & m_{12}^1(s) = \displaystyle\int_0^s \mathrm{d}s\,, \\ m_{21}^1(s) = \displaystyle\int_0^s \mathscr{f}(s)\,\mathrm{d}s\,, & m_{22}^1(s) = 0\,. \end{cases} \tag{12}$$

(1) G. SACERDOTI: CNEN Report (1965).

For $n > 1$ the matrix elements are given by the following recursive formulae:

$$\left\{\begin{array}{ll} m_{11}^{2(n+1)}(s) = \int_0^s m_{21}^{2n+1}(s)\,\mathrm{d}s\,, & m_{21}^{2n+1}(s) = \int_0^s \mathcal{f}(s)\, m_{11}^{2n}(s)\,\mathrm{d}s\,, \\ m_{12}^{2n+1}(s) = \int_0^s m_{22}^{2n}(s)\,\mathrm{d}s\,, & m_{22}^{2(n+1)}(s) = \int_0^s \mathcal{f}(s)\, m_{12}^{2n+1}(s)\,\mathrm{d}s\,. \end{array}\right. \tag{13}$$

Equations (10)-(13) provide a general powerful tool to compute the transfer matrix, although it is not possible to specify its speed of convergence *a priori*.

In the case of e.m. fields the function $\mathcal{f}(s)$ can be expressed as $\mathcal{f}(s) = hd(\mathrm{s})$, where $d(s)$ is a given function, and h is a constant parameter, so that eqs. (12) reduce to a series expansion in powers of hs. In the case of a quadrupolelike field, $d(s) = 1$, $\mathcal{f}(s) = h$ constant and positive, and it is easy to show that the elements M_{11}, M_{22} contain the Taylor expansion of $\cos(hs)$ and the elements M_{12}, M_{21} contain the taylor expansion of $\sin(hs)$ and the matrix $\boldsymbol{M}$ reproduces exactly the transfer matrix of a quadrupole.

4. – Transfer matrix for an accelerating cell.

A rather idealized example (but often useful to start the analysis of standing wave accelerating structures) is the «pill-box» cavity.

The field components of the fundamental mode TM_{010} are

$$E_z = E_0 J_0(\varkappa r)\sin(\omega t)\,, \qquad B_\phi = \frac{E_0}{c}\, J_1(\varkappa r)\cos(\omega t)\,. \tag{14}$$

If we look at a small region around the cell axis, the Bessel functions J_ι can be approximated by $J_0(kr) \sim 1$, $J_1(kr) \sim kr/2$, so that the conditions stated in sect. 2 are fulfilled. The infinitesimal transfer matrix is

$$\begin{pmatrix} r(z+\Delta z) \\ p_r(z+\Delta z) \end{pmatrix} = \begin{bmatrix} 1 & \dfrac{\Delta z}{p(z)} \\ -\left(\dfrac{\pi g}{2c}\right)^2 N(z)\cos(\varkappa z)\dfrac{\Delta z}{p(z)} & 1 \end{bmatrix} \begin{pmatrix} r(z) \\ p_r(z) \end{pmatrix}, \tag{15}$$

where we have introduced the accelerating gradient $g = 2eE_0/\pi$ in a cell of length $\lambda/2$ and the number of equivalent cells $N(z) = 2p(z)/cg\lambda$.

In the asymptotic case the particle momentum does not change appreciably within a cell, so it can be assumed constant and equal to its average value in the cell, *i.e.*

$$p(z) = p(0) + \frac{g\lambda}{4c}\,. \tag{16}$$

Integrals (13) concern now only elementary trigonometric functions and are straightforward. We give here the transfer matrix of a full cell:

$$M\left(z=\frac{\lambda}{2}\right)=\begin{bmatrix}(\cos\theta+\eta\sin\theta) & \xi\sin\theta \\ -\dfrac{1+\eta^2}{\xi}\sin\theta & (\cos\theta-\eta\sin\theta)\end{bmatrix}, \tag{17}$$

for the variables (r, p_r). The constants are

$$\begin{cases}\theta=\chi\ln\left[\dfrac{p(z)}{p(0)}\right], & \chi=\dfrac{\sqrt{\pi^2/8-1}}{2}, \\ \eta=-\sqrt{\dfrac{8}{\pi^2-8}}, & \xi=\dfrac{2}{\sqrt{\pi^2/8-1}}.\end{cases} \tag{18}$$

A comparison with numerical integration of eqs. (2), (3) gives the results shown in fig. 1. In this example and in the following we consider multicell resonant cavities at 500 MHz, with $g=5$ MeV/m.

