LNF–89/031

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Estratto da: Z. Phys. C Part. and Field 41, 631 (1989)

Z. Phys. C – Particles and Fields 41, 631–636 (1989)

Zeitschrift für Physik C Particles and Fields

Conformal invariance of σ-models and classical string physics

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Received 24 April 1988

Abstract. We show that the identification of the conformal anomaly of the general bosonic two-dimensional non-linear σ-model as the generating functional for on-shell string scattering amplitudes is correct up to $O(\alpha')$ terms. The absence, in the loop corrections to the spacetime effective action, of contributions from the explicit coupling to the dilaton field is suggested as a general feature for σ-models describing tree-level string physics.

1 Introduction and summary

The postulated correspondence between two-dimensional non-linear σ-models and string theories has been clarified in recent literature [1–7]. For compactification from higher-dimensional to ordinary $D=4$ spacetime, the solution of the equations of motion for the massless excitations of the string is needed. Hence, some effort has been devoted to deriving effective actions for massless particles, in the context of string theory. More precisely, two-dimensional non-linear σ models are believed to describe the classical physics of strings propagating in background fields. The crucial feature required for consistency of such description is provided by conformal invariance. The requirement of conformal invariance translates into a generalized condition on the β-functions of the σ-model, which can be implemented in perturbation theory, with the string parameter α' providing the loop expansion parameter. The conditions for conformal invariance of the σ-model are identified with the equations of motion for the massless string modes [1–7]. These equations can be derived from a single spacetime low-energy effective action, which reproduces the classical S-matrix elements for the massless states of the string theory [8,9].

The above ideas have been implemented in explicit calculations for the purely bosonic σ-model up to two-loops in perturbation theory [8], as well as for the $N=1$ and $N=2$ supersymmetric σ-models up to four loops [10–13]. In [8] the conformal invariance condition is used to obtain the conformal anomaly (or dilaton β-function), as an integrability condition from the β-function of the pure metric bosonic σ-model to the two-loop order. At the same time, it is suggested that the action function obtained integrating the dilaton β-function is equivalent to the generalized conformal invariance condition. This equivalence should hold on-shell for the massless modes of the string, i.e. modulo field redefinitions of the graviton and dilaton fields. The two-loop effective action derived for the bosonic σ-model in [8] generates the bosonic string tree-level three and four-particle scattering amplitudes.

A similar philosophy motivated the work of [12, 13] dealing with the $N=1$ supersymmetric σ-model (the $N=2$ being a particular case of the $N=1$ model). In [13] the procedure of [7] is extended to obtain a consistency condition which, if satisfied by the loop correction to the graviton β-function, ensures the existence of the integrability condition which permits to identify the dilaton β-function to the corresponding loop order. Then, it is shown that the four-loop graviton β-function [14] satisfies the abovementioned consistency condition. This allows to obtain the five-loop dilaton β-function of the $N=1$ supersymmetric σ-model [12, 13]. The latter is used to construct the on-shell effective action which generates the $O(\alpha'^3)$ classical string S-matrix elements with external graviton and dilaton fields. This result agrees with the four-loop 1PI generating functional for the tree-level string four-particle amplitudes [15, 16]. The explicit form of the $O(\alpha'^3)$ low-energy effective action was computed also in [12], based upon the integrability condition [8].*

The two-loop β-functions β^g_{ij} and β^b_{ij} governing the renormalization of the target space metric tensor g_{ij} and torsion potential b_{ij} of the bosonic two-

* We should notice that the overall sign of the $O(\alpha'^3)$ terms in (12) and (17) of [12] should be reversed from + to −, as a result of an algebraic error

dimensional non-linear σ-model with Wess-Zumino-Witten-like terms have been recalculated recently [17, 18], correcting an error in previous computations [19]. A consistency condition for β^g_{ij} and β^b_{ij} was presented in a recent letter [20]. The consistency condition, together with the requirement of vanishing conformal anomaly, suffice to establish the existence of an integrability condition for the generalized conformal invariance equations of [8]. The conditions for ultraviolet finiteness of the σ-model were further restricted in [20], to include only field redefinitions which guarantee that the conformal anomaly vanishes [21].

In the present work we discuss the conditions for ultraviolet finiteness of the σ-model in the case of a generic vector function on the manifold V_i. We derive the consistency condition for the loop corrections to β^g_{ij} and β^b_{ij}, showing that it is satisfied by the two-loop corrections of [17, 18]. This allows us to evaluate the two-loop conformal anomaly, and to construct, up to $O(\alpha')$ terms, the 1PI generating functional for the massless particle string tree S-matrix.

