ISTITUTO NAZIONALE DI FISICA NUCLEARE - ISTITUTO NAZIONALE DI FISICA NUCLEARE - ISTITUTO NAZIONALE DI FISICA NUCLEARE - ISTITUTO NAZIONALE DI FISICA NUCLEARE - ISTITUTO NAZIONALE DI

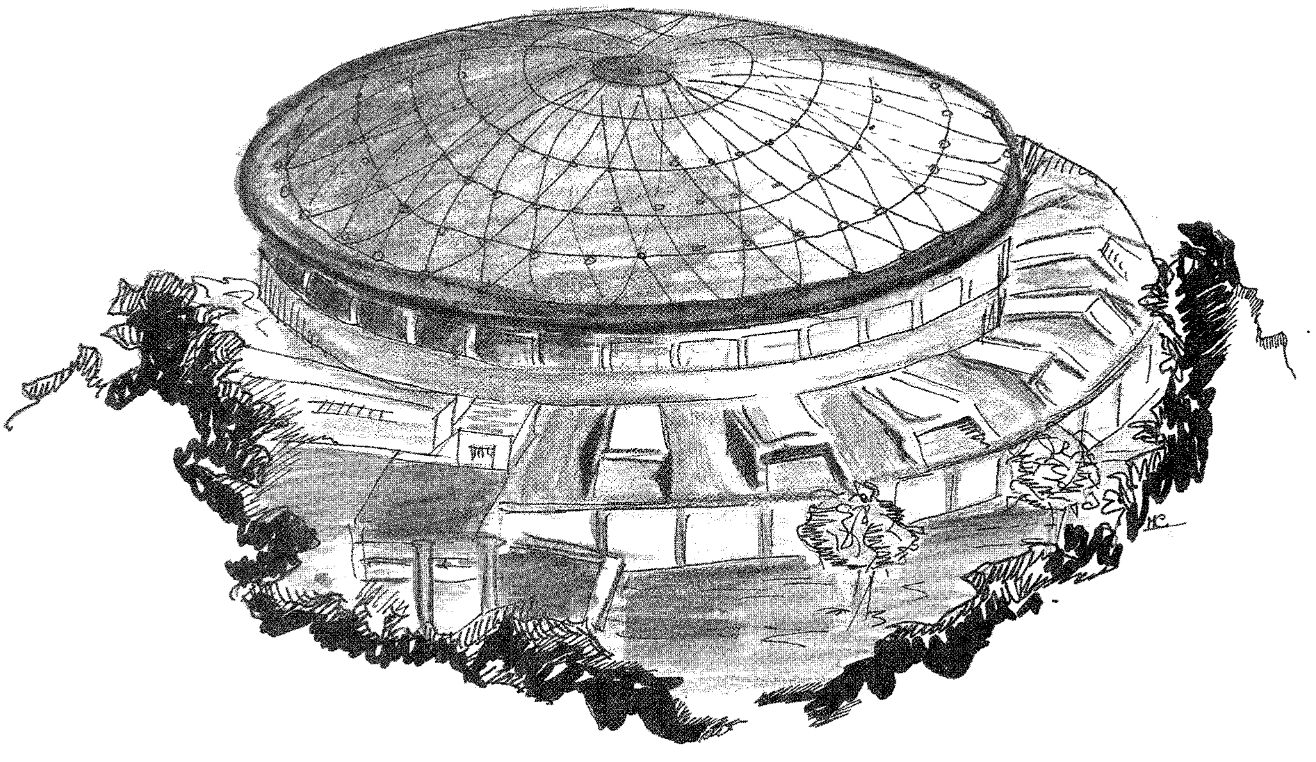
Laboratori Nazionali di Frascati

To be Submitted to Il Nuovo Cimento

LNF-89/047(PT)
24 Luglio 1989

C. Basili, E. Etim and M. Pallotta:

CESÀRO SUMMABILITY OF THE FOURIER INTEGRAL REPRESENTATIONS OF THE BLOCH-NORDSIECK DISTRIBUTION AND ITS DERIVATIVES



Servizio Documentazione
dei Laboratori Nazionali di Frascati
P.O. Box, 13 - 00044 Frascati (Italy)

INFN - Laboratori Nazionali di Frascati
Servizio Documentazione

LNF-89/047(PT)
24 Luglio 1989

CESÀRO SUMMABILITY OF THE FOURIER INTEGRAL REPRESENTATIONS OF THE BLOCH-NORDSIECK DISTRIBUTION AND ITS DERIVATIVES

C. Basili(*), E. Etim and M. Palletta
INFN - Laboratori Nazionali Frascati, P.O.Box 13, I-00044-Frascati, (Italy)

ABSTRACT

The Fourier integral representations for the Bloch-Nordsieck distribution $\rho(\omega)$ and its derivatives $\rho^{(n)}(\omega)$ $(n = 1, 2, \dots)$ are not Cauchy-convergent. The theory therefore contains ambiguities. Consequently the equations, characterising the theory and which relate these derivatives to the discrete translations $\rho(\omega\text{-}mE)$ $(m = 0, 1, 2,\dots n)$ of $\rho(\omega)$ are not satisfied by these representations as they stand. The equations are consistently satisfied only by the regularised forms of the corresponding integrals. This regularization is carried out in this paper using the method of Cesàro summability. This allows to compute these derivatives $\rho^{(n)}(\omega)$ $(n = 0, 1, 2,\dots)$ and to investigate their behaviour as functions of ω in the entire half-line $0 \le \omega < \infty$.

1. - INTRODUCTION

The Bloch-Nordsieck method(1) is a semi-classical non-perturbative theory of the emission of soft radiation by a charged source. It is infrared convergent. It was proposed initially to solve the specific problem of eliminating the infrared divergence introduced by the physically unrealistic counting of soft photons by perturbation theory(1,2). For this specific application, the

(*) - Consiglio Nazionale delle Ricerche - ISRDS, Via Cesare de Lollis, 12 - 00185 Roma, Italy

theory is soluble in closed form for the probability density $\rho(\omega)$ for the emission of soft photons of total energy ω by a charged source of energy E, with $\omega \leq E$[3,4].

More recently the Bloch-Nordsieck method has been proposed for other uses, e.g. for the description of the energy spectrum of hadrons generated in the hard collisions between quarks[5]. The emission of these hadrons is thus considered to be "soft" compared to the "hardness" of the quark-quark and quark-gluon interactions. Used in this way the Bloch-Nordsieck energy distribution is required to manifest properties which were unthinkable in the restricted framework of radiative corrections. Such properties include its asymptotic behaviour for $\omega \rightarrow \infty$ and therefore requires the extension of $\rho(\omega)$ from the "classical" region $0 \leq \omega \leq E$ to the "non-classical" $E < \omega < \infty$. The classical region exists in reference to the fact that classically, a charged particle of energy E cannot emit radiation of total energy $\omega > E$. Quantum mechanically, however, the probability for the emission a photon of energy $\omega > E$ is not zero. To explain this is a problem of physics. It will not be discussed here. Related to this asymptotic behaviour is an enquiry[6] in the application of the Bloch-Nordsieck theory to soft hadron physics. The enquiry is if the energy spectrum $\rho(\omega)$ satisfies KNO scaling[7] and, if not, does it depart from this scaling law in the direction consistent with experiments? To answer these questions one needs better knowledge of the properties of $\rho(\omega)$ than is usually required in the applications of infrared radiative corrections. It is at this point that problems arise: (i) $\rho(\omega)$ is not known in closed form for all values of ω. (ii) In attempting to establish this knowledge by evaluating $\rho(\omega)$ and its derivatives numerically from their Fourier integral representations one discovers that these integrals do not converge. These are serious problems. But they are not a consequence of anything wrong with the physics. The problems arise from the wrong expectation of finding the values of integrals which have not been properly defined. The Fourier integral representations for $\rho(\omega)$ and its derivatives $\rho^{(n)}(\omega)$ $(n = 1, 2, \ldots)$ are not Cauchy-convergent. This gives rise to ambiguities in the theory. The problem therefore is one of regularisation. Its solution is the purpose of this paper. In sect. 2 we review briefly the Bloch-Nordsieck theory and formulate the problem to be solved. In sect. 3 we present the solution, namely that the Fourier integral representation for $\rho(\omega)$ and its derivatives can be regularised by Cesàro summability[8-10]. Sect. 4 concludes the paper.

