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Laboratori Nazionali di Frascati

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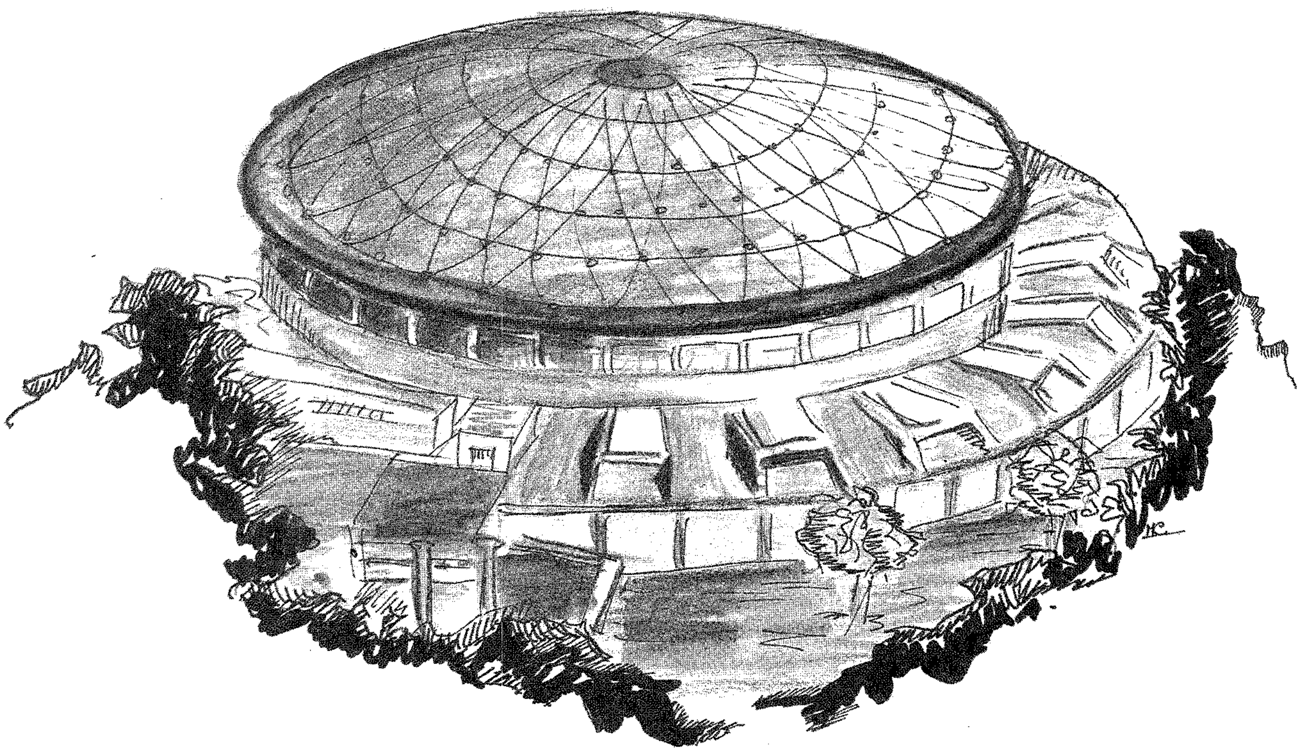

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SPECIAL CASES OF THE REGULARISATION OF THE CONFORMAL INVERSION TRANSFORM BY CESÀRO SUMMABILITY

C. Basili (*), E. Etim and M. Pallotta
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ABSTRACT

Hankel transforms are representations of the conformal inversion operator in Hilbert space. Their action on eigenfunctions of the dilatation operator exists and is consistent, however, only in the sense of analytically regularised integrals. Special cases of this action reduce to Fourier transforms, more specifically to sine and cosine transforms, of these eigenfunctions. These transforms are, in general, not Cauchy-convergent. It is shown here that they may be analytically regularised by Cesàro summability.

Regularisation is another name for summability. The basic idea of summability is that before an integral can be assumed to have a value it must first be given a definition. In other words, for any given integral, the first question about it should be "How is its value to be defined?" rather than "What is its value?"(1). And there can be different definitions, giving rise, in principle, to different values for the same integral. Uniqueness of the value of the integral is not the issue. The issue is consistency. The purpose of regularisation is to eliminate ambiguities. An ambiguity may be the classic one of outright divergence of the integral such as the ultraviolet divergences in quantum field theory. It may be failure to respect symmetry constraints of a theory as a result of naive manipulations of the integrals involved. This is the case of the famous Adler-Bell-Jackiw anomaly(2,3). Or more simply, the behaviour of the

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integral does not allow for the application of the standard Cauchy criteria for the existence of limits in order to draw conclusions about its convergence. For lack of a better name one says in this case that the integral is Cauchy - ambiguous. In all cases, the ambiguity is eliminated and consistency restored by a suitable procedure of regularisation.

In this paper we consider, basically, generalisations of the well known integral

$$\int_0^\infty dx \, \frac{\sin(x)}{x} = \frac{\pi}{2} \tag{1}$$

whose value is easily established by the method of residues. The generalisations are

$$A_+(\mu) := \int_0^\infty dx \;\; x^\mu \sin(x) \tag{2.a}$$

$$A_-(\mu) := \int_0^\infty dx \;\; x^\mu \cos(x) \tag{2.b}$$

where the parameter μ can have any value, real or complex. For $\mu = -1$, $A_+(\mu)$ reduces to the integral in eq. (1). The integrals in eqs. (2) are Cauchy-ambiguous for positive integer values of μ, i.e. $\mu = 0, 1, 2.$ This is easy to see by integration by parts. They are outright divergent for negative integer values of μ.

This behaviour does not come about quite as obviously as the behaviour of the integrands for $x \to o$ would suggest. It will be seen later that $A_+(\mu)$ diverges for even negative integer values of μ i.e. $\mu = -2N$ ($N = 1, 2, \ldots$) and $A_-(\mu)$ for odd negative integer values of μ, i.e. $\mu = -(2N+1)$ ($N = 0, 1, 2, \ldots$). The existence of these singularities is fundamentally different from the existence of the ambiguities at $\mu = 0, 1, 2, \ldots$. The singularities are those of analytic functions and cannot be eliminated by any method of regularisation. If it were possible to regulate them away, then the corresponding analytic functions would have no singularities. By Liouville's theorem these functions would then be constants, independent of μ. This would lead to a contradiction, for there are different values of μ for which the integrals are Cauchy-convergent to different values. On the other hand, the ambiguities at $\mu = 0, 1, 2, \ldots$ and at other values of μ, do not have a fundamental origin. They arise from the inapplicability of a particular definition of convergence. They do not exist in a wider theory of convergence(1). The application of this wider theory constitutes regularisation.