We plan our work as follows. In Sect. 2 the application of the ideas outlined above is reviewed, for the cases of both the pure metric bosonic and the $N=1$ supersymmetric σ-models. We carry out such analysis not merely for introductory purposes. It is our aim to illustrate the rather suggestive and simple interplay of various conditions for conformal invariance, pointing out that they are equivalent in a perturbation theory with α' as the loop expansion parameter, modulo the trace anomaly on the string world-sheet [21]. Section 3 is devoted to the derivation of the consistency condition for the generalized bosonic σ-model with Wess-Zumino-Witten-like terms. The feature of non-vanishing torsion spoils the criterium of perturbative equivalence for conformal invariant conditions on the flat world-sheet, pointed out in Sect. 2. In order to achieve conformal invariance on the flat world-sheet, one must restrict the form of the field redefinitions allowed for the σ-model with torsion. Such a restriction coincide with the one which ensures that conformal invariance is maintained in the curved world-sheet calculations [21]. In Sect. 4 we prove that the two-loop β-functions β^g_{ij}, β^b_{ij} satisfy the consistency condition of Sect. 3. As a consequence, the β-functions possess an integrability condition which allows us to calculate the $O(\alpha')$ conformal anomaly β^ϕ and the string tree-level effective action $I_{\text{spacetime}}$. We discuss in Sect. 5 the results and draw the conclusions of the present work. We remark that $D\phi$ terms in the loop corrections to β^ϕ can be eliminated on-shell, i.e. by suitably redefining the quantum corrections, for all conformally invariant two-dimensional σ-models which describe classical string physics. Such $D\phi$ terms are not present in the loop corrections to $I_{\text{spacetime}}$, provided surface terms originating from integrating by parts are negligible. Finally, we exhibit in the appendix the algebraic manipulations showing that the consistency condition is satisfied by the two-loop corrections of [17, 18].

2 Conformal anomaly of σ-models

We start from the generalized equation for ultraviolet-finite σ-models with vanishing torsion [8]

$$R_{ij} + T_{ij} = D_i V_j + D_j V_i, \tag{2.1}$$

where T_{ij} represents the corrections to contributions of leading order in α'. In writing (2.1) for a generic vector function V on the manifold, we are setting the "flat-space" trace of the stress tensor to zero [4]. In order to achieve full conformal invariance, one must also make sure that the conformal anomaly, i.e. the Schwinger term in the stress-tensor algebra, vanishes. This is guaranteed by the restriction [21]

$$V_i = \tfrac{1}{2} D_i \phi \tag{2.2}$$

which gives

$$R_{ij} + T_{ij} - D_i D_j \phi = 0. \tag{2.3}$$

Carrying out manipulations on flat string world-sheet for the torsion-free σ-model, we can recover (2.2), at least perturbatively in α', by a consistency condition on T_{ij} given in (2.11) below. Taking the divergence of (2.1) and using the Bianchi identity, as well as (2.1) itself, we get

$$\tfrac{1}{2} D_i R + D_j T_{ij} = D^2 V_i + D_i D_j V_j + \tfrac{1}{2} D_i (V^2) + V_j D_j V_i - T_{ij} V_j. \tag{2.4}$$

By taking the curl of (2.4), we obtain

$$(D^2 + 2 V_j D_j) D_{[k} V_{i]} + 2 R_{klij} D_{[l} V_{j]} = D_{[k}(D_{|j|} T_{i]j} + 2 T_{i]j} V_j) - T_{[k|j}(D_{j|} V_{i]} - D_{i]} V_j). \tag{2.5}$$

We can solve (2.5) perturbatively in α'. Recalling that T_{ij} vanishes by construction in the limit $\alpha' \to 0$, we have

$$(D^2 + 2 V^\circ{}_j D_j) D_{[k} V^\circ{}_{i]} + 2 R_{klij} D_{[l} V^\circ{}_{j]} = 0 \tag{2.6}$$

for the contribution V° of leading order in α'. In compact Euclidean spacetimes (2.6) has the unique solution

$$D_{[k} V^\circ{}_{i]} = 0. \tag{2.7}$$

For spacetimes with trivial cohomology (2.7) implies

$$V^\circ_i = \tfrac{1}{2} D_i \phi^\circ \tag{2.8}$$

for some scalar function ϕ° on the manifold. Using (2.8), we write the expansion in α' of the vector function V

$$V_i = \tfrac{1}{2} D_i \phi^\circ + W_i, \tag{2.9}$$

where W_i contains the corrections to the leading order in α'. From (2.5) we get

$$[D^2 + (D_j \phi) D_j] D_{[k} W_{i]} + 2 R_{klij} D_{[l} W_{j]} = D_{[k}(D_{|j|} T_{i]j} + T_{i]j} D_j \phi). \tag{2.10}$$

