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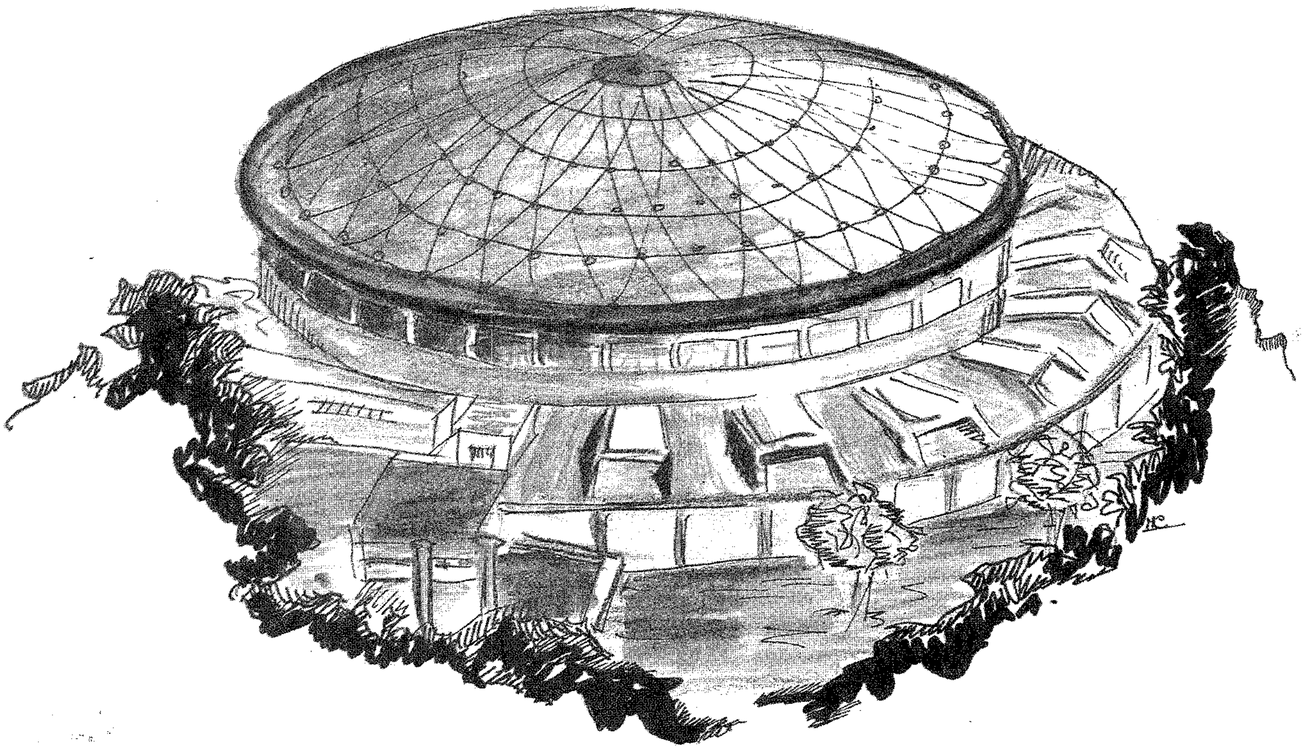

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NONCOMPACT FORMULATION OF ABELIAN GAUGE THEORIES ON THE LATTICE AND CHARGE QUANTIZATION

G. De Franceschi
INFN - Laboratori Nazionali di Frascati, Via E. Fermi, 40 - 00044 FRASCATI (Roma).
Dipartimento di Fisica dell'Università La Sapienza - Roma

F. Palumbo
INFN - Laboratori Nazionali di Frascati, Via E. Fermi, 40 - 00044 FRASCATI (Roma).

ABSTRACT

It is pointed out that in the Hamiltonian formulation of gauge theories on a lattice consistency of the Gauss constraint requires that the gauge group be compact. As a consequence the electric charge is quantized irrespective of the compactness of the transverse gauge field.