Before coming to this, let us remark that the integrals in eqs. (2.a) and (2.b) are special cases of Hankel transforms. Hence their regularisation re-enters as part of the regularisation of Hankel transforms discussed in ref. (4). To see this we recall that (5)

$$\sin(x) = \left(\frac{\pi x}{2}\right)^{1/2} J_{1/2}(x) \tag{3.a}$$

$$\cos(x) = \left(\frac{\pi x}{2}\right)^{1/2} J_{-1/2}(x) \tag{3.b}$$

where $J_{\pm 1/2}(x)$ is the Bessel function of order $\nu = \pm 1/2$. Substituting from eqs. (3) into (2), one gets the Hankel transforms

$$A_+(\mu) = \left(\frac{\pi}{2}\right)^{1/2} \int_0^\infty dx \; x^{\mu+1/2} \; J_{1/2}(x) \tag{4.a}$$

$$A_-(\mu) = \left(\frac{\pi}{2}\right)^{1/2} \int_0^\infty dx \; x^{\mu+1/2} \; J_{-1/2}(x) \tag{4.b}$$

As Hankel transforms, the integral in eq. (4.a) exists for $-2 < \mathrm{Re}(\mu) < -1$ while that in eq. (4.b) exists for $\mathrm{Re}(\mu)$ pinched at $\mathrm{Re}(\mu) = -1$ i.e. $-1 - \varepsilon < \mathrm{Re}(\mu) < -1 + \varepsilon$. The values of the integrals are[4,6]

$$\int_0^\infty dx\, x^{\mu+1/2} \; J_{1/2}(x) = 2^{\mu+1/2} \frac{\Gamma\left(1+\frac{\mu}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{\mu}{2}\right)} \tag{5.a}$$

$$\int_0^\infty dx \; x^{\mu+1/2} J_{-1/2}(x) = 2^{\mu+1/2} \frac{\Gamma\left(\frac{1}{2}+\frac{\mu}{2}\right)}{\Gamma\left(-\frac{\mu}{2}\right)} \tag{5.b}$$

Since the gamma function $\Gamma(z)$ is an analytic function of its argument with simple poles at $z = -N$ $(N = 0, 1, 2, \ldots)$, one uses this analyticity property in eqs. (5), (4) and (2) to define the integrals $A_\pm(\mu)$ in the whole μ-plane as

$$A_+(\mu) := \int_0^\infty dx \; x^\mu \; \sin(x) = \sqrt{\pi}\, 2^\mu \frac{\Gamma\left(1+\frac{\mu}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{\mu}{2}\right)} \tag{6.a}$$

$$A_-(\mu) := \int_0^\infty dx \; x^\mu \; \cos(x) = \sqrt{\pi}\, 2^\mu \frac{\Gamma\left(\frac{1}{2}+\frac{\mu}{2}\right)}{\Gamma\left(-\frac{\mu}{2}\right)} \tag{6.b}$$

The function $A_+(\mu)$ is therefore analytic in the μ-plane with singularities (simple poles) at

$$\mu = -2N \qquad (N = 1, 2, \ldots) \tag{7.a}$$

and zeroes at

$$\mu = 2N + 1 \qquad (N = 0, 1, 2, \ldots) \tag{7.b}$$

The singularities arise from the poles of the gamma function $\Gamma(1 + \mu/2)$ in the numerator of eq. (5.a) while the zeroes arise from the poles of $\Gamma(1/2 - \mu/2)$ in the denominator. Similarly the function $A_-(\mu)$ is analytic in the μ-plane with simple poles at

$$\mu = -(2N + 1) \qquad (N = 0, 1, 2, \ldots) \tag{8.a}$$

and zeroes at

$$\mu = -2N \qquad (N = 0, 1, 2, \ldots) \tag{8.b}$$

Using the Legendre duplication formula [7]

$$\Gamma(2z) = 2^{2z-1}\,\pi^{-1/2}\,\Gamma(z)\,\Gamma(\tfrac{1}{2}+z) \tag{9}$$

and the identities [8]

$$\Gamma(z)\;\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{10.a}$$

$$\Gamma(\tfrac{1}{2}+z)\;\Gamma(\tfrac{1}{2}-z) = \frac{\pi}{\cos(\pi z)} \tag{10.b}$$

eqs. . (6.a) and (6.b) can be re-expressed as

$$A_+(\mu) := \int_0^\infty dx\; x^\mu \sin(x) = \frac{-\pi}{2\,\Gamma(-\mu)\,\sin\left(\frac{\pi\mu}{2}\right)} \tag{11.a}$$

$$A_-(\mu) := \int_0^\infty dx\; x^\mu \cos(x) = \frac{\pi}{2\,\Gamma(-\mu)\,\cos\left(\frac{\pi\mu}{2}\right)} \tag{11.b}$$

Eqs. (11.a) and (11.b) constitute the regularisation of the integrals $A_\pm(\mu)$ by analytic continuation. We now wish to show that the regularisation of $A_\pm(\mu)$, and yielding the results in eqs. (11.a) and (11.b), may be obtained directly be Cesàro summability of the integrals involved[4]. This corresponds to a definition of convergence more general than that of Cauchy. It includes Cauchy convergecne as a special case [1]. For simplicity let us rewrite the integrals in eqs. (11.a) and (11.b) in the combined form

$$A_\pm(\mu) := \int_0^\infty dx\; x^\mu\, f_\pm(x) \tag{12}$$

with

$$f_+(x) := \sin(x) \tag{13.a}$$

$$f_-(x) := \cos(x) \tag{13.b}$$

Let

$$A_{\pm L}(\mu) := \int_0^L dx\; x^\mu\, f_\pm(x) \tag{14}$$

be the cut-off versions of the original integrals with the cut-off L. We choose an increasing sequence $L_1 < L_2 < \ldots < L_N$ $(N = 1, 2, \ldots)$ of these cut-offs and denote the corresponding cut-off integrals by

$$A_{\pm N}(\mu) \;:=\; \int_0^{L_N} dx \;\, x^{\mu}\; f_{\pm}(x) \qquad\qquad N = 1, 2, \ldots \tag{15}$$