Let us assume that T_{ij} satisfies the condition

$$D_j T_{ij} + T_{ij} D_j \phi = D_i S \tag{2.11}$$

for some scalar S. Then, the rhs of (2.10) vanishes, so that the solution to (2.10) is uniquely determined, order by order in perturbation theory, by

$$D_{[k} W_{i]} = 0. \tag{2.12}$$

Equations (2.9, 2.12) imply that V_i may be expressed as in (2.2).

All this indicates that a crucial feature for consistency of the procedure is to check that the condition (2.11) holds. If this is so, then we should expect that the generalized conformal invariance equation (2.3) have a condition of integrability requiring constancy of the conformal anomaly β^ϕ, including loop effects. In fact, using manipulations analogous to (2.4), we get from (2.3)

$$D_i[-\tfrac{1}{2}R + D^2\phi + \tfrac{1}{2}(D\phi)^2] = D_j T_{ij} + T_{ij} D_j \phi = D_i S. \tag{2.13}$$

The last passage in (2.13) clarifies that the existence of an integrability condition proceeds from (2.11). This allows us to extract the form of the associated β^ϕ, by integrating (2.13) (setting the constant of integration equal to zero, as required for the existence of flat empty spacetime)

$$\beta^\phi = -R + 2D^2\phi + (D\phi)^2 - 2S = 0. \tag{2.14}$$

The spacetime equations of motion for the graviton and dilaton fields given by (2.3, 2.14) may be derived from a single spacetime effective action

$$\begin{aligned} I_{\text{spacetime}} &= -\int d^D X g^{1/2} e^\phi \beta^\phi \\ &= \int d^D X g^{1/2} e^\phi [R + (D\phi)^2 + 2S], \end{aligned} \tag{2.15}$$

where $D = 26$ for the pure metric bosonic σ-model, while $D = 10$ for $N = 1$ and $N = 2$ supersymmetric σ-models. The action (2.15) represents the 1PI generating functional for the massless particle tree S-matrix of string theory. Recalling the results of [8, 12, 13], we conclude that (2.11) is satisfied by the loop corrections to all torsion-free σ-models [22, 14]. The scalar functions corresponding to (2.11) read, respectively [8]

$$S = \tfrac{1}{8}\alpha' R_{abcd} R_{abcd} \tag{2.16}$$

for the pure metric bosonic σ-model and [12, 13]

$$\begin{aligned} S = \alpha'^3(1/16)\zeta(3)[&-R_{habk}R_{pabq}R_{hcdp}R_{qcdk} \\ &-\tfrac{1}{2}R_{hkab}R_{pqab}R_{hcdp}R_{qcdk} \\ &+(1/9)D_jD_j(R_{h(ba)k}R_{acdk}R_{bcdh}) \\ &+(1/9)(D_j\phi)D_j(R_{h(ba)k}R_{acdk}R_{bcdh})] \end{aligned} \tag{2.17}$$

for the supersymmetric case. Recalling (2.14, 2.15), one may exhibit loop-corrected expression for β^ϕ and $I_{\text{spacetime}}$. The last two terms on the rhs of (2.16) can be removed by a field redefinition of the superfield action of the $N = 1$ supersymmetric σ-model, which modifies the counterterm T_{ij} as follows [14]

$$T_{ij} \to T_{ij} - (2/3)D_iD_j[R_{hkab}R_{hcdb}(R_{kdca} + R_{kacd})] \tag{2.18}$$

The result (2.17) becomes, then

$$\begin{aligned} S = \alpha'^3(1/16)\zeta(3)(&-R_{habk}R_{pabq}R_{hcdp}R_{qcdk} \\ &-\tfrac{1}{2}R_{hkab}R_{pqab}R_{hcdp}R_{qcdk}). \end{aligned} \tag{2.19}$$

Notice that the contribution to $I_{\text{spacetime}}$ from the last two terms in the rhs of (2.17) is explicitely eliminated by performing an integration by parts. The loop corrections to the action (2.15) agree with the results from the low-energy scattering of strings [8, 12, 13].