1. Recently it has been observed that the Gauss constraint, when enforced as a condition on the states, may have some special implications due to global properties of the operator which represents it[(1)]. The consequence on non abelian gauge theories has been analyzed in ref. (2). In this letter we show that one such implication is present also in the abelian case where, under the standard assumptions of quantization, consistency of the Gauss constraint requires that the longitudinal gauge field be the gradient of an angle which is the gauge variable. The gauge group must therefore be compact irrespective of the compactness of the transverse field. This has the consequence that the electric charge is quantized irrespective of the compactness of the gauge field.

We construct on the lattice the Hamiltonian with noncompact transverse fields and the transfer matrix starting from the Hamiltonian. We find that Euclidean invariance is broken in two ways, one common to all the formulations based on a Hamiltonian, and another specific to the present way of constructing the Hamiltonian. At the end of the paper we will make some remarks concerning this latter breaking.

2. Let us start by writing the Hamiltonian density in the continuum, by separating the transverse from the longitudinal gauge field and adopting the convention of summation over repeated indices.

Putting the electric charge g = 1, the Hamiltonian density is

$$\mathcal{H} = \frac{1}{2} E_T^2 - \frac{1}{2} \partial_k E_k \Delta^{-1} \partial_h E_h + \frac{1}{4} F_{hk}^2 \quad + i \psi \gamma_k [\partial_k + i (A_{Tk} + A_{Lk})] \psi + m \overline{\psi} \psi. \quad (1)$$

In the above equation the transverse fields are defined by

$$\begin{aligned} A_{Th} &= (\delta_{hk} - L_{hk}) A_k \\ E_{Th} &= (\delta_{hk} - L_{hk}) E_k, \end{aligned} \quad (2)$$

L_{hk} being the longitudinal projection operator

$$(L_{hk} A_k)(x) = - \int d^3 y\, \partial_h^x \partial_k^y \Delta^{-1} (x - y) A_k (y). \quad (3)$$

Anticipating that the longitudinal gauge field must the gradient of an angle, we cannot define it in terms of A_k, and we must introduce a new variable φ with its conjugate momentum π

$$A_{Lk} = \partial_k \varphi$$
$$E_{Lk} = \partial_k \Delta^{-1} \pi. \tag{4}$$

The independent variables are therefore A_h, φ, with conjugate momenta E_h, π, respectively, but the Hamiltonian does not depend on

$$\bar{A}_{Lh} = L_{hk} A_k$$
$$\bar{E}_{Lh} = L_{hk} E_k. \tag{5}$$

Finally the magnetic strength is

$$F_{hk} = \partial_h A_k - \partial_k A_h. \tag{6}$$

The Gauss constraint reads

$$(\pi + \psi^* \psi) \Psi = 0. \tag{7}$$

The Gauss operator $\pi + \psi^* \psi$ generates the gauge transformations

$$\varphi \to \varphi + \lambda$$
$$\psi \to e^{-i\lambda} \psi. \tag{8}$$

Now in order to put the Hamiltonian on a lattice we must define in a gauge-invariant way the product of matter fields at different points. We use the splitting point procedure in the form

$$\psi^+ (x)\, e^{-i \int_y^x dx_k A_{Lk}}\, \psi (y). \tag{9}$$

Notice that in the exponent only the longitudinal field appears. Because of Eq. (4), Eq. (9) can be rewritten

$$\psi^+ (x)\, e^{-i\varphi(x)}\, e^{+i\varphi(y)}\, \psi (y). \tag{10}$$

It is therefore convenient to introduce the gauge-invariant field

$$\chi = e^{+i\varphi}\,\psi. \tag{11}$$

This allows us to rewrite the Hamiltonian density in terms of gauge-invariant fields

$$\mathcal{H} = \frac{1}{2} E_T^2 - \frac{1}{2}\pi\,\Delta^{-1}\,\pi + \frac{1}{4} F_{hk}^2 + i\,\bar{\chi}\,\gamma_k\,[\partial_k + i\,A_{Tk}]\,\chi + m\,\bar{\chi}\,\chi. \tag{12}$$

Such an expression can be directly transferred on a lattice because replacing continuous derivatives by discrete derivatives does not spoil the gauge invariance.