Next, we form the hierarchy of arithmetic means

$$C^{(0)}_{\pm N}(\mu) \;:=\; A_{\pm N}(\mu) \tag{15.a}$$

$$C^{(k)}_{\pm N}(\mu) \;:=\; \frac{1}{N}\sum_{n=1}^{N} C^{(k-1)}_{\pm n}(\mu) \qquad\qquad k = 1, 2, \ldots \tag{15.b}$$

and investigate their limits

$$C^{(0)}_{\pm}(\mu) \;:=\; \lim_{N\to\infty} C^{(0)}_{\pm N}(\mu) \tag{16.a}$$

$$C^{(k)}_{\pm}(\mu) \;:=\; \lim_{N\to\infty} C^{(k)}_{\pm N}(\mu) \qquad\qquad k = 1, 2, \ldots \tag{16.b}$$

The $C^{(k)}_{\pm N}(\mu)$ $(k = 0, 1, 2, \ldots)$ are known as Cesàro means. The cut-off integrals $A_{\pm N}(\mu)$ and their Cesàro means $C^{(k)}_{\pm N}(\mu)$ are easily generated with the help of a computer. By the same means, one investigates their behaviour as functions of L_N $(N = 1, 2, 3 \ldots)$ and consequently the manner of approach to the limits defined in eqs. (16) when they exist. By definition, if the limit in eq. (16.a) exists then

$$C^{(0)}_{\pm}(\mu) \;:=\; A_{\pm}(\mu) \tag{17.a}$$

and the original integrals $A_{\pm}(\mu)$ are said to be Cauchy convergent. In this case the limits $C^{(k)}_{\pm N}(\mu)$ of all the Cesàro means $C^{(k)}_{\pm N}(\mu)$ $(k = 1,2,\ldots)$ also exist and one has

$$C^{(k)}_{\pm}(\mu) \;:=\; A_{\pm}(\mu) \qquad\qquad k = 1, 2, \ldots . \tag{17.b}$$

This follows from eqs. (15.a), (15.b) and (17.a). On the other hand if the limit in eq(16.a) does not exist, one of the limits in eq(16.b) may nevertheless exist. Let $\bar{k}$ be the minimum order of the Cesàro mean for which this happens. We then have the following situation: the limits

$$C^{(\bar{k}-k)}_{\pm}(\mu) \;:=\; \lim_{N\to\infty} C^{(\bar{k}-k)}_{\pm N}(\mu) \qquad\qquad k = 1, 2, ..\bar{k} \tag{18}$$

do not exist, but the limits

$$C_{\pm}^{(\bar{k}+k)}(\mu) \;:=\; \lim_{N\to\infty} C_{\pm N}^{(\bar{k}+k)}(\mu) \qquad k=0,1,2,.. \tag{19}$$

all exist and one has the equality

$$C_{\pm}^{(\bar{k}+k)}(\mu) \;=\; C_{\pm}^{(\bar{k})}(\mu) \tag{20}$$

In this case one defines the values of the original integrals $A_{\pm}(\mu)$ in the following extended sense

$$A_{\pm}(\mu) \;:=\; C_{\pm}^{(\bar{k})}(\mu) \tag{21}$$

and one says that the corresponding integrals are Cesàro summable to $C_{\pm}^{(\bar{k})}(\mu)$ or, equivalently Cesàro summable at order $\bar{k}$. Eq. (21) is nothing else but a condition of analytic continuation. Consequently Cesàro summability regularises the integrals $A_{\pm}(\mu)$ analytically. It is easily checked that the values of the integrals obtained in this way agree with eqs. (11.a) and (11.b). The results of the check are presented graphically in Figs. (1) and (2) for $\mu = 0$, 1 and 3/2. These plots give the behaviour of the Cesàro means $C_{\pm N}^{(k)}(\mu)$ as a function of L_N for $\mu = 0$, 1, 3/2 and $k = 0, 1, 2, 3$. They also illustrate the rapidity of the approach of the $C_{\pm N}^{(k)}(\mu)$ to their limits, when these exist. The limits when they exist, agree in all cases i.e. for $\mu = 0$, 1, 3/2, with the values of $A_{\pm}(\mu)$ given by eqs. (11.a) and (11.b).

It is useful to see the explicit values of the integrals $A_{\pm}(\mu)$ in eqs. (11) for a few values of μ. Here are some of these integrals:

<u>$\mu = -1$:</u>

$$\int_0^\infty dx\,\frac{\sin(x)}{x} \;=\; \frac{\pi}{2} \tag{22.a}$$

$$\int_0^\infty dx\,\frac{\cos(x)}{x} \;=\; \infty \tag{22.b}$$

<u>$\mu = -1/2$:</u>

$$\int_0^\infty dx\,\frac{\sin(x)}{\sqrt{x}} \;=\; \sqrt{\frac{\pi}{2}} \tag{23.a}$$

$$\int_0^\infty dx\,\frac{\cos(x)}{\sqrt{x}} \;=\; \sqrt{\frac{\pi}{2}} \tag{23.b}$$

$\underline{\mu = 0:}$

$$\int_0^\infty dx \ \sin(x) \ = \ 1 \tag{24.a}$$

$$\int_0^\infty dx \ \cos(x) \ = \ 0 \tag{24.b}$$

$\underline{\mu = 1/2:}$

$$\int_0^\infty dx \ \sqrt{x} \ \sin(x) \ = \ \sqrt{\frac{\pi}{8}} \tag{25.a}$$

$$\int_0^\infty dx \ \sqrt{x} \ \cos(x) \ = \ \sqrt{\frac{\pi}{8}} \tag{25.b}$$

$\underline{\mu = 1:}$

$$\int_0^\infty dx \ x \ \sin(x) \ = \ 0 \tag{26.a}$$

$$\int_0^\infty dx \ x \ \cos(x) \ = \ -1 \tag{26.b}$$

$\underline{\mu = 3/2:}$

$$\int_0^\infty dx \ x^{3/2} \ \sin(x) \ = \ \frac{3}{8} \ \sqrt{2\pi} \tag{27.a}$$

$$\int_0^\infty dx \ x^{3/2} \ \cos(x) \ = \ -\frac{3}{8} \ \sqrt{2\pi} \tag{27.b}$$