2. - THE BLOCH-NORDSIECK DISTRIBUTION

The Bloch-Nordsieck energy distribution $\rho(\omega)$ is the probability density for the emission of soft photons of total energy ω by a classical charged source of energy E. Let n be the number of these photons defining a particular soft photon configuration. Let ω_i, with $\omega_i \leq E$, be the energy of a single photon in an energy cell labelled by i and containing n_i photons of equal energy. The n_i constitute a partition of n, i.e.

$$n = \sum_i n_i \tag{1}$$

and the products $n_i \, \omega_i$ a partition of ω, i.e.

$$\omega = \omega_n = \sum_i n_i \, \omega_i \, \theta(E - \omega_i) \tag{2}$$

The step function $\theta(E - \omega_i)$ ensures that in each act of photon emission, the emitted energy ω_i is always less than or equal to E, the energy of the charged source. Note that this restriction applies to the energy of a single photon ω_i but not to the total energy loss ω. For given n, eq. (2) only restricts ω to satisfy the inequality

$$\omega \leq nE \qquad ; \qquad n = 1, 2, \ldots \tag{3}$$

Thus for $n \geq 2$, ω may be grater than E. Nevertheless the average of ω taken with the energy density $\rho(\omega)$ is less than E, in agreement with the classical expectation.

To come to a definition of $\rho(\omega)$ we note that the distribution of the cell occupation numbers n_i is Poissonian, i.e.

$$P_{n_i} = e^{-<n_i>} \frac{<n_i>^{n_i}}{n_i \, !} \tag{4}$$

where $< n_i >$ is the average of n_i. Accordingly the distribution P_n of the sum of the n_i in eq. (1) is also Poissonian, i.e.

$$P_n = \Pi_i \, P_{n_i} = e^{-<n>} \frac{<n>^n}{n \, !} \tag{5}$$

The first part of the equality in eq. (5) expresses the statistical independence of the cells i. The second part of the equality follows from application of the multinomial theorem. From eqs. (2) and (5) we have, for the probability density $\rho(\omega)$, the definition

$$\rho(\omega) := \sum_{n=0}^{\infty} P_n \, \delta(\omega - \omega_n) \tag{6}$$

It is clear from this definition and from the properties of the Poisson distribution that $\rho(\omega) \to 0$ for $\omega \to \infty$. This is so because from eq. (3) $\omega \to \infty$ requires large $n \to \infty$. But $P_n \to 0$ for $n \to \infty$.

Substituting the Fourier integral representation

$$\delta(\omega - \omega_n) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt \; e^{i(\omega - \omega_n)t} \tag{7}$$

of the Dirac delta function $\delta(\omega - \omega_n)$ in eq. (6) one gets for $\rho(\omega)$ the following integral representation

$$\rho(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt\, e^{i\omega t}\; G(Et) \tag{8}$$

where

$$G(Et) := e^{-h(Et)} \tag{9.a}$$

$$h(Et) := \sum_i \langle n_i \rangle \left(1 - e^{-i\omega_i t}\right) = ß \int_0^{Et} \frac{du}{u}\,(1 - e^{-iu}) \tag{9.b}$$

β is a parameter proportional to the square of the charge of the emitting particle. It depends, in general, also on the energy E. G (Et) is the characteristic function of the probability distribution $\rho(\omega)$. Changing variable i.e. $\tau = \omega t$ in the integral in eq. (8), $\rho(\omega)$ may be re-expressed as

$$\rho(\omega) = \frac{1}{2\pi\omega}\int_{-\infty}^{+\infty} d\tau\, e^{i\tau}\; G\left(\frac{E\tau}{\omega}\right) \tag{10}$$

Taking the derivative of both sides of (10) with respect to ω, one gets[3]

$$\frac{d\rho(\omega)}{d\omega} = \frac{(ß - 1)}{\omega}\,\rho(\omega) - \frac{ß}{\omega}\,\rho(\omega - E) \tag{11}$$

Iterating eq. (11) one gets the general equation

$$\rho^{(n+1)}(\omega) = \frac{(ß - 1 - n)}{\omega}\,\rho^{(n)}(\omega) - \frac{ß}{\omega}\,\rho^{(n)}(\omega - E) \quad ; \quad n \geq 0 \tag{12}$$

where

$$\rho^{(n)}(\omega) := \frac{d^n \rho(\omega)}{d\omega^n} \tag{13}$$

Eq(12) is to be solved with the boundary condition

$$\rho(\omega < 0) = 0 \tag{14}$$

The temptation [11] is to do this iteratively e.g. by rewriting eq. (11) as the integral equation

$$\rho(\omega) = A\,\theta(E - \omega)\,\omega^{ß - 1} - ß\,\theta(\omega - E)\,\omega^{ß - 1}\int_E^{\omega} dy\, y^{-ß}\,\rho(y - E) \tag{15}$$

where the constant A is defined by

$$A := \lim_{\omega \to 0}\left[\omega^{1 - ß}\,\rho(\omega)\right] \tag{16}$$

The boundary condition in eq. (14) is incorporated in eq. (15) through the step functions $\theta(E - \omega)$ and $\theta(\omega - E)$. The iteration of eq. (15) leads therefore to a system of piece-wise solutions for $\rho(\omega)$ in the intervals $0 \leq \omega \leq mE$ $(m = 1, 2, \ldots)$. In this way, one continues $\rho(\omega)$ from the fundamental interval $0 \leq \omega \leq E$ into the "non-classical region" $E < \omega < \infty$. The first two iterates are

$$\rho_0(\omega) = A\,\omega^{\beta - 1} \quad ; \quad 0 \leq \omega \leq E \tag{17.a}$$

$$\rho_1(\omega) = A\,\omega^{\beta - 1}\,\theta(E - \omega) - A\,\omega^{\beta - 1}\left(\frac{\omega}{E} - 1\right)^{\beta} {}_2F_1\left(\beta, \beta, \beta + 1, 1 - \frac{\omega}{E}\right)\theta(\omega - E)$$
$$0 \leq \omega \leq 2E \tag{17.b}$$

${}_2F_1\,(a, b; c; z)$ is the hypergeometric function. The iteration quickly becomes cumbersome because of difficult integrals involving the hypergeometric function. The integrals are over finite intervals and are Cauchy-convergent. Apart from this, there is no other advantage in resorting to this iterative solution. One may solve for $\rho(\omega)$ directly from the Fourier integral representation in eq. (8). The crucial point is that the Fourier integral representations for $\rho(\omega)$ and its derivatives are not Cauchy-convergent. The convergence problem gets worse for the derivatives of $\rho(\omega)$. This is a serious problem because, according to eq. (12), derivatives of $\rho(\omega)$ can be expressed in terms of discrete translations $\rho(\omega - m\,E)$ $(m = 0, 1, 2, \ldots)$ of $\rho(\omega)$ and vice versa. Stated differently, this means that the Fourier integral representations

$$\frac{d^n \rho(\omega)}{d\omega^n} := \rho^{(n)}(\omega) = \frac{i^n}{2\pi}\int_{-\infty}^{+\infty} dt\, t^n\, e^{i\omega t}\, G(Et) \quad ; \quad n = 0, 1, 2, \ldots \tag{18}$$

for $\rho^{(n)}(\omega)$ $(n = 0, 1, 2, \ldots)$ satisfy eq. (12) only when these integrals are suitably regularised. The regularisation of these integrals is the problem which is solved in this paper. This is carried out in the next section.