We define on the lattice dimensionless fields according to

$$\begin{aligned} &A_h(\mathbf{x}) \rightarrow a^{-1}\,A_h(\mathbf{r}) \\ &E_h(\mathbf{x}) \rightarrow a^{-2}\,E_h(\mathbf{r}) \\ &\varphi(\mathbf{x}) \rightarrow \varphi(\mathbf{r}) \\ &\pi(\mathbf{x}) \rightarrow a^{-3}\,\pi(\mathbf{r}) \\ &\psi(\mathbf{x}) \rightarrow a^{-3/2}\,\psi(\mathbf{r}), \end{aligned} \tag{13}$$

where a is the lattice spacing and **r** a lattice vector. Notice that the gauge fields are defined on the sites, not on the links.

We define derivatives according to

$$\Delta_k f(\mathbf{r}) = f(\mathbf{r} + \hat{k}) - f(\mathbf{r}) \tag{14}$$

but the laplascian

$$\Delta f(\mathbf{r}) = \sum_{k=1}^{3} f(\mathbf{r} + \hat{k}) + f(\mathbf{r} - \hat{k}) - 2 f(\mathbf{r}) \neq \sum_{k=1}^{3} \Delta_k\,\Delta_k. \tag{15}$$

The inverse laplascian is

$$\Delta^{-1}(\mathbf{r}) = \sum_{\mathbf{p}} G(\mathbf{p})\, e^{i\frac{2\pi}{N}\mathbf{p}\cdot\mathbf{r}}, \quad -\frac{N-1}{2} \le p_h \le \frac{N-1}{2}, \tag{16}$$

where N is the (odd) number of links and

$$G(\mathbf{p}) = \begin{cases} \{2N^3 \sum_{h=1}^{3} \cos\frac{2\pi}{N} p_h - 1\}^{-1}, & \mathbf{p} \neq 0 \\ 0, & \mathbf{p} = 0. \end{cases} \tag{17}$$

The inverse laplascian satisfyies the relation

$$\Delta\Delta^{-1}(\mathbf{r}-\mathbf{r}') = \Delta^{-1}(\mathbf{r}-\mathbf{r}')\,\Delta = \delta_{\mathbf{r},\mathbf{r}'} - \frac{1}{N^3}. \tag{18}$$

Δ^{-1} is the inverse of Δ only on functions with vanishing zero-momentum components. We assume all the variables A_i, E_i, φ and π to have vanishing zero-momentum components. The zero-momentum components of the gauge field should be explicitely added to A_T, but we will disreguard them.

The operator

$$L_{hk}(\mathbf{r},\mathbf{r}') = \Delta_h(\mathbf{r})\,\Delta_k(\mathbf{r}')\,\Delta^{-1}(\mathbf{r}-\mathbf{r}') \tag{19}$$

conserves its projector property, which allows us to define transverse fields on the lattice

$$A_{Th}(\mathbf{r}) = [\delta_{hk}\,\delta_{\mathbf{r},\mathbf{r}'} - L_{hk}(\mathbf{r},\mathbf{r}')]\,A_k(\mathbf{r}')$$

$$E_{Th}(\mathbf{r}) = [\delta_{hk}\,\delta_{\mathbf{r},\mathbf{r}'} - L_{hk}(\mathbf{r},\mathbf{r}')]\,E_k(\mathbf{r}'). \tag{20}$$