In conclusion we recall the physical context underlying these integrals. As shown in ref(4) the conformal inversion operator is represented in Hilbert space by a Hankel transform. Let R denote the inversion operator and $D\lambda$ a dilatation by the parameter λ in the configuration space of the variable x. One has, by definition

$$x' \ := \ Rx \ = \ \frac{1}{x} \tag{28}$$

$$x'' \ := \ D_\lambda x \ = \ \lambda x \tag{29}$$

and consequently

$$R \, D_\lambda \ = \ D_{1/\lambda} R \tag{30}$$

Let

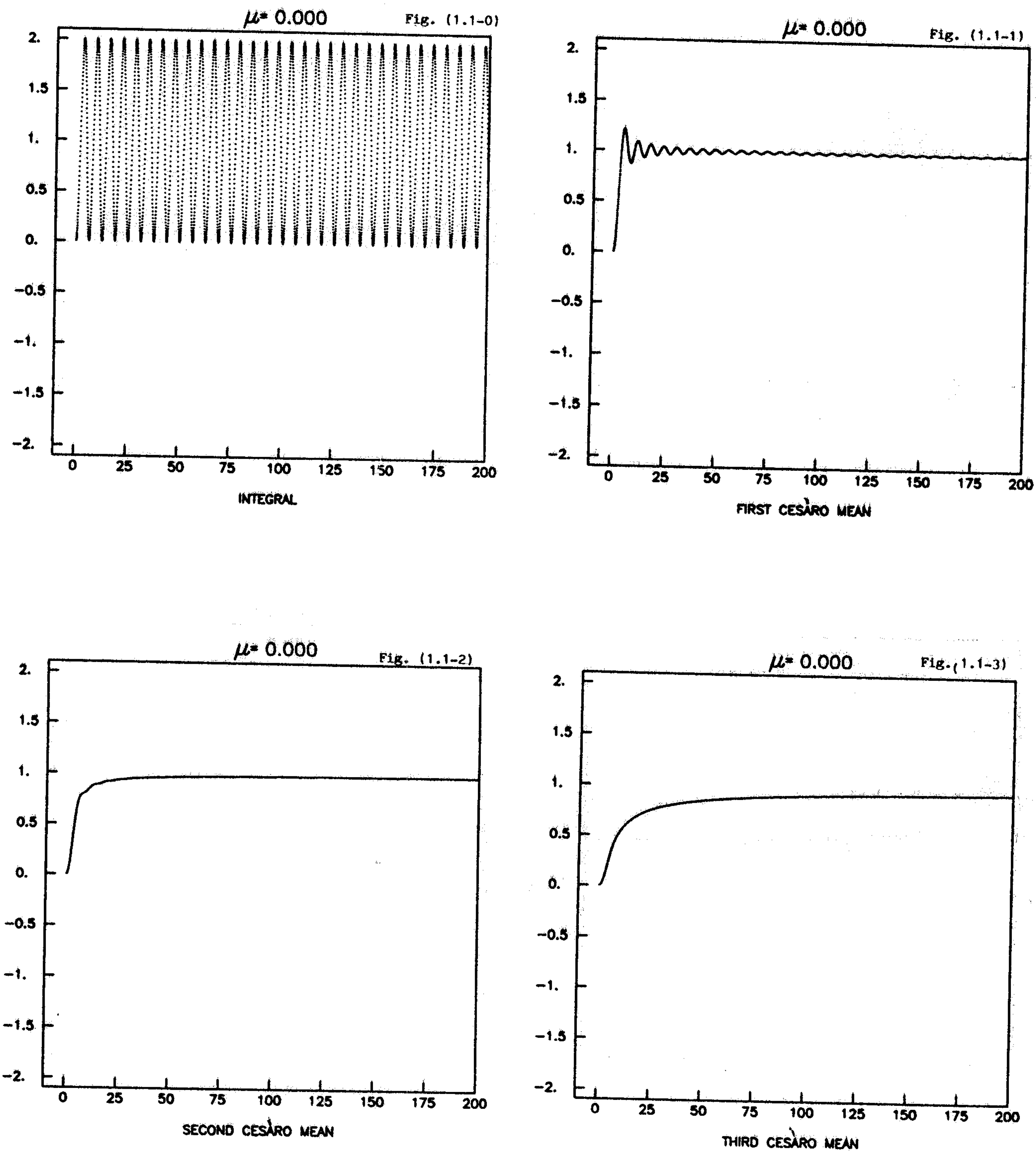
$$D \ := \ \left[\frac{dD_\lambda}{d\lambda} \right]_{\lambda = 1} \tag{31}$$

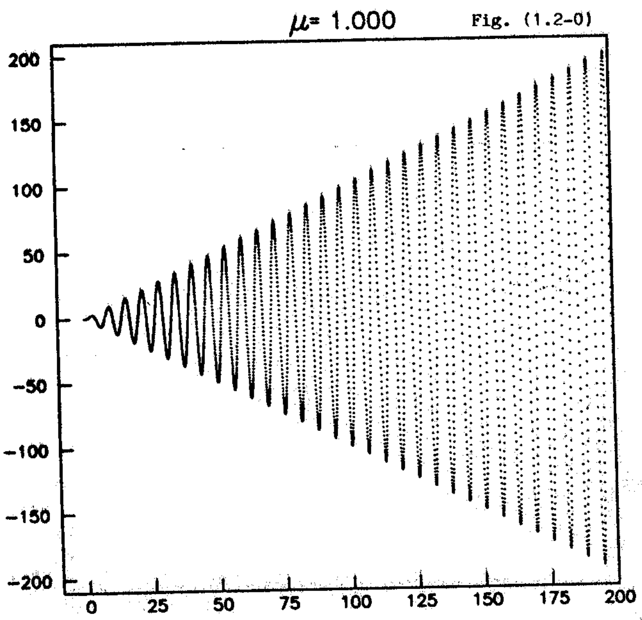
stand for the infinitesimal generator of dilatations. Then from eqs. (30) and (31) one finds that R and D anticommute, i.e.

$$R\,D + D\,R = 0 \tag{32}$$

These are well known results. There is nothing ambiguous about eq(32) and no infinities involved. However the representation of eq(32) in Hilbert space is not ambiguity - or infinity - free. This is related to the fact that the operator D is not bounded. The consequence is that the representation of eq(32) in Hilbert space does not exist except in the generalised sense of analytically regularised integrals. This regularisation is what we have been concerned with in this paper. The integrals involved need not necessarily have direct physical applications. The operators e.g. R and D, which the elements of the integrals represent have direct physical meanings. For instance, the function x^{μ} in the integrals considered is an eigenfunction of the dilatation operator D with eigenvalue μ. The functions sin(x) and cos(x), as we have seen, are related to the Bessel functions $J_{\pm 1/2}(x)$ [cf. eqs. (3)]. The resulting Hankel transforms represent in Hilbert space the action of the operator R on eigenfunctions of D. A consistent mathematical implementation of this action requires regularisation. This is our thesis. The analytic regularisaton of the integrals considered in this paper is a practical application of this thesis.