3 The consistency condition

Next we focus on the bosonic σ-model with torsion terms. We introduce the torsion potential b_{ij}, as well as $H_{ijk} = 3D_{[i}b_{jk]}$. Defining the tensor A_{ij} as the loop correction to $\beta^b{}_{ij}$, in analogy to (2.1), we have for the ultraviolet-finite σ-model [8]

$$R_{ij} - \tfrac{1}{4}H_{ikl}H_{jkl} + T_{ij} = D_iV_j + D_jV_i, \tag{3.1}$$

$$-\tfrac{1}{2}D_kH_{ijk} + A_{ij} = V_kH_{ijk}. \tag{3.2}$$

Neglecting, for the time being, the trace anomaly due to the curved two-dimensional metric, we go on taking the divergence of (3.1) and use the Bianchi identity, as well as (3.1), (3.2) and (A.20) from the appendix

$$\begin{aligned} &\tfrac{1}{2}D_iR + D_jT_{ij} - (1/24)D_i(H^2) - \tfrac{1}{2}A_{jl}H_{ijl} + \tfrac{1}{4}H_{iml}H_{mlj}V_j \\ &= D_i(D_jV_j + \tfrac{1}{2}V^2) + D^2V_i + V_jD_jV_i - T_{ij}V_j. \end{aligned} \tag{3.3}$$

The curl of (3.3) can now be considered, giving the expression

$$\begin{aligned} &(D^2 + 2V_jD_j)D_{[k}V_{i]} + 2R_{klij}D_{[l}V_{j]} \\ &= D_{[k}(D_{|j|}T_{ij} + 2T_{i]j}V_j - \tfrac{1}{2}H_{i]jl}A_{jl}) \\ &\quad + (\tfrac{1}{4}H_{jml}H_{ml[k} - T_{j[k})(D_{|j|}V_{i]} - D_{i]}V_j). \end{aligned} \tag{3.4}$$

Notice the presence in the rhs of (3.4) of a torsion term of leading order in α', which makes it impossible to solve (3.4) perturbatively in α' for a generic vector function V, even assuming that the world-sheet is flat.

Since the one-loop trace anomaly for curved world-sheet vanishes, provided one restricts V_i in (3.1, 3.2) to be the gradient of a scalar, we will make the assumption (2.2) in the following. We expect that this restriction suffices to guarantee the vanishing of the trace anomaly, even to higher loops. Thus, a sufficient condition for the theory to enjoy full conformal invariance is obtained restricting (3.1) and (3.2) as follows

$$\beta^g{}_{ij} = R_{ij} - \tfrac{1}{4}H_{ikl}H_{jkl} + T_{ij} - D_iD_j\phi = 0, \tag{3.5}$$

$$\beta^b{}_{ij} = -\tfrac{1}{2}D_kH_{ijk} + A_{ij} - \tfrac{1}{2}H_{ijk}D_k\phi = 0. \tag{3.6}$$

Carrying out manipulations similar to the ones which lead to (3.3), we obtain

$$\begin{aligned} &D_i[-\tfrac{1}{2}R + D^2\phi + \tfrac{1}{2}(D\phi)^2 + (1/24)H^2] \\ &= D_jT_{ij} + T_{ij}D_j\phi - \tfrac{1}{2}H_{ijl}A_{jl}. \end{aligned} \tag{3.7}$$

Thus, checking the validity of the following consistency condition

$$D_j T_{ij} + T_{ij} D_j \phi - \tfrac{1}{2} H_{ijl} A_{jl} = D_i S \tag{3.8}$$

for some scalar function S on the manifold, ensures that (3.5) and (3.6) admit an integrability condition

$$D_i[-\tfrac{1}{2}R + D^2\phi + \tfrac{1}{2}(D\phi)^2 + (1/24)H^2 - S] = 0. \tag{3.9}$$

Notice that (3.9) is completely general, as long as T_{ij} and A_{ij} remain unspecified. When considering loop corrections to the β-functions satisfying (3.8), one can integrate (3.9) to obtain

$$-R + 2(D^2\phi) + (D\phi)^2 + (1/12)H^2 - 2S = \text{const.} \tag{3.10}$$

Equation (3.10), with S defined by (3.8), represents the generalization to higher loops of the equation of motion of the dilaton field, for the σ-model with torsion. For the existence of the flat spacetime solution we must set the constant to zero. We identify the conformal anomaly, including higher orders in α', as

$$\beta^\phi = -R + 2D^2\phi + (D\phi)^2 + (1/12)H^2 - 2S. \tag{3.11}$$

This generalizes the arguments of [8] to the case in which a background antisymmetric field is present.