Finally we can write Hamiltonian and Gauss constraint

$$H = \frac{1}{a}\sum_{\mathbf{r}} \frac{1}{2}E_T^2 - \frac{1}{2}\pi\,\Delta^{-1}\,\pi + \frac{1}{4}F_{hk}^2 + i\,\bar{\chi}\,\gamma_k\,(\partial_k + i\,A_{Tk})\,\chi + am\,\bar{\chi}\chi, \tag{21}$$

$$[\pi(\mathbf{r}) + \chi^*(\mathbf{r})\,\chi(\mathbf{r})]\,\Psi = 0. \qquad (22)$$

It remains to define the scalar product. The definition is obvious for functions of φ, but there is a difficulty concerning functions of A_k, because the Hamiltonian does not depend on $\bar{A}_L$, which is noncompact. Such a difficulty can be overcome by imposing an auxiliary constraint commuting with Hamiltonian and Gauss constraint, in order to freeze the dependence of the states on $\bar{A}_L$.

We find convenient to impose

$$\Big[\frac{1}{2}\mathcal{E}^2(\mathbf{r}) + \frac{\alpha^2}{2}\mathcal{A}^2(\mathbf{r}) - \frac{\alpha}{2}\Big]\,\Psi = 0, \qquad (23)$$

where

$$\mathcal{A}(\mathbf{r}) = \sum_{\mathbf{r}'} \Delta_k(\mathbf{r}')\,(-\Delta)^{-1/2}(\mathbf{r} - \mathbf{r}')\,A_k(\mathbf{r}')$$

$$\mathcal{E}(\mathbf{r}) = \sum_{\mathbf{r}'} \Delta_k(\mathbf{r}')\,(-\Delta)^{-1/2}(\mathbf{r} - \mathbf{r}')\,E_k(\mathbf{r}'). \qquad (24)$$

Since the operator appearing in Eq. (23) is positive semidefinite, we can actually replace it by

$$H_c\,\Psi = 0, \qquad (25)$$

where

$$H_c = \sum_{\mathbf{r}} \frac{1}{2}\mathcal{E}^2(\mathbf{r}) + \frac{\alpha}{2}\mathcal{A}^2(\mathbf{r}) - \frac{\alpha}{2}. \qquad (26)$$

Moreover, if we take the limit $\alpha \to \infty$ we can forget about the new constraint confining ourselves to the eigenstates of the Hamiltonian $H + H_c$.

We are now in the position to discuss the Gauss constraint. If the spinor field is quantized in the standard way, the spectrum of the charge density operator is integral, and the same must be true for consistency of the operator π. If π is realized as $-i\,\partial/\partial\varphi$ this is only possible if φ is compact and we assume periodic boundary conditions. This is the reason why we have not defined the longitudinal gauge field according to eq. (5), which gives a noncompact variable.

The compactness of φ has the well known consequence of the quantization of the electric charge(3).

3. The transfer matrix which is obtained from our Hamiltonian is

$$Z = \prod_{\mathbf{r},t} \int d\psi(\mathbf{r},t) \int d\psi^*(\mathbf{r},t) \int_{-\pi}^{+\pi} d\varphi(\mathbf{r},t) \int_{-\infty}^{+\infty} dA_0(\mathbf{r},t) .$$

$$\prod_k \int_{-\infty}^{+\infty} dA_k(\mathbf{r},t) \exp\left(i \sum_{\mathbf{r},t} \mathcal{L}_G + \mathcal{L}_F\right). \tag{27}$$

In the above equation t is the discrete lattice time and A_0 a new variable originating from the Gauss constraint. The gauge field Lagrangian density $\mathcal{L}_G$ is

$$\mathcal{L}_G = \frac{1}{2}\frac{a}{\tau} \{[\Delta_0 A_k(\mathbf{r},t)]^2 + [\Delta_k(\Delta_0 \varphi(\mathbf{r},t) - A_0(\mathbf{r},t))]^2\}$$

$$-\frac{\tau}{a}\left[\frac{1}{4} F_{hk}^2(\mathbf{r},t) + \frac{\alpha^2}{2} \mathcal{A}^2(\mathbf{r},t)\right], \tag{28}$$

where τ is the time lattice spacing, and Δ_0 the time derivative.