FIG. 1 - Plots of the integral $A_+(\mu)$ in eq. (2.a) as a function of cut-off and the first three Cesaro means of the integral for μ=0,1,3/2.





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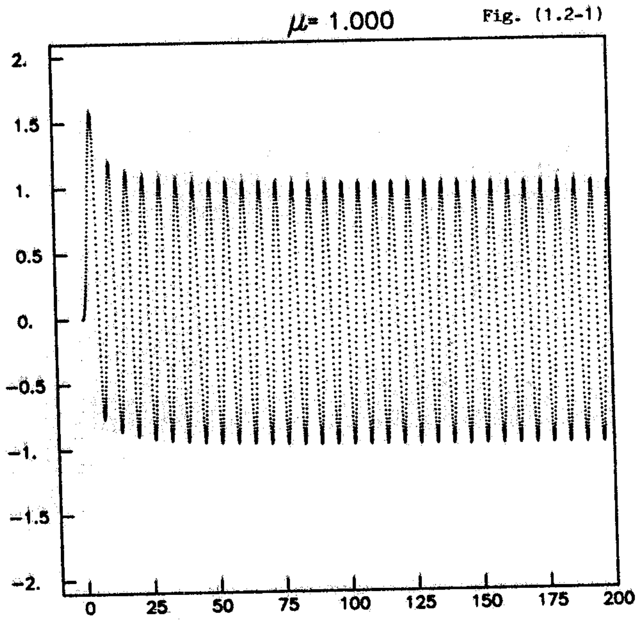

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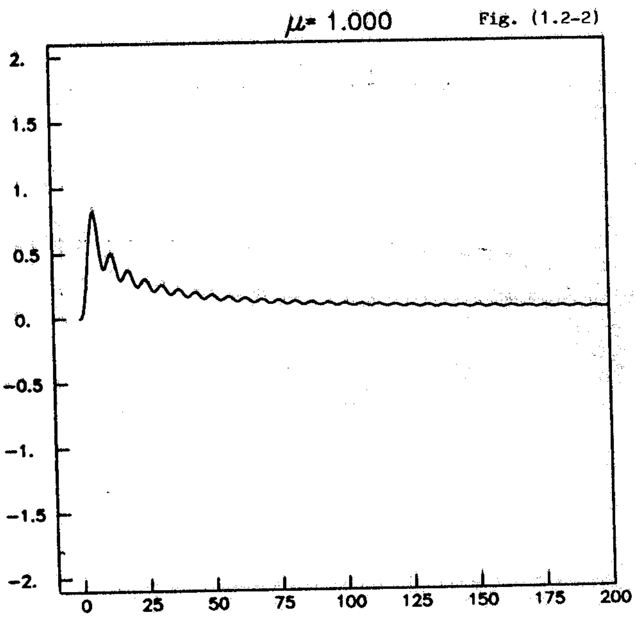
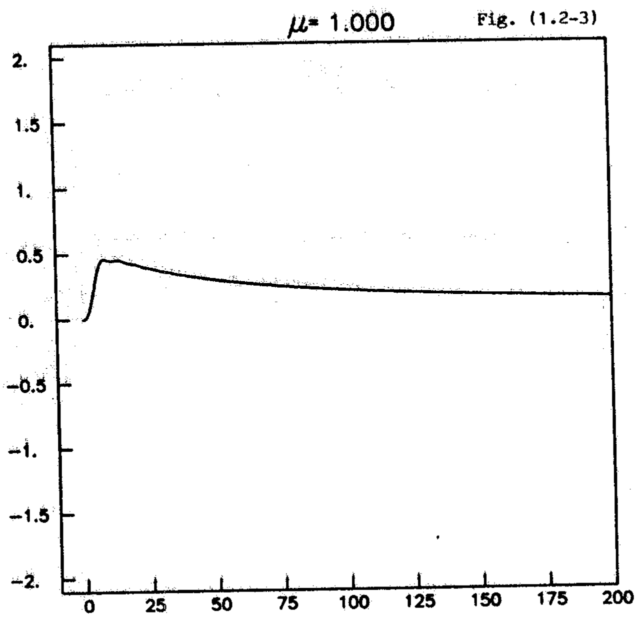