4 The $O(\alpha')$ dilaton β-function

Turning our attention to the two-loop corrections of [17, 18], we notice that the $O(\alpha')$ contributions to the tensors in (3.5, 3.6) have the explicit form

$$\begin{aligned} T_{ij} = -\tfrac{1}{4}\alpha'[&-2R_{ilkm}R_{jlkm} - \tfrac{1}{2}H_{hkj}H_{hlm}R_{iklm} \\ &- R_{kijl}H_{khm}H_{lhm} + (D_l H_{jmk})(D_m H_{ilk}) \\ &- \tfrac{1}{2}H_{hki}H_{hlm}R_{jklm} - \tfrac{1}{4}H_{jml}H_{imk}H_{kpq}H_{lpq} \\ &- \tfrac{1}{4}H_{pil}H_{pmk}H_{qjk}H_{qml}], \end{aligned} \tag{4.1}$$

$$\begin{aligned} A_{ij} = -\tfrac{1}{4}\alpha'[&(D_k H_{pq[i})R_{j]kpq} - \tfrac{1}{4}H_{km[i}(D_{j]}H_{kpq})H_{mpq} \\ &+ \tfrac{1}{4}H_{km[i}(D_{|k|}H_{j]pq})H_{mpq} \\ &- \tfrac{1}{2}(D_m H_{ijk})H_{kpq}H_{mpq}]. \end{aligned} \tag{4.2}$$

Performing an algebraic exercise, whose details can be found in the appendix, one can show that the consistency condition (3.8) is satisfied by (4.1, 4.2)

$$\begin{aligned} &D_j T_{ij} + T_{ij} D_j \phi - \tfrac{1}{2} H_{ijl} A_{jl} \\ &= -\tfrac{1}{4}\alpha' D_i\{-\tfrac{1}{2}R_{abcd}R_{abcd} + \tfrac{1}{4}H_{akj}H_{abm}R_{kjbm} \\ &\quad + (1/16)[H_{kpq}H_{mpq}H_{kab}H_{mab} \\ &\quad - (1/3)H_{qab}H_{pak}H_{qmk}H_{pmb}]\} \end{aligned} \tag{4.3}$$

Thus, we conclude that the conformal invariance equations (3.5, 3.6) admit indeed to $O(\alpha')$ an integrability condition requiring the constancy of a scalar function, which is identified with the conformal anomaly of the σ-model and the dilaton equation of motion. Comparing (4.3) with (3.8), we have

$$\begin{aligned} S = -\tfrac{1}{4}\alpha'\{&-\tfrac{1}{2}R_{abcd}R_{abcd} + \tfrac{1}{4}H_{akj}H_{abm}R_{kjbm} \\ &+ (1/16)[H_{kpq}H_{mpq}H_{kab}H_{mab} \\ &- (1/3)H_{qab}H_{pak}H_{qmk}H_{pmb}]\}. \end{aligned} \tag{4.4}$$

Substituting (4.4) into (3.11), we obtain the two-loop expression of the conformal anomaly

$$\begin{aligned} \beta^\phi = -R &+ 2D^2\phi + (D\phi)^2 + (1/12)H^2 \\ &+ \tfrac{1}{2}\alpha'\{-\tfrac{1}{2}R_{abcd}R_{abcd} + \tfrac{1}{4}H_{akj}H_{abm}R_{kjbm} \\ &+ (1/16)[H_{kpq}H_{mpq}H_{kab}H_{mab} \\ &- (1/3)H_{qab}H_{pak}H_{qmk}H_{pmb}]\}. \end{aligned} \tag{4.5}$$

The spacetime equations of motions (3.5, 3.6, 4.5) for the massless string modes may be derived from the following spacetime effective action

$$\begin{aligned} &I_{\text{spacetime}} \\ &= -\int d^{26}X g^{1/2} e^\phi \beta^\phi = \int d^{26}X g^{1/2} e^\phi\{R + (D\phi)^2 \\ &\quad - (1/12)H^2 - \tfrac{1}{2}\alpha'[-\tfrac{1}{2}R_{abcd}R_{abcd} \\ &\quad + \tfrac{1}{4}H_{akj}H_{abm}R_{kjbm} + (1/16)H_{kpq}H_{mpq}H_{kab}H_{mab} \\ &\quad - (1/48)H_{qab}H_{pak}H_{qmk}H_{pmb}]\}. \end{aligned} \tag{4.6}$$