The fermion field Lagrangian density $\mathcal{L}_F$ is

$$\mathcal{L}_F = \frac{\tau}{a} i \bar{\psi}(\mathbf{r},t) e^{-iA_0(\mathbf{r},t)} \{\gamma^0 [\psi(\mathbf{r},t) e^{iA_0(\mathbf{r},t)} - \psi(\mathbf{r},t-1)]$$

$$+ \gamma_k [(1 + i A_{Tk}(\mathbf{r},t-1)) \psi(\mathbf{r},t-1) - e^{-i\varphi(\mathbf{r},t-1) + i\varphi(\mathbf{r}-\hat{k},t-1)} \psi(\mathbf{r}-\hat{k},t-1)]\}.$$

$$+ \tau m \bar{\psi}(r,t) \psi(r,t). \tag{29}$$

These Lagrangians are invariant under the transformations

$$A_0(\mathbf{r}, t) \to A_0(\mathbf{r}, t) + \lambda(\mathbf{r}, t) - \lambda(\mathbf{r}, t-1)$$

$$\varphi(\mathbf{r}, t) \to \varphi(\mathbf{r}, t) + \lambda(\mathbf{r}, t)$$

$$\psi(\mathbf{r}, t) \to e^{-i\lambda(\mathbf{r}, t)}\, \psi(\mathbf{r}, t). \tag{30}$$

It is easy to check that in the formal continuum limit a, $\tau \to 0$, $\alpha \to \infty$, by putting $A_0(\mathbf{x}, x_0) = (1/\tau)\, A_0(\mathbf{r}, t)$ one gets the continuum path integral in the Coulomb gauge.

It is also obvious that the above action behaves as the compact one as far as charge quantization is concerned, because the gauge variable is compact.

Eq. (27) has been obtained by expressing the transfer matrix in terms of matrix elements of the evolution operator times a δ-function to account for the Gauss constraint

$$Z = \prod_t U(t), \tag{31}$$

$$U(t) = <\prod_{\mathbf{r}} \psi^*(\mathbf{r}, t)\, \varphi(\mathbf{r}, t) \prod_k A_k(\mathbf{r}, t) \,|\, \delta_G(t)\, e^{-i\tau H}$$

$$|\prod_{\mathbf{r}} \psi(\mathbf{r}, t-1)\, \varphi(\mathbf{r}, t-1) \prod_k A_k(\mathbf{r}, t-1)>, \tag{32}$$

where

$$\delta_G(t) = \prod_{\mathbf{r}} \frac{1}{2\pi} \int_{-\pi}^{+\pi} dA_0(\mathbf{r}, t)\, e^{-i A_0(\mathbf{r}, t)(\hat{\psi}^* \hat{\psi} + \pi)}. \tag{33}$$

In the above equation $\hat{\psi}$ is the fermionic destruction operator while ψ is the corresponding Grassmann variable and the fermionic states are in the olomorfic representation. Kronecker δ-functions rather than Dirac δ-functions appear because the spectrum of the Gauss operator is integral. As a consequence the integration variable A_0 is compact. It will be decompactified, however, in the course of manipulations on the transfer matrix.

The matrix element of the fermionic exponential is

$$< \psi^*(\mathbf{r}, t) \,|\, e^{-iA_0(\mathbf{r}, t)\, \hat{\psi}^*(\mathbf{r})\, \hat{\psi}(\mathbf{r})} \,|\, \psi(\mathbf{r}, t-1) > =$$

$$= \exp\{\psi^*(\mathbf{r}, t)\, \psi(\mathbf{r}, t-1) \exp[-i A_0(\mathbf{r}, t)]\}. \tag{34}$$