SECOND CESÀRO MEAN



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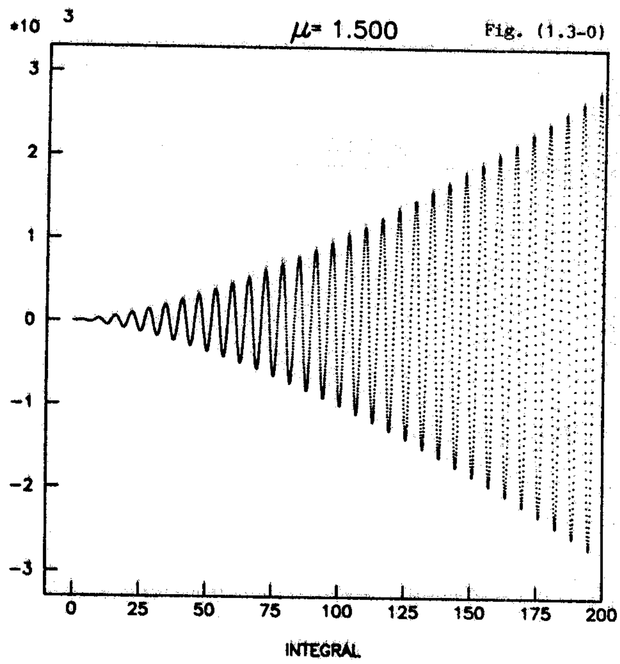
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Fig. (1.3-0)
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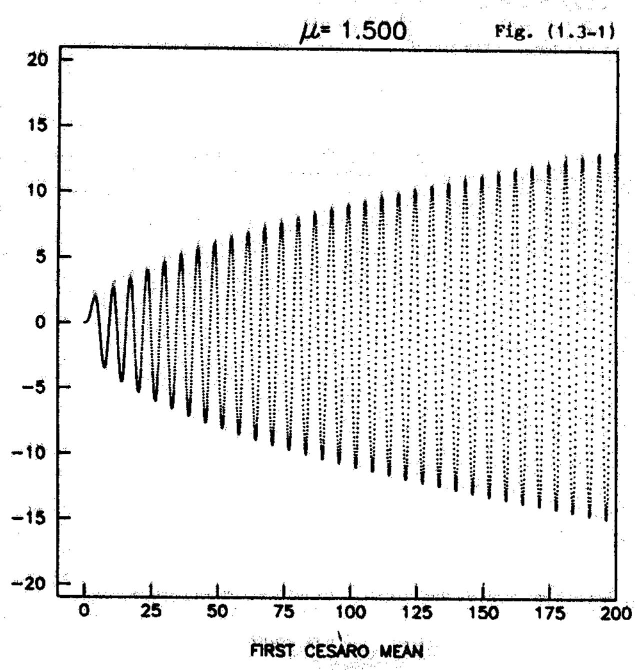
INTEGRAL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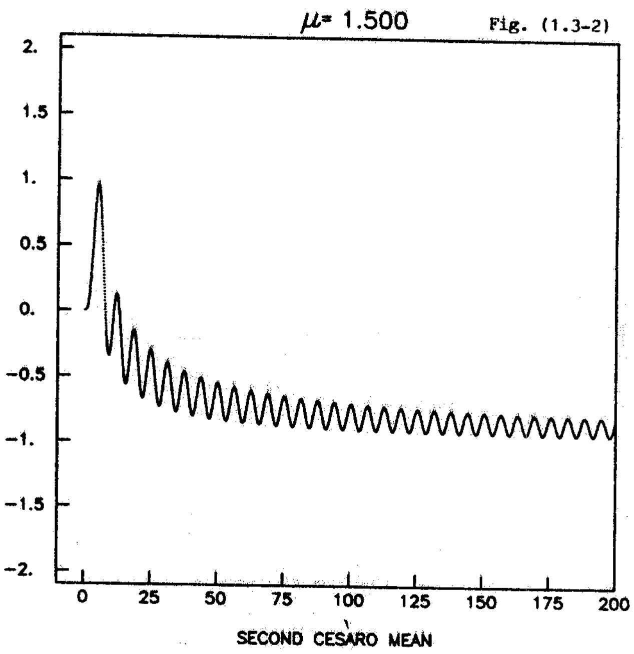
FIRST CESÀRO MEAN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SECOND CESÀRO MEAN

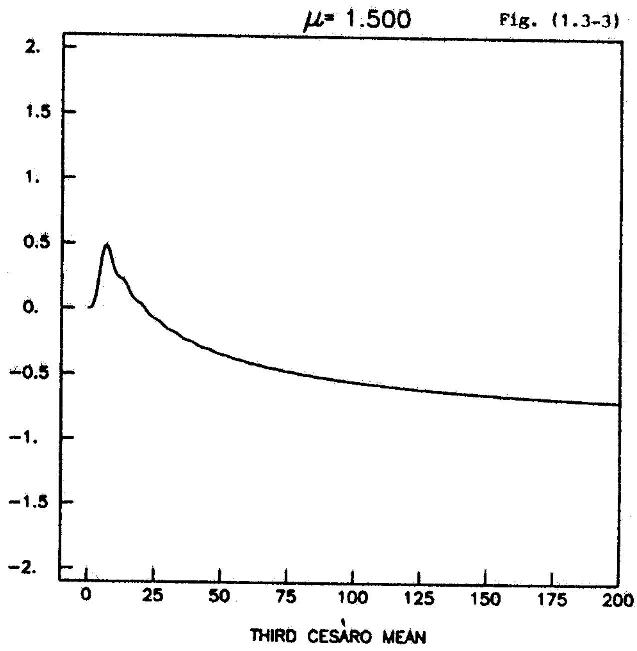
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Fig. (1.3-3)
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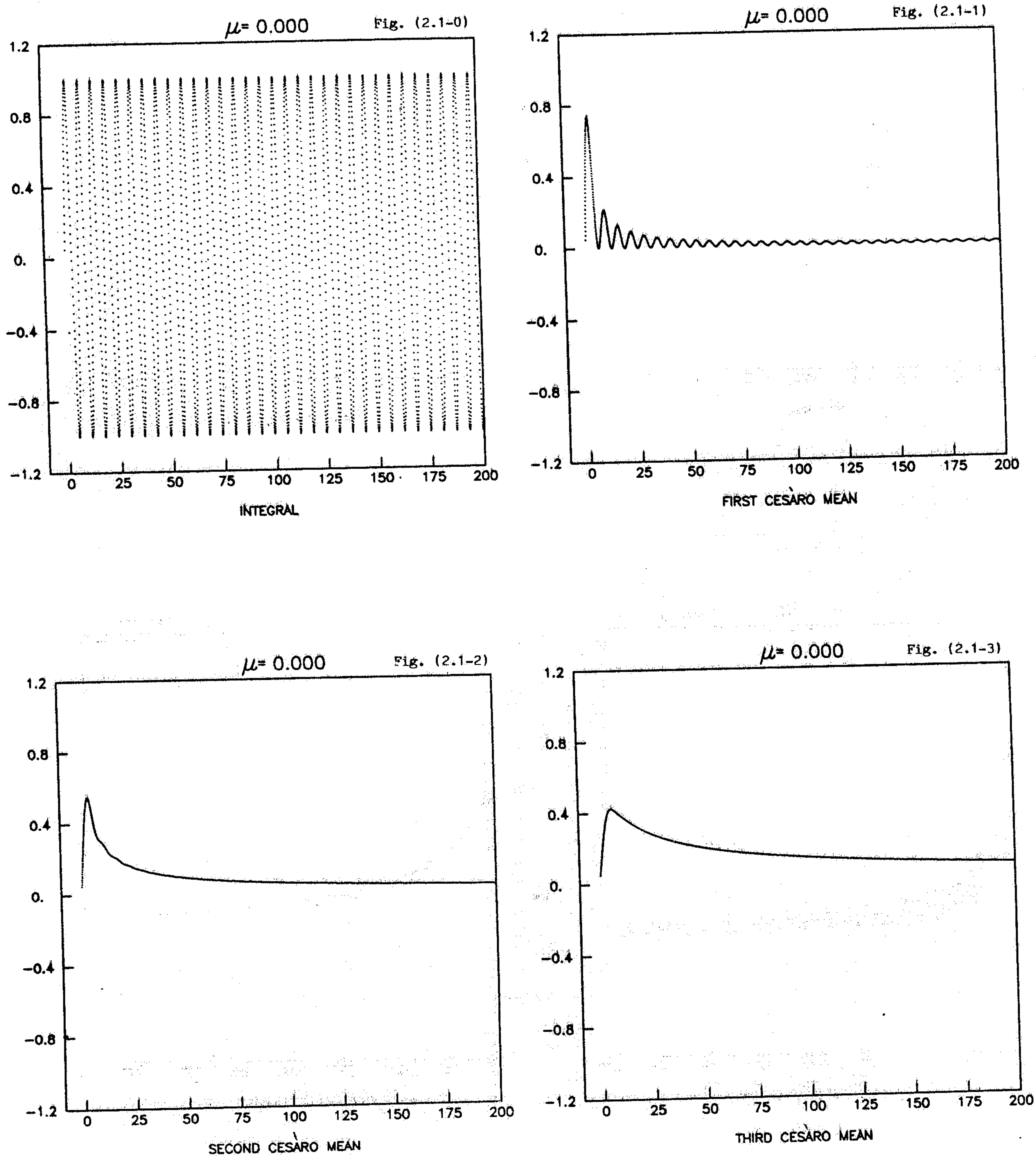
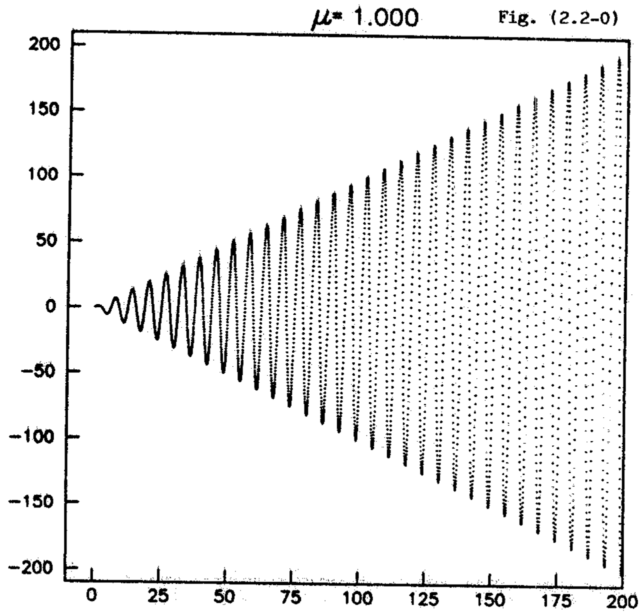
THIRD CESÀRO MEAN