3. - CESÀRO SUMMABILITY OF $\rho(\omega)$ AND ITS DERIVATIVES

The purpose of this section is to regularise the Cauchy non-convergent integrals in eq. (18) and show that the regularised integrals satisfy eq. (12). The method of regularisation is by Cesàro summability(8). This method of summability has been applied in refs(9) and (10) to regularise Hankel transforms and of sine and cosine transforms of powers. The integrals in eq. (18) can also be expressed essentially in terms of cosine transforms of powers. We will therefore not go into details in the description of this method of summability. The first step in the summability procedure is to express the characteristic function G(Et = z) in eq. (18) in terms of its real and imaginary parts, i.e.

$$G(z) = e^{-R(z)} [\cos(S(z)) - i \sin(S(z))] \qquad (19)$$

where, from eqs. (9.a) and (9.b),

$$R(z) := ß \int_0^z \frac{du}{u} (1 - \cos(u)) \qquad (20.a)$$

$$S(z) := ß \int_0^z \frac{du}{u} \sin(u) \qquad (20.b)$$

R(z) and S(z) are the real and imaginary parts, respectively, of the function h(z) in eq. (9.b). One notes from eqs. (20) that R(z) and S(z) are, respectively, even and odd in z, i.e..

$$R(-z) = R(z) \qquad (21.a)$$

$$S(-z) = - S(z) \qquad (21.b)$$

Eqs(19)- (21) allow to rewrite eq. (18) in the form

$$\rho^{(n)}(\omega) = \frac{(-)^{n/2}}{\pi} \int_0^\infty dt\ t^n\ e^{-R(Et)} \cos(\omega t - S(Et)) \quad ; \quad n \text{ even} \qquad (22.a)$$

$$\rho^{(n)}(\omega) = \frac{(-)^{\frac{n+1}{2}}}{\pi} \int_0^\infty dt\ t^n\ e^{-R(Et)} \sin(\omega t - S(Et)) \quad ; \quad n \text{ odd} \qquad (22.b)$$

Now since from the boundary condition in eq. (14), one has

$$\rho^{(n)}(-\omega) = 0 \quad ; \quad n = 0, 1, 2, \ldots \qquad (23)$$

eqs. (22.a) and (22.b) reduce to

$$\rho^{(n)}(\omega) = \frac{2(-)^{n/2}}{\pi} \int_0^\infty dt\ t^n\ e^{-R(Et)} \cos(S(Et))\ \cos(\omega t) \quad ; \quad n \text{ even} \qquad (24.a)$$

$$\rho^{(n)}(\omega) = \frac{2(-)^{\frac{n-1}{2}}}{\pi} \int_0^\infty dt\ t^n\ e^{-R(Et)} \sin(S(Et))\ \cos(\omega t) \quad ; \quad n \text{ odd} \qquad (24.b)$$

Eqs(24.a) and (24.b) can be represented as Hankel transforms, i.e.

$$\rho^{(n)}(\omega) = \omega^{1/2} \int_0^\infty dt\ t^{n+1/2}\ J_{-1/2}(\omega t)\ f_n^{(e,0)}(Et) \tag{25}$$

where the even and odd functions $f_n^{(e)}(z)$ and $f_n^{(0)}(z)$, respectively, are defined by

$$f_n^{(e)}(z) = (-)^{n/2} \left(\frac{2}{\pi}\right)^{1/2} e^{-R(z)} \cos(S(z)) \tag{26.a}$$

$$f_n^{(e)}(z) = (-)^{\frac{n-1}{2}} \left(\frac{2}{\pi}\right)^{1/2} e^{-R(z)} \sin(S(z)) \tag{26.b}$$

and $J_{-1/2}(z)$ is the Bessel function of order $\nu = -1/2$. The Bessel function arises from the relation

$$\cos(z) = \left(\frac{\pi z}{2}\right)^{1/2} J_{-1/2}(z) \tag{27}$$

Eq(25) is thus in the form in which the summability procedure described in refs(9) and (10) can be applied directly, even though the $f^{(e,0)}(z)$ are not powers. The summability procedure is straightforward. One introduces an increasing set $T_1 < T_2 < \ldots T_N$ $(N = 1, 2, \ldots)$ of cut-offs and considers the corresponding integrals

$$\rho_N^{(n)}(\omega) = \omega^{1/2} \int_0^{T_N} dt\ t^{n+1/2}\ J_{-1/2}(\omega t)\ f_n^{(e,0)}(Et) \quad ; \quad N = 1, 2, \ldots \tag{28}$$

Next one forms the arithmetic means

$$C_{Nn}^{(0)}(\omega) := \rho_N^{(n)}(\omega) \tag{29.a}$$

$$C_{Nn}^{(k)}(\omega) := \frac{1}{N} \sum_{m=1}^{N} C_{mn}^{(k-1)}(\omega) \quad ; \quad k \geq 1 \tag{29.b}$$

and considers their corresponding limits

$$C_n^{(0)}(\omega) := \lim_{N \to \infty} C_{Nn}^{(0)}(\omega) \tag{30.a}$$

$$C_n^{(k)}(\omega) := \lim_{N \to \infty} C_{Nn}^{(k)}(\omega) \tag{30.b}$$

The $C_{Nn}^{(k)}(\omega)$ $(k = 1, 2, \ldots)$ are the so-called Cesàro means of the cut-off integrals $\rho_N^{(n)}(\omega)$. If the integrals $\rho^{(n)}(\omega)$ were Cauchy-convergent then we would have, by definition

$$\rho^{(n)}(\omega) = C_n^{(0)}(\omega) \tag{31}$$

These integrals are not Cauchy-convergent and one has therefore to investigate the convergence of the sequence of higher Cesàro means $C_{Nn}^{(k)}(\omega)$ $(k \geq 1)$.Let the limit $C_n^{(k)}(\omega)$ of $C_{Nn}^{(k)}(\omega)$ exist for some $k = \bar{k} \geq 1$. In this case one defines $\rho^{(n)}(\omega)$ by

$$\rho^{(n)}(\omega) := C_n^{(\bar{k})}(\omega) \tag{32}$$

and one then says that $\rho^{(n)}(\omega)$ is Cesàro summable at order $\bar{k}$. It follows from eqs. (29) that, in this case,

$$C_n^{(k+\bar{k})}(\omega) = C_n^{(\bar{k})}(\omega) \quad ; \quad k = 0, 1, 2, \ldots \tag{33}$$

while the limits $C_n^{(\bar{k}-k)}(\omega)$ $(k = 1, 2, \ldots \bar{k})$ do not exist.