This action is the 1PI generating functional for the massless particles tree scattering amplitudes. In fact, (4.6) agrees on-shell with the action of [9] which generates all bosonic string tree-level three-particle amplitudes correct to $O(\alpha')$. Four-particle scattering amplitudes, which correspond to quartic H-field interactions, are computed for the heterotic string in [23]. A low-energy effective action, restricted to the case $D_i\phi = 0$, appears in [18], but the coefficient of the term $(R_{abcd})^2$ differs from the one in (4.6), as well as from the result holding for the pure metric σ-model [8]. We believe that (4.6) is correct because it reproduces the result from the low-energy scattering of strings.

5 Conclusions

The results of the present work show that the identification of the on-shell effective action with the conformal anomaly suggested in [8] is correct including $O(\alpha')$ terms. In the context of the $N = 1$ and $N = 2$ supersymmetric σ-models we see from (2.15, 2.19) that no terms containing the dilaton field appear in the $O(\alpha'^3)$ corrections to the spacetime effective action [12, 13]. This feature is present in (4.6) as well, showing that the explicit coupling to the dilaton field does not contribute to the $O(\alpha')$ corrections to $I_{\text{spacetime}}$. Thus, in computing the two-loop corrections to all the β-functions of the bosonic σ-model with torsion, one can assume that the world-sheet is flat. It is interesting to conjecture that a general proof of this statement could be derived for all conformally invariant two-dimensional σ-models describing classical string physics.

Added note. After the completion of our work, we became aware of the work of [24], which analyses the renormalization scheme dependence of the β-functions, and also performs the perturbative check of the equivalence of the $O(\alpha')$ terms in the equations of motion for the string massless modes with the two-loop corrections to the conformal anomaly. The presence of an additional $\alpha'(D\phi)^4$ term in the effective action (4.6) is remarked, although the coefficient of such term is seen to cancel out, after a rescaling of the field variables.

Acknowledgement. We wish to thank the Department of Physics and Astronomy at University of Maryland for hospitality extended during the undertaking of this work.

Appendix

Algebraic manipulations for the two-loop β-functions

In the following we outline the proof of (4.3). The two-loop corrections of (4.1, 4.2) can be written as

$$T_{ij} = -\tfrac{1}{4}\alpha' \sum_{I=1}^{6} T^{(I)}{}_{ij}, \tag{A.1}$$

$$A_{ij} = -\tfrac{1}{4}\alpha' \sum_{J=1}^{2} A^{(J)}{}_{ij}, \tag{A.2}$$

where

$$T^{(1)}{}_{ij} \equiv -2R_{ilkh}R_{jikh}, \tag{A.3}$$

$$T^{(2)}{}_{ij} \equiv -\tfrac{1}{2}H_{hkj}H_{hlp}R_{iklp}, \tag{A.4}$$

$$T^{(3)}{}_{ij} \equiv -R_{kijl}H_{kpq}H_{lpq}, \tag{A.5}$$

$$T^{(4)}{}_{ij} \equiv \tfrac{1}{2}(D_lH_{ihk})(D_lH_{jhk}) - (1/6)(D_iH_{klh})(D_jH_{klh}) = (D_lH_{jhk})(D_hH_{ilk}), \tag{A.6}$$

$$T^{(5)}{}_{ij} \equiv -\tfrac{1}{2}H_{hki}H_{hlp}R_{jklp}, \tag{A.7}$$

$$T^{(6)}{}_{ij} \equiv -\tfrac{1}{4}H_{jhl}H_{ihk}H_{kpq}H_{lpq} - \tfrac{1}{4}H_{hil}H_{hpk}H_{qjk}H_{qpl}, \tag{A.8}$$

$$A^{(1)}{}_{ij} \equiv (D_lH_{pq[i})R_{j]lpq}, \tag{A.10}$$

$$A^{(2)}{}_{ij} \equiv -\tfrac{1}{4}H_{lh[i}(D_{j]}H_{lpk})H_{hpk} + \tfrac{1}{4}H_{lh[i}(D_{|l|}H_{j]pk})H_{hpk} - \tfrac{1}{2}(D_lH_{ijk})H_{kpq}H_{lpq}. \tag{A.11}$$