FIG. 2 - Plots of the integral A_(μ) in eq. (2.b) as a function of cut-off and the first three Cesaro means of the integral for μ=0,1,3/2.

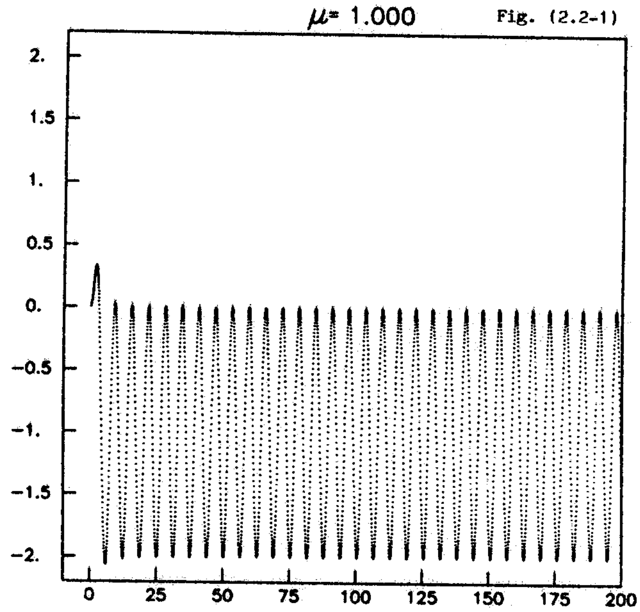


μ= 1.000 Fig. (2.2-0)



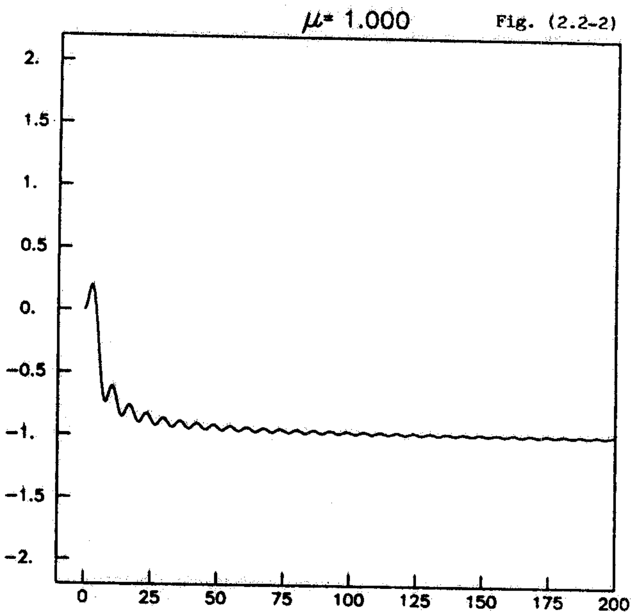
INTEGRAL

μ= 1.000 Fig. (2.2-1)



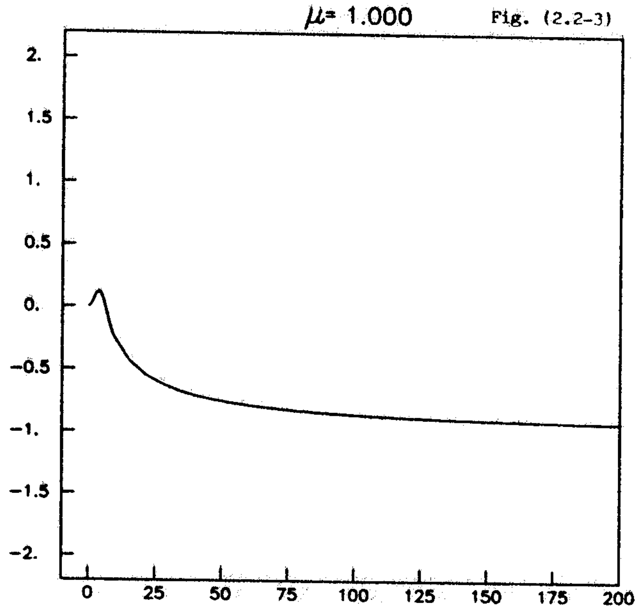
FIRST CESÀRO MEAN

μ= 1.000 Fig. (2.2-2)