In Figs. (1) - (4) we show the behaviour of $C_{Nn}^{(k)}(\omega)$ for $n = 0, 1, 2, 3$ and their approach to the limits $C_n^{(k)}(\omega)$. These correspond to $\rho(\omega)$ (i.e. $n = 0$) and its first three derivatives (i.e. $n = 1, 2, 3$). As can be seen from these plots, the Fourier integral representations for $\rho^{(n)}(\omega)$ $(n = 0, 1, 2, 3)$ are not Cauchy-convergent i.e. the corresponding limits $C_n^{(0)}(\omega)$ $(n = 0, 1, 2, 3)$ do not exist. We have instead that the limits of the higher Cesàro means $C_{Nn}^{(k)}(\omega)$ $(k \geq 1)$ exist for $N \to \infty$. Thus one defines $\rho^{(n)}(\omega)$ according to eq. (32) as the limits of these higher Cesàro means. The integrals so regularised satisfy eq. (12). This is important by itself. But it is even more so since the derivatives $\rho^{(n)}(\omega)$ are not Cesàro-convergent at the same order. We have checked explicitly that eq. (12) is satisfied by the regularised integrals for $\rho^{(n)}(\omega)$ $(n = 0, 1, 2, 3)$ also by expressing these derivatives in terms of the discrete translations $\rho(\omega - mE)$ $(m = 0, 1, 2, \ldots n)$ of $\rho(\omega)$. One has, in formulae,

$$\frac{d\rho(\omega)}{d\omega} = \frac{(\beta - 1)}{\omega}\rho(\omega) - \frac{\beta}{\omega}\rho(\omega - E) \tag{34.1}$$

$$\frac{d^2\rho(\omega)}{d\omega^2} = \frac{(\beta - 1)(\beta - 2)}{\omega^2}\rho(\omega) - \frac{\beta}{\omega}\left[\frac{(\beta - 2)}{\omega} + \frac{(\beta - 1)}{\omega - E}\right]\rho(\omega - E) + \frac{\beta^2}{\omega(\omega - E)}\rho(\omega - 2E)$$

(34.2)

$$\frac{d^3\rho(\omega)}{d\omega^3} = \frac{(\beta-1)(\beta-2)(\beta-3)}{\omega^3}\,\rho(\omega) -$$

$$-\frac{\beta}{\omega}\left[\frac{(\beta-2)(\beta-3)}{\omega^2} + \frac{(\beta-3)(\beta-1)}{\omega(\omega-E)} + \frac{(\beta-1)(\beta-2)}{(\omega-E)^3}\right]\rho(\omega-E) +$$

$$+\frac{\beta^2}{\omega}\left[\frac{(\beta-3)}{\omega(\omega-E)} + \frac{(\beta-2)}{(\omega-E)^2} + \frac{(\beta-1)}{(\omega-E)(\omega-2E)}\right]\rho(\omega-2E) -$$

$$-\frac{\beta^3}{\omega(\omega-E)(\omega-2E)}\,\rho(\omega-3E) \qquad (34.3)$$

The functions $\rho^{(n)}(\omega)$ $(n = 0, 1, 2, 3)$ are plotted in Figs (5) - (8) as functions of ω.The derivatives $\rho^{(n)}(\omega)$ $(n = 1, 2, 3)$ are discontinuous at $\omega = mE$ $(m = 1, 2, \ldots n)$ as can be seen from eqs. (34.1) - (34.3) and from Figs (6) - (8). The nature of these discontinuities depends on the value of ß. This too is illustrated in Figs.(6) - (8). The regularisation of the Fourier integral representations of $\rho^{(n)}(\omega)$ $(n = 0, 1, 2,\ldots)$ has thus allowed to investigate the behaviour of these functions over the whole half-line $0 \le \omega \le \infty$. For instance, $\rho(\omega)$ exists and is non-zero also for $\omega > E$. It tends to zero asymptotically for $\omega \to \infty$ in agreement with eq. (6).

4. - CONCLUSIONS

We have been concerned in this paper with a consistency problem in the theory of the Bloch-Nordsieck energy - distribution $\rho(\omega)$. The theory predicts that the derivatives $\rho^{(n)}(\omega)$ $(n = 1, 2,\ldots)$ of $\rho(\omega)$ are expressible in terms of the discrete translations $\rho(\omega - mE)$ $(m = 0, 1, 2,\ldots n)$ of $\rho(\omega)$. These relationships between $\rho^{(n)}(\omega)$ $(n = 1, 2, \ldots)$ and $\rho(\omega - mE)$ $(m = 0, 1, 2, \ldots n)$ are not satisfied, however, by the Fourier integral representations of these functions as they stand. The reason is that these integral representations do not converge. They have therefore to be regularised. They have been regularised in this paper using the method of Cesàro summability. The regularised integrals satisfy the relationships between the derivatives $\rho^{(n)}(\omega)$ $(n = 1, 2,\ldots)$ and the discrete translations $\rho(\omega - mE)$ $(m = 0,1,2,\ldots n)$. This restores consistency in the theory.

Quite apart from this, the regularisation of the integral representation for $\rho(\omega)$ allows to determine $\rho(\omega)$ consistently for all ω, i.e. $0 \le \omega \le \infty$, and not only for $0 \le \omega \le E$. The determination of $\rho(\omega)$ for $\omega > E$ is required by phenomenological applications (5,6) which simulate "soft" hadronic processes by the Bloch-Nordsieck mechanism. These applications have their merits. It is therefore important to furnish them with the accurate determination of $\rho(\omega)$ for $0 \leq \omega \leq \infty$. The behaviour of $\rho(\omega)$ as a function of $\omega > E$ has remained unknown for a very long time. This ignorance has led to approximations and speculations[6]. All this is unnecessary. $\rho(\omega)$ is well determined by summability and easily computed for all values of ω.

Having thus settled the question about the exact determination of $\rho(\omega)$ and its derivatives $\rho^{(n)}(\omega)$ $(n = 0,1,2,\ldots)$, the problem now arises as to the physical implications of the relationships between $\rho^{(n)}(\omega)$ $(n = 1, 2,\ldots)$ and $\rho(\omega - mE)$ $(m = 0,1,2,\ldots n)$. It is shown in a forthcoming publication that these relationships arise from the breaking of a basic symmetry, namely, scale invariance inherent in the physical problem. To anticipate this a little, we call attention to eq. (17.a) where the first iterate $\rho_o(\omega) = A\omega^{\beta-1}$, valid for $0 \leq \omega \leq E$, is a dilatation eigenfunction; that is, $\rho_o(\lambda\omega) = \lambda^{\beta-1}\ \rho_o(\omega)$. This symmetry is broken by the higher order iterates $\rho_m(\omega)$ $(m = 1, 2,\ldots)$ defined in $0 \leq \omega \leq (m+1)E$. The $\rho_m(\omega)$ are defined piece-wise in the sub-intervals $(m' - 1) \leq \omega \leq m'E$ $(m' = 1, 2,\ldots m)$ and in each sub-interval it is a product of $\rho_o(\omega)$ and a function of the translated variable $\omega - m'E$ $(m' = 1, 2, m)$ (cf eq. (17.b)). The translated variables $\omega - mE$ introduce new energy scales $E_m := mE$ $(m = 1, 2,)$ and lead consequently to the breaking of dilatation invariance. The relationships between the derivatives $\rho^{(n)}(\omega)$ $(n = 1,2,\ldots)$ and $\rho(\omega - mE)$ $(m = 1,2,\ldots)$ express basically this fact. It is therefore important that these relationships between $\rho^{(n)}(\omega)$ and $\rho(\omega - mE)$ should be consistently satisfied. Therein lies the importance of the regularisation programme carried out in this paper.