From (3.5) and the Bianchi identity, we get

$$D_jR_{jabc} = -R_{jabc}D_j\phi + \tfrac{1}{4}D_{[b}(H_{c]kl}H_{akl} + O(\alpha'), \tag{A.12}$$

while (3.6) gives

$$D_kH_{ijk} = -H_{ijk}D_k\phi + O(\alpha'). \tag{A.13}$$

The use of (A.12, A.13), neglecting higher orders in α', eliminates entirely the ϕ-dependence of the lhs of (4.3). Using (A.12, A.13) and the definitions (A.4–A.7), one readily obtains

$$D_jT^{(1)}{}_{ij} + T^{(1)}{}_{ij}D_j\phi = -2(D_jR_{iabc})R_{jabc} - R_{iabc}D_b(H_{ckl}H_{akl}), \tag{A.14}$$

$$D_jT^{(2)}{}_{ij} + T^{(2)}{}_{ij}D_j\phi = -\tfrac{1}{2}R_{iabc}(D_lH_{kbc})H_{kal} - \tfrac{1}{2}H_{hkj}H_{hlm}D_jR_{iklm}, \tag{A.15}$$

$$D_jT^{(3)}{}_{ij} + T^{(3)}{}_{ij}D_j\phi = R_{iabc}D_b(H_{akl}H_{ckl}) - \tfrac{1}{4}D_{[k}(H_{i]ab}H_{lab})H_{kcd}H_{lcd}, \tag{A.16}$$

$$D_jT^{(4)}{}_{ij} + T^{(4)}{}_{ij}D_j\phi = \tfrac{1}{4}(D_mH_{ilk})H_{lab}H_{jab}H_{jmk} + R_{hmlj}H_{jhk}D_{[m}H_{k]il} + R_{hljm}H_{ihk}(D_jH_{mlk}) - \tfrac{1}{2}R_{iabc}(D_lH_{kbc})H_{akl}. \tag{A.17}$$

Adding up (A.14–A.17), we obtain the cancellation of terms with the structure $R_i \cdots H \cdots D \cdot H \cdots$, where the dots stand for dummy indices. The partial result reads

$$\sum_{I=1}^{4} [D_jT^{(I)}{}_{ij} + T^{(I)}{}_{ij}D_j\phi] = -\tfrac{1}{2}D_i(R_{jabc}R_{jabc}) - \tfrac{1}{2}H_{hkj}H_{hlm}D_jR_{iklm} + R_{hmlj}H_{jhk}D_{[m}H_{k]il} + R_{hljm}H_{ihk}(D_jH_{mlk}) + \tfrac{1}{4}H_{lab}H_{jab}H_{jmk}D_mH_{ilk} - \tfrac{1}{4}H_{kcd}H_{lcd}D_{[k}(H_{i]ab}H_{lab}). \tag{A.18}$$

From (A.8) and (A.12), we get

$$D_jT^{(5)}{}_{ij} + T^{(5)}{}_{ij}D_j\phi = -\tfrac{1}{2}R_{jklm}D_j(H_{hki}H_{hlm}) - \tfrac{1}{4}D_l(H_{mab}H_{kab})H_{hki}H_{hlm}. \tag{A.19}$$

Using the Bianchi identity and the closure property of the antisymmetric tensor H_{ijk}

$$D_jH_{ikl} - D_iH_{jkl} - D_kH_{ijl} - D_lH_{ikj} = 0, \tag{A.20}$$

one can rewrite the second term on the rhs of (A.18) as follows

$$-\tfrac{1}{2}H_{hkj}H_{hlm}D_jR_{iklm} = -\tfrac{1}{2}(D_hH_{ikj} - 2D_kH_{ihj})H_{hlm}R_{kjlm} + \tfrac{1}{4}D_i(H_{hkj}H_{hlm}R_{kjlm}). \tag{A.21}$$

Summing (A.18) with (A.19) and recalling (A.21), one has

$$\sum_{I=1}^{5} [D_jT^{(I)}{}_{ij} + T^{(I)}{}_{ij}D_j\phi] = D_i(-\tfrac{1}{2}R_{abcd}R_{abcd} + \tfrac{1}{4}H_{hkj}H_{hlm}R_{kjlm}) + R_{hljm}H_{ihk}D_jH_{mlk} - \tfrac{1}{2}R_{jklm}H_{hki}D_jH_{hlm} + \tfrac{1}{4}H_{lab}H_{jab}H_{jmk}D_mH_{ilk} - \tfrac{1}{4}H_{kcd}H_{lcd}D_{[k}(H_{i]ab}H_{lab}) - \tfrac{1}{4}D_l(H_{mab}H_{kab})H_{hki}H_{hlm}. \tag{A.22}$$