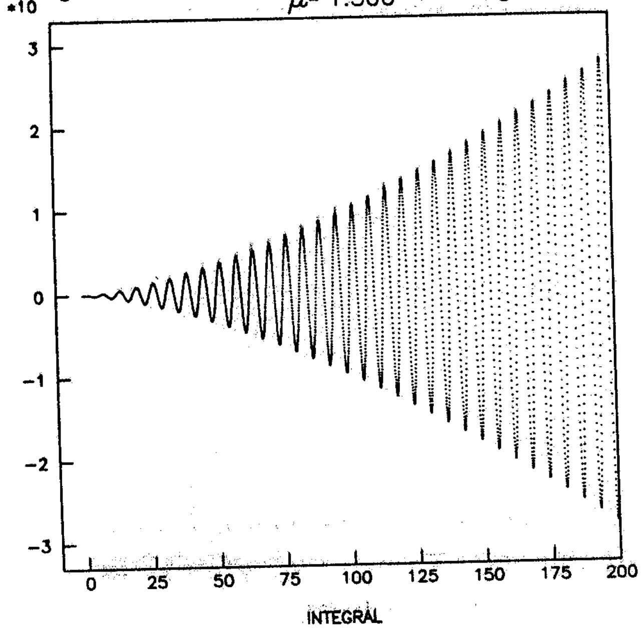
SECOND CESÀRO MEAN

μ= 1.000 Fig. (2.2-3)

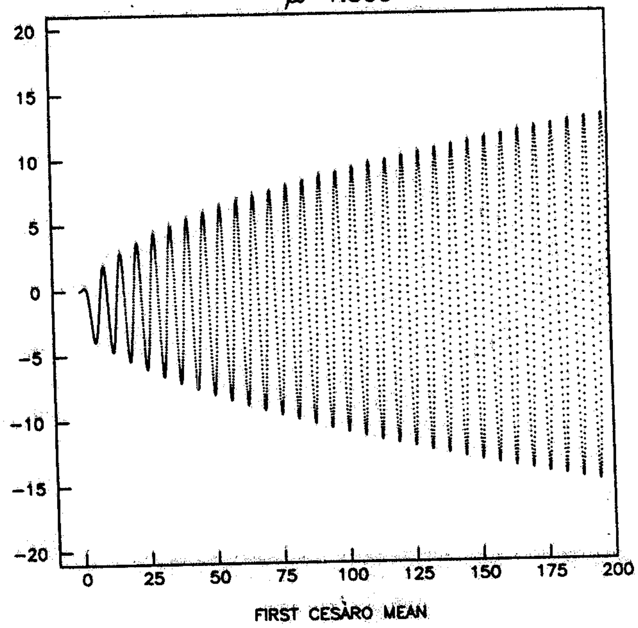


THIRD CESÀRO MEAN

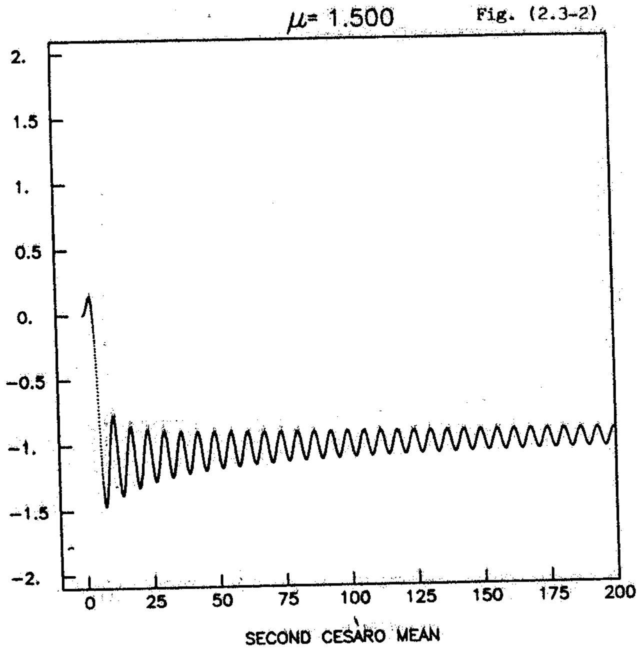
μ= 1.500 Fig. (2.3-0)



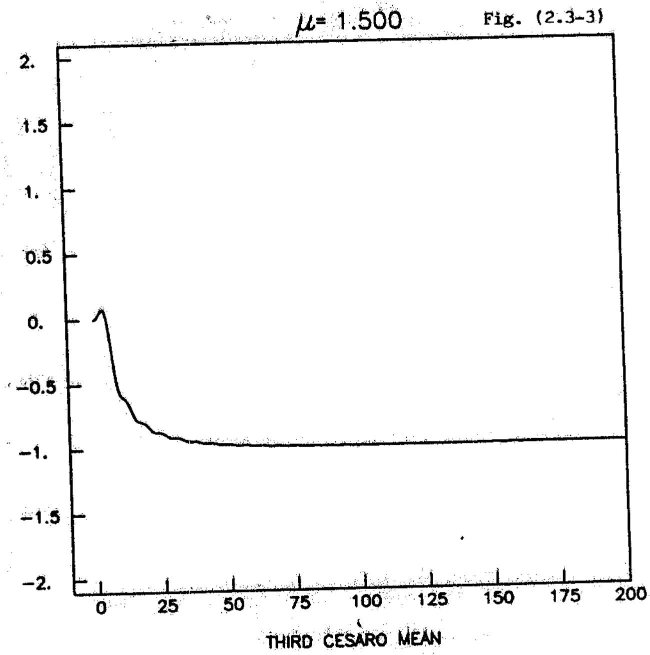
μ= 1.500 Fig. (2.3-1)



μ= 1.500 Fig. (2.3-2)



μ= 1.500 Fig. (2.3-3)



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8. See Ref(7) p. 3, eqs. (6) and (7).