By standard methods we obtain, to first order in τ

$$U(t) = \int \prod_{r} dA_0(\mathbf{r}, t)\, U_G(t)\, U_F(t). \tag{35}$$

U_F (t) is the exponential of the sum over spatial sites of the fermion action density Eq. (29), while

$$U_G(t) = \prod_{\mathbf{r},k} \int_{-\infty}^{+\infty} dE_k(\mathbf{r}) \sum_{n(\mathbf{r})=-\infty}^{+\infty} < A(\mathbf{r}, t)\, \varphi(\mathbf{r}, t) \mid E_k(\mathbf{r})\, n(\mathbf{r}) >.$$

$$< E_k(\mathbf{r})\, n(\mathbf{r}) \mid A(\mathbf{r}, t-1)\, \varphi(\mathbf{r}, t-1) > \exp \{-i\, A_0(\mathbf{r}, t)\, n(\mathbf{r})$$

$$-\frac{\tau}{a} \frac{1}{2} E_k^2(\mathbf{r}) + \frac{1}{2} [[(-\Delta)^{-1/2} n](\mathbf{r})]^2 - \frac{1}{4} F_{hk}^2(\mathbf{r}, t-1) - \frac{\alpha^2}{2} \mathcal{A}^2(\mathbf{r}, t-1)\}. \tag{36}$$

The continuous eigenvalues E_k and the discrete eigenvalues n lable the complete set of eigenstates of E_k and π respectively.

By application of the Poisson summation formula U_G can be rewritten

$$U_G(t) = \prod_{r} \sum_{m(\mathbf{r})} \exp -i \{[\frac{1}{2} \frac{a}{\tau} (\Delta_0 A(\mathbf{r}, t))^2 + \frac{1}{2} \frac{a}{\tau} [\Delta_k (\Delta_0 \varphi(\mathbf{r}, t)$$

$$- A_0(\mathbf{r}, t) + 2\pi m(\mathbf{r}))]^2 - \frac{\tau}{a} [\frac{1}{4} F_{hk}(\mathbf{r}, t-1) - \frac{\alpha^2}{2} \mathcal{A}^2(\mathbf{r}, t-1)]\}. \tag{37}$$

The resulting action is analogous to the Villain[4] one as far as the variable φ is concerned.

The sum over m can be eliminated, however, by extending the integrals over A_0 from minus to plus infinity. This completes the derivation of Eq. (27).

4. Euclidean invariance is always broken when one starts by the Hamiltonian formulation, but here the breaking is even worse due to the use of gauge invariant variables, as it is shown by Eqs. (28) and (29).

Moreover the use of gauge-invariant variables allows us to introduce a different coupling for transverse and longitudinal fields. We can in fact replace Eq. (12) by

$$\mathcal{H} = \frac{1}{2} E_T^2 - \frac{1}{2} g_1^2 \pi \Delta^{-1} \pi + \frac{1}{4} F_{hk}^2 + i \bar{\chi} \gamma_k (\partial_k + ig_2 A_{Tk}) \chi + m \bar{\chi} \chi,$$

with $g_1 \neq g_2$ without spoiling gauge invariance. Since the equality of g_1 and g_2 is not required by rotational invariance it appears as a fine tuning.

This latter difficulty might be not so serious. Nielsen[5] and collaborators have in fact shown that gauge invariant theories have an infrared stable fixed point which is Lorentz invariant. This has been shown by treating the breaking of Lorentz invariance as a small perturbation. As a consequence such a result may be used to show that if we start with g_1 close to g_2 we get in the continuum limit for low energy phenomena $g_1 = g_2$, i.e. the stability of the fine tuning.

We see no way, however, to treat the explicit breaking of Euclidean invariance as small, and therefore Nielsen's result cannot be used to guarantee that, if a continuum limit exists, it is Lorentz-invariant. We have nothing to say about this fundamental problem. Before been able to solve it, however, one has to properly define the Hamiltonian formalism. In this respect we think that we have pointed out one additional difficulty of this formalism due to the Gauss contraint along with one possible solution.

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