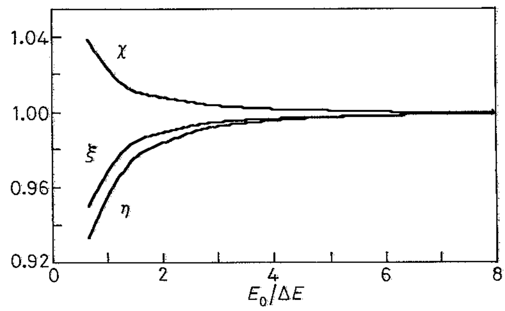

Fig. 1. – Normalized values of η, χ and ξ (numerical/analytical).

It is worth noting that the asymptotic (analytical) matrix gives fairly good results when the injection energy is at least 6 times larger than the energy gain per cell.

This result was already found out for an r.f. cavity with a purely sinusoidal field, which corresponds to the lowest-order term in the development of a periodic field into space harmonics [2]. For many practical cases this is a good

[2] E. E. CHAMBERS: *Proceedings of the 1968 Summer Study on Superconducting Devices and Accelerators*, BNL Report 50155 (1968).

approximation to the real field in a RF cavity. We have, for $kr \ll 1$,

$$\text{(19)} \qquad \begin{cases} E_z(z,\ t) = A_0 \sin(\varkappa z)\sin(\omega t)\,, \\ E_r(r,\ z,\ t) = -A_0 \dfrac{\varkappa r}{2}\cos(\varkappa z)\sin(\omega t)\,, \\ B_\phi(r,\ z,\ t) = A_0 \dfrac{\varkappa r}{2c}\sin(\varkappa z)\cos(\omega t)\,. \end{cases}$$

Using the same procedure we get the transfer matrix for a small increment Δz:

$$\text{(20)} \qquad \begin{pmatrix} r(z+\Delta z) \\ p_r(z+\Delta z) \end{pmatrix} = \begin{bmatrix} 1 & \dfrac{\Delta z}{p(z)} \\ -\pi\left(\dfrac{g}{c}\right)^2 N(z)\sin(2\varkappa z)\dfrac{\Delta z}{p(z)} & 1 \end{bmatrix} \begin{pmatrix} r(z) \\ p_r(z) \end{pmatrix}.$$

In the asymptotic case, the method produces a matrix which is formally identical to the (17), but with different values for the 3 constants:

$$\text{(21)} \qquad \chi = \frac{1}{\sqrt{8}}\,, \qquad \eta = -\sqrt{2}\,, \qquad \xi = \sqrt{8}\,.$$

This result was already obtained by Chambers (2), who made use of a completely different approach. Also in this case the analytical method provides a good result for $\mathcal{E}(0)/\Delta\,\mathcal{E} > 6$.

The analytical solution is not restricted to the end of the cell, in fact the method can be applied to produce the transfer matrix at any position z along the cell axis.

5. – Conclusions.

We have presented an analytical approach to the study of radial motion in typical standing wave accelerating structures. A simple perturbation method has been used to derive a general analytic expression for the radial transfer matrix in the presence of the fundamental accelerating modes. The results have been checked against accurate numerical calculations in the important case of asymptotic behaviour, *i.e.* $\mathcal{E}(0)/\Delta\mathcal{E} \gg 1$ where the agreement is very good.

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R. Boni, S. Spataro and F. Tazzioli (LNF) are thanked for useful discussions.