FIG. 1 - Plot of the function $\rho_N(\omega)$ (given by the integral in eq (28) for n=O) as a function of cut - off T_N and its first three Cesàro means.

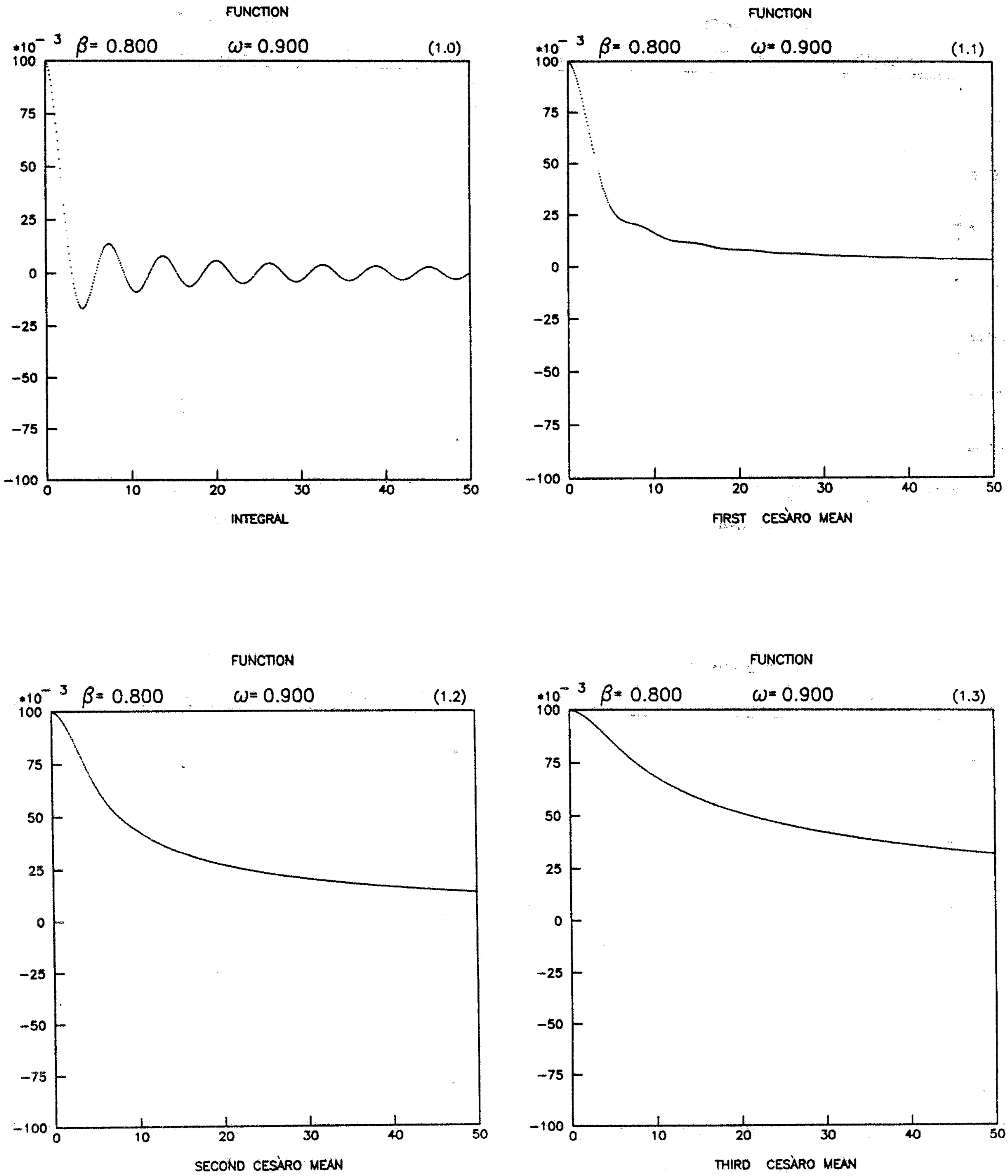


FIG. 2 - Same as Fig. 1 but for the first derivative, $\rho_N^{(1)}(\omega)$, of $\rho_N(\omega)$ with respect to ω.

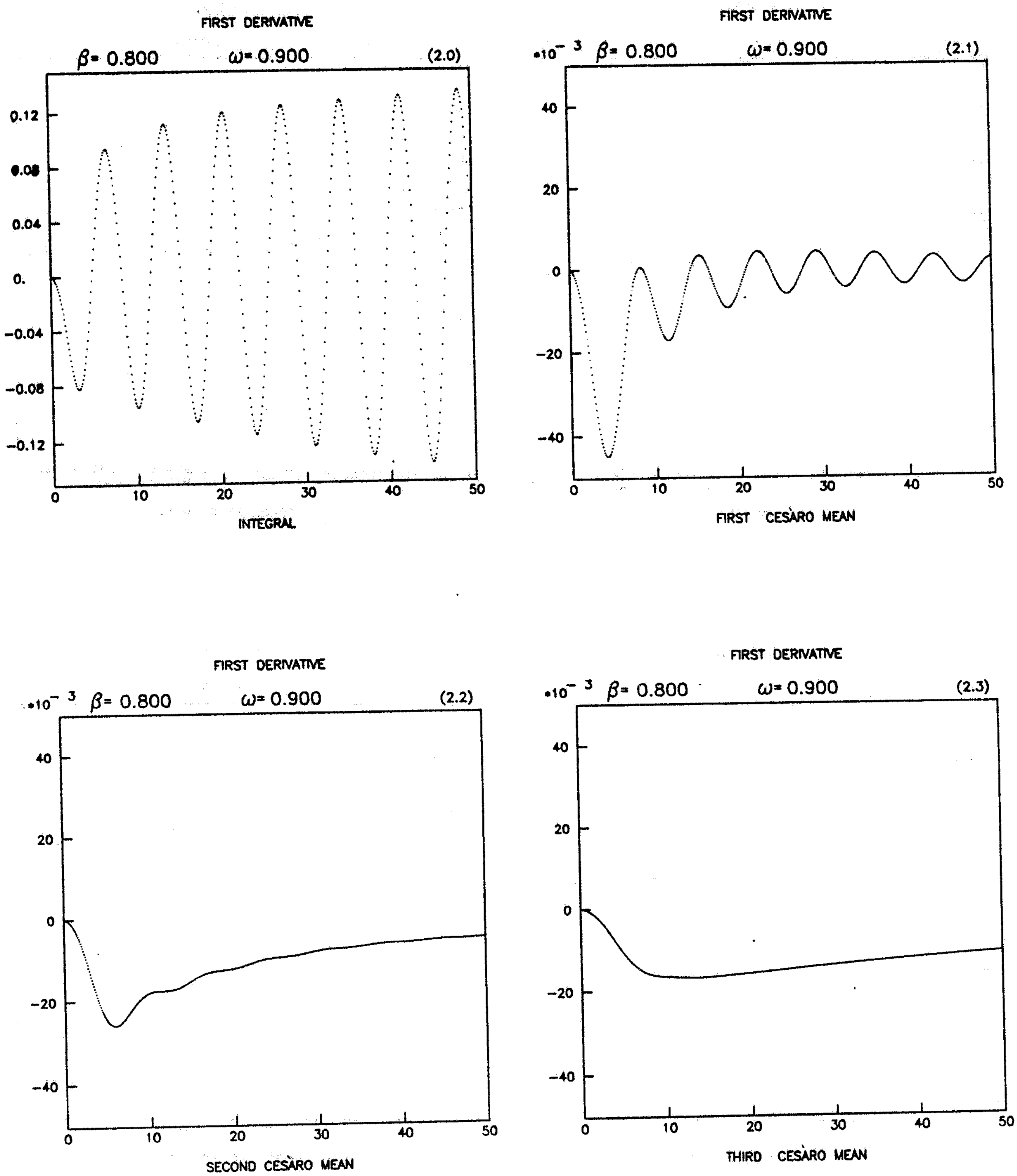


FIG. 3 - Same as Fig. 1 but for the second derivative, $\rho_N^{(2)}(\omega)$, of $\rho_N(\omega)$ with respect to ω.