The definition (A.10) gives

$$-\tfrac{1}{2}H_{ijk}A^{(1)}{}_{jk} = -H_{ihk}(D_jH_{lmh})R_{kjlm}. \tag{A.23}$$

The cancellation of unwanted $R - H - DH$ terms is achieved by adding (A.22, A.23)

$$\sum_{I=1}^{5} [D_jT^{(I)}{}_{ij} + T^{(I)}{}_{ij}D_j\phi] - \tfrac{1}{2}H_{ijk}A^{(1)}{}_{jk} = D_i(-\tfrac{1}{2}R_{abcd}R_{abcd} + \tfrac{1}{4}H_{hkj}H_{hlm}R_{kjlm}) + \tfrac{1}{4}H_{lab}H_{jab}H_{jmk}D_mH_{ilk} - \tfrac{1}{4}H_{kcd}H_{lcd}D_{[k}(H_{i]ab}H_{lab}) - \tfrac{1}{4}D_l(H_{mab}H_{kab})H_{hki}H_{hlm}. \tag{A.24}$$

Using (A.13) and the definition (A.9), we get

$$D_jT^{(6)}{}_{ij} + T^{(6)}{}_{ij}D_j\phi = -\tfrac{1}{4}H_{jml}D_j(H_{imk}H_{kcd}H_{lcd}) - \tfrac{1}{4}H_{cjk}D_j(H_{dil}H_{dmk}H_{cml}). \tag{A.25}$$

Combining the terms appearing on the rhs of (A.24) with the terms on the rhs of (A.25), one can prove the following relations

$$\tfrac{1}{4}H_{lab}H_{jab}H_{jmk}D_mH_{ilk}-\tfrac{1}{4}H_{kcd}H_{lcd}(D_{[k}H_{i]ab})H_{lab} \\ -\tfrac{1}{4}H_{jml}(D_jH_{imk})H_{kcd}H_{lcd}=0, \quad (A.26)$$

$$\tfrac{1}{4}H_{kcd}H_{lcd}H_{kab}D_iH_{lab}-\tfrac{1}{4}H_{cjk}(D_jH_{dil})H_{dmk}H_{cml} \\ =(1/16)D_i[H_{kcd}H_{lcd}H_{kab}H_{lab} \\ -(1/3)H_{cjl}H_{djk}H_{cmk}H_{dml}]. \quad (A.27)$$

Also, we read from the definition (A.11)

$$-\tfrac{1}{2}H_{ijk}A^{(2)}{}_{jk}=\tfrac{1}{4}H_{ijk}H_{lmj}(D_kH_{lha})H_{mha} \\ -\tfrac{1}{4}H_{ijk}H_{lmj}(D_lH_{kha})H_{mha}+\tfrac{1}{4}H_{ijk}(D_lH_{jka})H_{acd}H_{lcd}. \quad (A.28)$$

Collecting all the remaining terms for a final cancellation, we define

$$C_i\equiv-\tfrac{1}{4}H_{kcd}H_{lcd}H_{iab}D_kH_{lab}-\tfrac{1}{4}D_l(H_{mab}H_{kab})H_{hki}H_{hlm} \\ -\tfrac{1}{4}H_{jml}H_{imk}D_j(H_{kcd}H_{lcd}) \\ -\tfrac{1}{4}H_{cjk}H_{dil}D_j(H_{dmk}H_{cml}) \\ +\tfrac{1}{4}H_{ijk}H_{lmj}(D_kH_{lha})H_{mha} \\ -\tfrac{1}{4}H_{ijk}H_{lmj}(D_lH_{kha})H_{mha} \\ +\tfrac{1}{4}H_{ijk}(D_lH_{jka})H_{acd}H_{lcd}. \quad (A.29)$$

It is easy to check that

$$\tfrac{1}{4}H_{ijk}H_{lmj}(D_{[k}H_{l]ha})H_{mha}-\tfrac{1}{4}H_{cjk}H_{dil}D_j(H_{dmk}H_{cml})=0 \quad (A.30)$$

which implies

$$C_i=0. \quad (A.31)$$

The proof of (4.3) is completed, by recalling (A.1, A.2) and casting together (A.24–A.31)

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