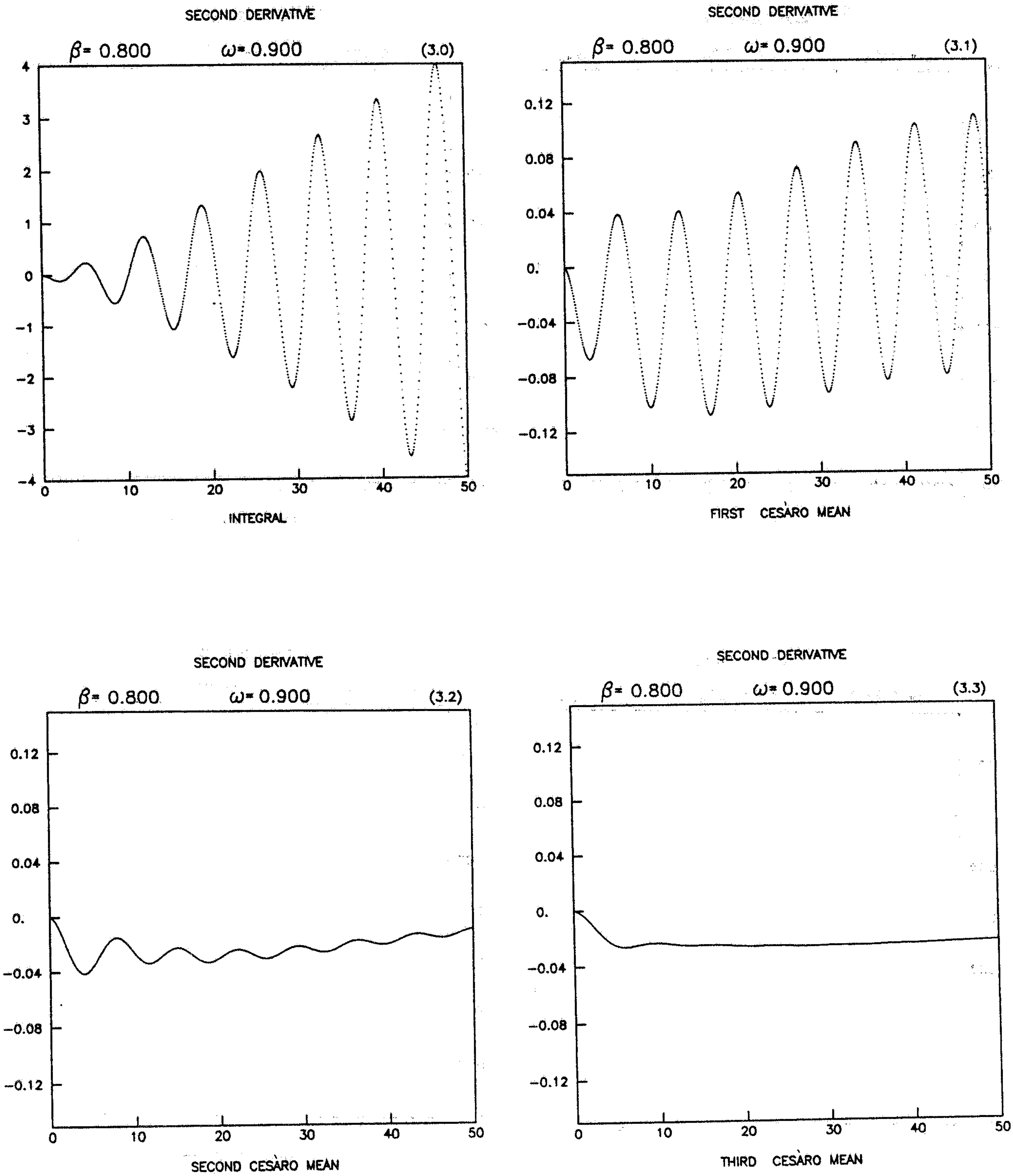


FIG. 4 - Same as Fig. 1 but for the third derivative, $\rho_N^{(3)}(\omega)$, of $\rho_N(\omega)$ with respect to ω.

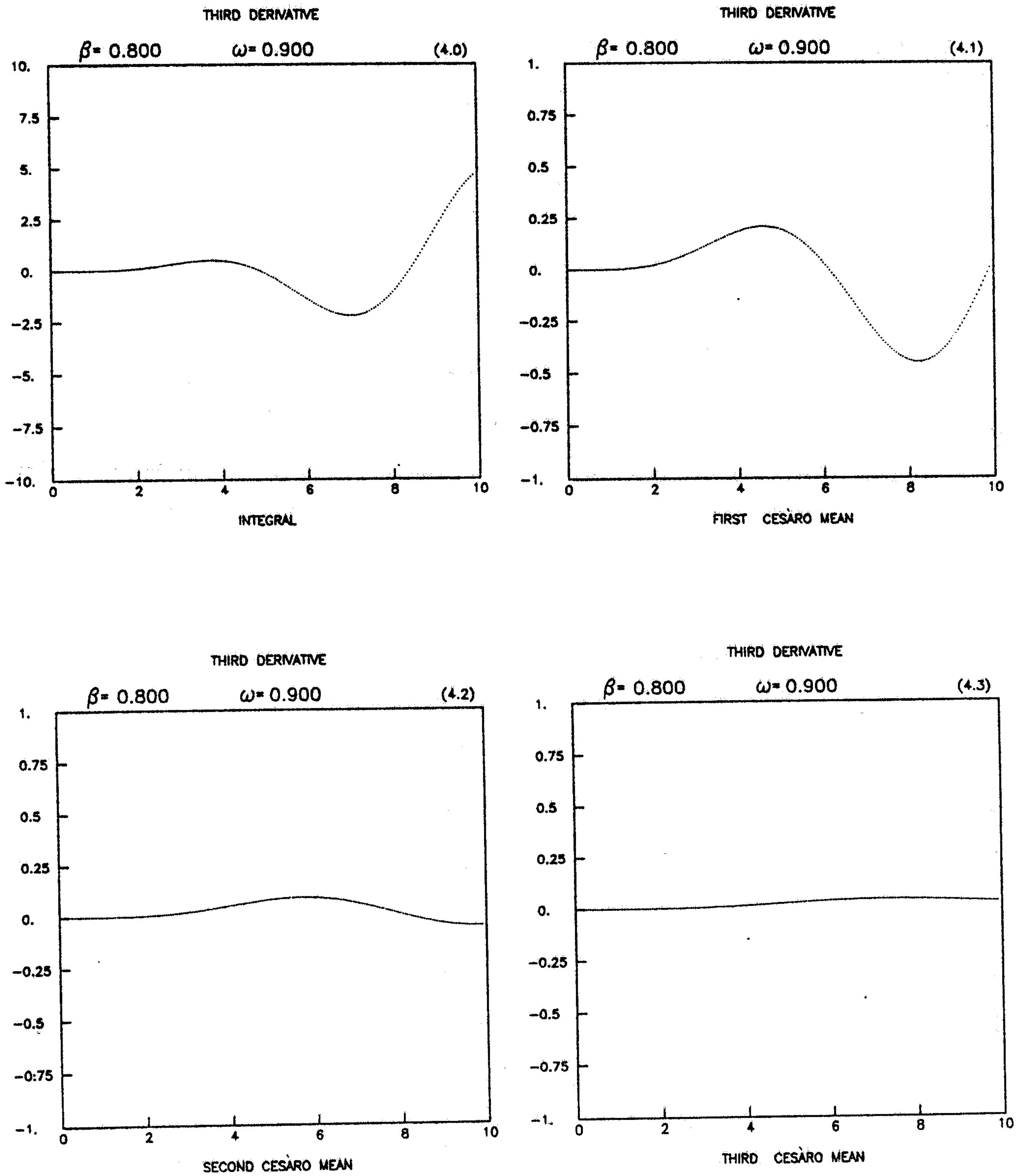


FIG. 5 - Plot of $\rho(\omega)$ as a function of ω.

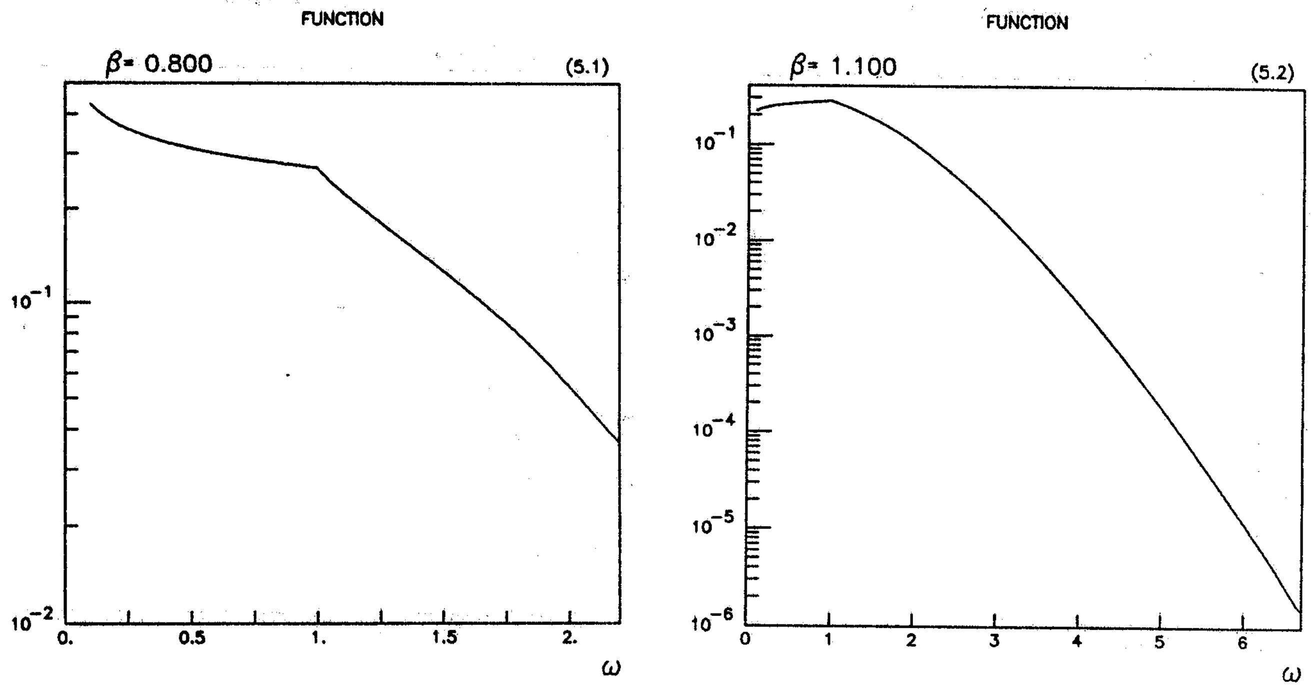


FIG. 6 - Plot of the first derivative, $\rho^{(1)}(\omega)$, of $\rho(\omega)$ as a function of ω.

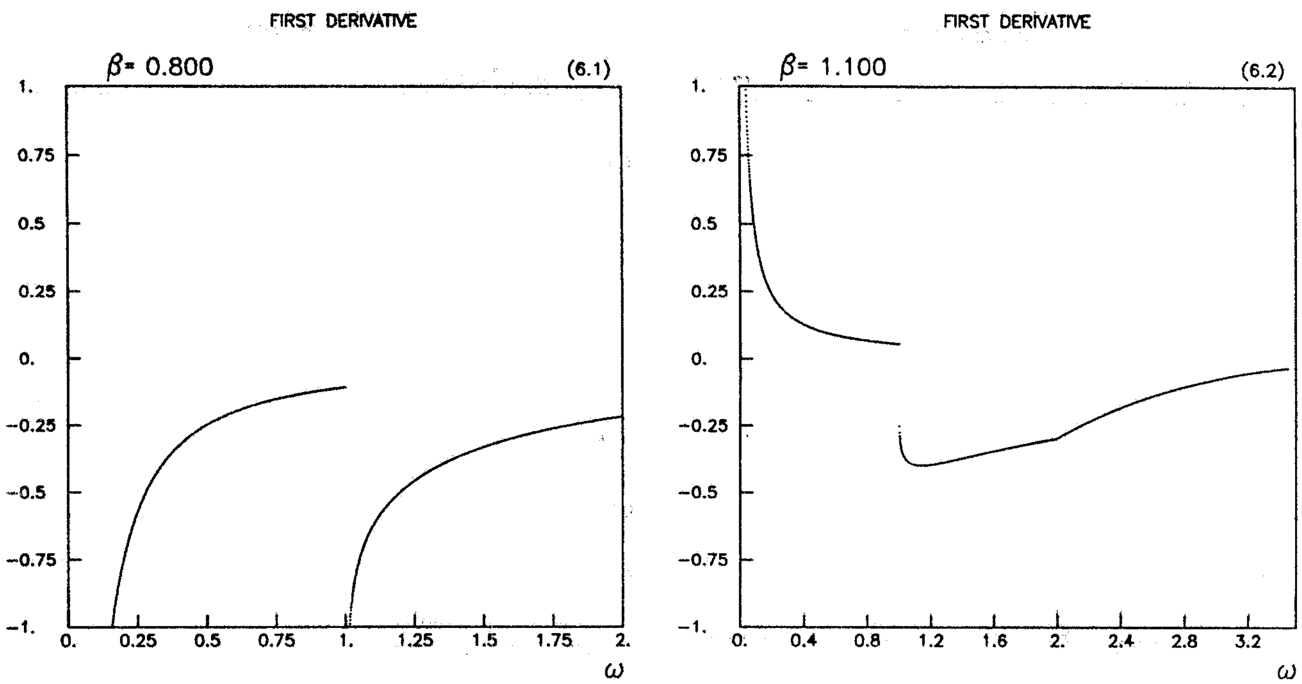


FIG. 7 - Plot of the second derivative, $\rho^{(2)}(\omega)$, of $\rho(\omega)$ as a function of ω.

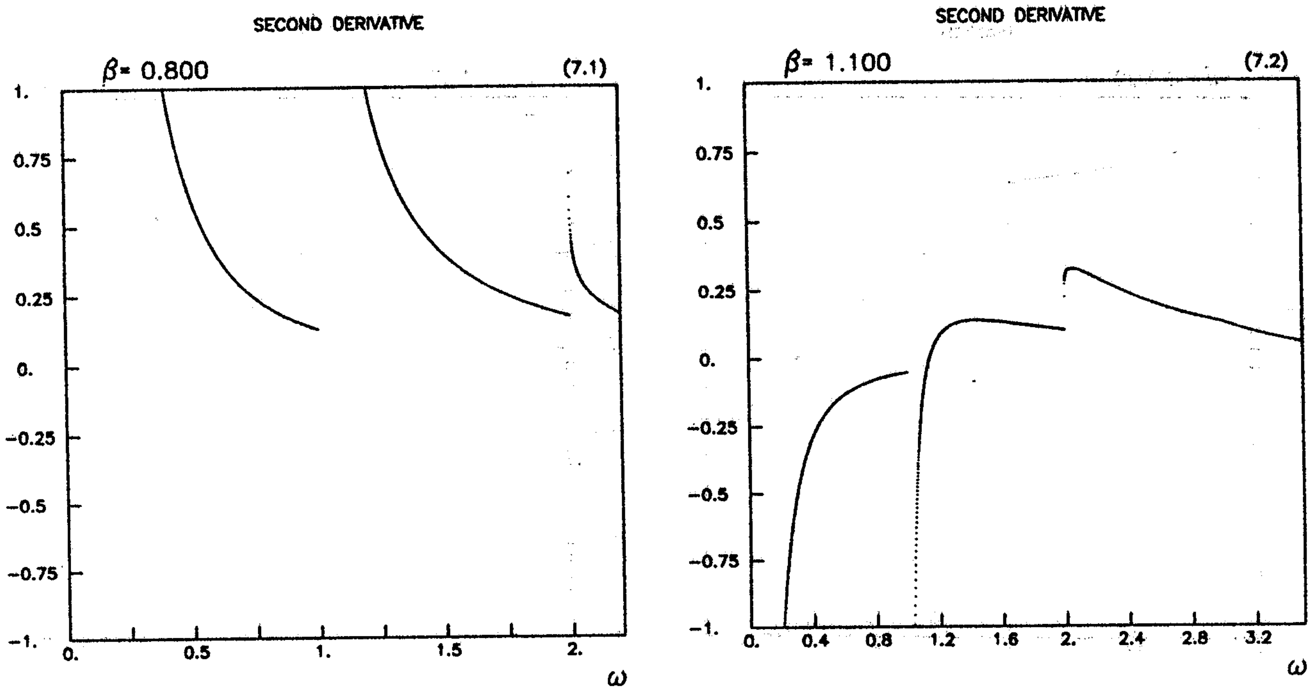
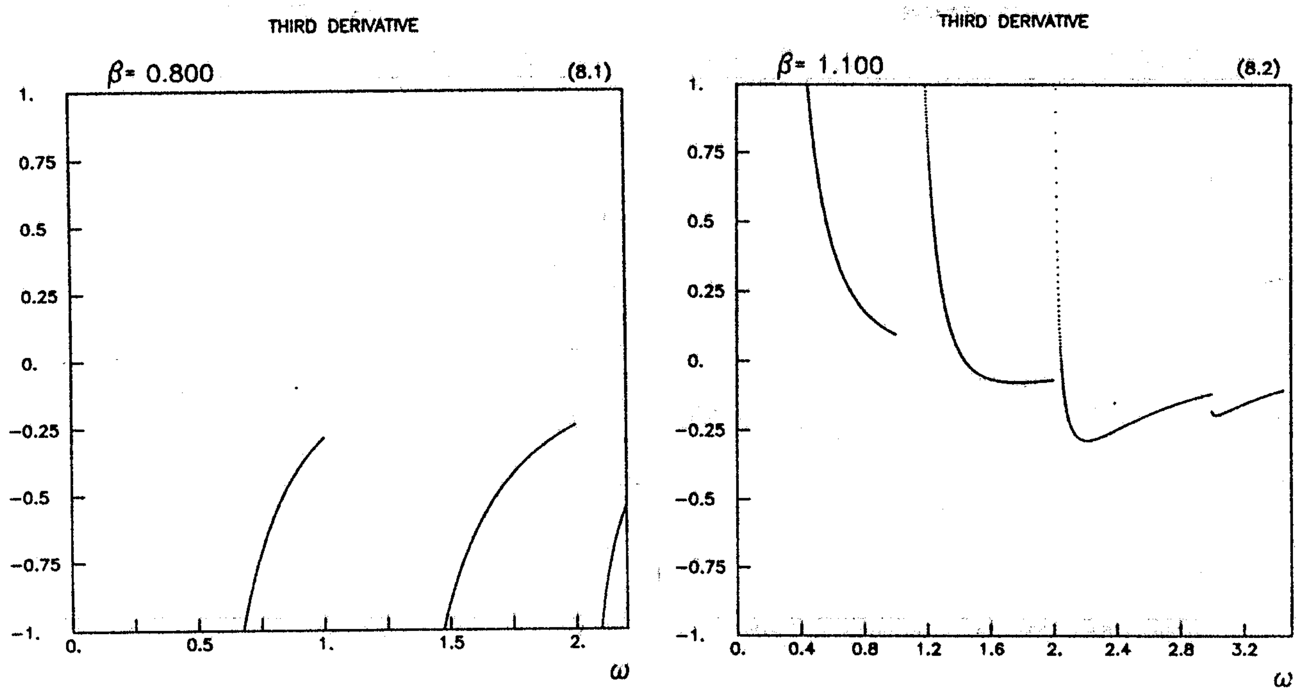


FIG. 8 - Plot of the third derivative, $\rho^{(3)}(\omega)$, of $\rho(\omega)$ as a function of ω.



REFERENCES

1. F. Bloch and A. Nordsieck; Phys. Rev. 52, 54 (1937).
2. J.M. Jauch and F. Rohrlich; The Theory of Photons and Electrons (Addison-Wesley Publishing Company Inc., Reading, Massachusetts, (1959)) ch. 16.
3. E. Etim, G. Pancheri and B. Touschek; Nuovo Cimento 51B, 276 (1967).
4. D.R. Yennie, S.C. Frautschi and H. Suura; Ann. Phys. (N.Y.)13, 379 (1961).
5. G. Pancheri and Y.N. Srivastava; Phys. Lett. 159B, 69(1985); 182B, 199 (1986).
6. G. Pancheri, Y.N. Srivastava and M. Pallotta, Phys. Lett. 151B (1985) 453. See also ref. (5).
7. Z. Koba, H.B. Nielsen and P. Olesen; Nucl. Phys. B40, 317 (1972).
8. G.H. Hardy; Divergent Series (Oxford University Press, London, (1956)) ch. 1. pp. 5-13; ch. 5 pp. 94-112
9. C. Basili, E. Etim and M. Pallotta, Frascati Preprint LNF-89/022(PT).
10. C. Basili, E. Etim and M. Pallotta; Frascati Preprint LNF-89/046(PT).
11. G. Pancheri and Y. Srivastava; Phys. Rev. D15, 2915 (1977).