



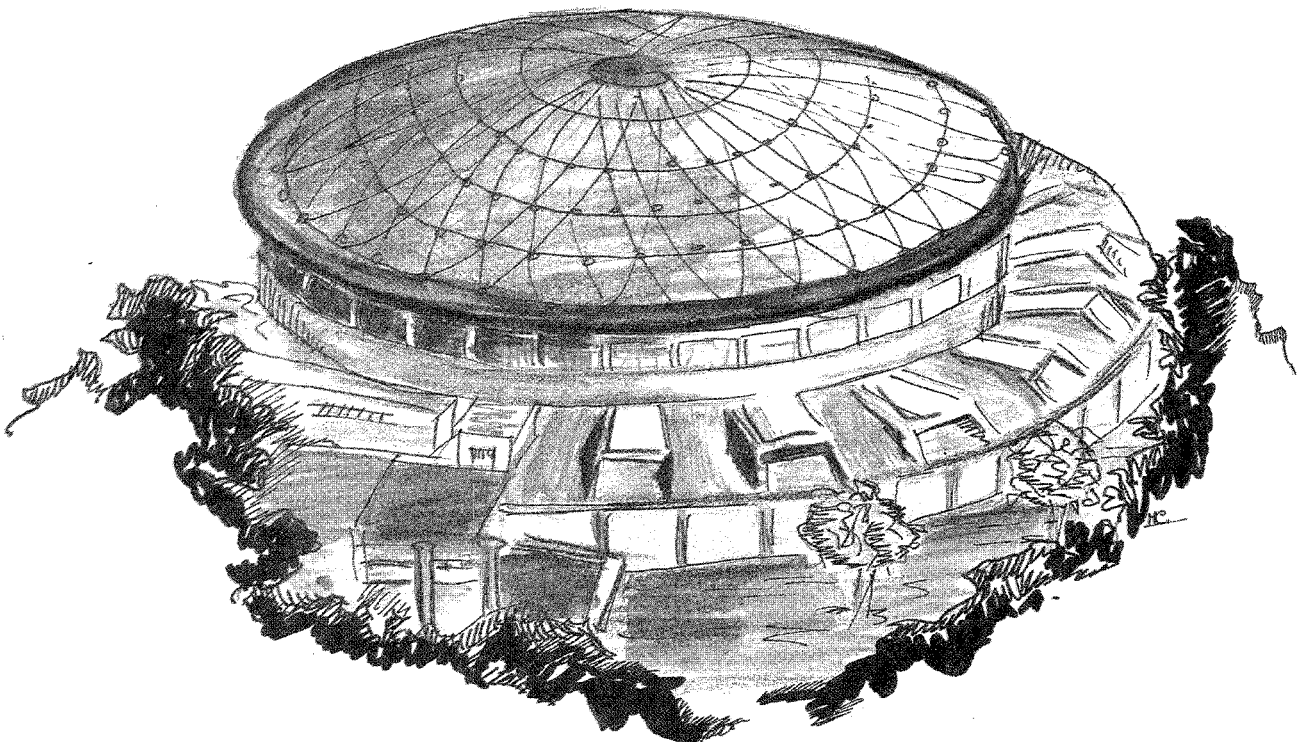
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INTEGRALS BY CESÀRO SUMMABILITY**



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**ANALYTIC REGULARISATION OF INVERSION TRANSFORM INTEGRALS
BY CESÀRO SUMMABILITY**

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ABSTRACT

The conformal inversion operator is represented in Hilbert space by Hankel transforms. These transforms occur in a wide variety of problems, e.g. high energy diffraction scattering, conformal invariant quantum field theory, the improvements of Borel summability of QCD asymptotic expansions, etc. The convergence of these transforms is generally not guaranteed everywhere, hence the need for their regularisation and analytic continuation. It is proposed in this paper that Cesàro summability is not only very simple but also most effective as a method of regularisation and analytic continuation of these integrals. Its power and utility are greatly enhanced by the availability of fast computing facilities. The hierarchy of Cesàro means of the transform integrals is generated, and their convergence investigated, numerically. In all cases we obtain results in agreement with analytic continuation. Cesàro summability can therefore be applied directly to effect the analytic continuation involved in QCD sum rules and not only to the Borel improvements of the relevant integrals. This would provide an additional check of the results obtained via Borel summability.

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1. - INTRODUCTION

The regularisation of a divergent integral^(*) is not concerned with the value, per se, of the integral but rather with how such a value may be consistently defined. Different definitions may give rise to different values. The uniqueness of the value of the integral is not the issue. The uniqueness problem is addressed by renormalisation techniques. A given method of regularisation is, therefore, a consistent procedure whereby a value is assigned to the divergent integral^(*).

An integral may depend on a set of parameters (e.g. coupling constants, momentum variables, etc.) which vary in a certain domain. The integral may then be well defined and convergent^(°) for values of the parameters in some sub-domain but divergent outside or on its boundaries. A problem of regularisation with a constraint then arises, namely that the procedure should also yield the values of the integral in its sub-domain of convergence in parameter space. Regularisation methods which satisfy such constraints are said to be regular⁽¹⁾. In these cases, one also speaks of this extended definition of the integral as summability⁽¹⁾. The best known of such methods is analytic regularisation⁽²⁾, based on the principle of analytic continuation. Actually summability⁽¹⁾ is what the problem of redefining the integral is all about, whether or not there are constraints. Regularisation is indeed another name for summability and analytic regularisation, a regular method of summability. Regularisation is, therefore, not always a totally unphysical procedure. There are many functions defined by integrals related to real physical processes which are analytic in their dependent variables. The analytic regularisation of such integrals, by whatever method, uniquely extends the definition of the integral and, hence, the physical process into a larger domain. Uniqueness is guaranteed by the uniqueness of analytic continuation. Such methods of regularisation are therefore of direct significance to physics.

The purpose of this paper is to apply one such method of regularisation, to a class of well known integrals in physics represented by the Hankel transform⁽³⁾

$$F_{\nu}(x) := \int_0^{\infty} dy J_{\nu}(xy) f(y) \quad (1)$$

The procedure of regularisation is through Cesàro summability⁽¹⁾. This is one of the simplest methods of summability and will be defined later. Hankel transforms form a sub-set of integrals with oscillating integrands which can be analytically regularised through Cesàro summability. We concentrate on Hankel transforms in this paper because they are more

(*) or series.

(°) i.e. in the ordinary sense of Riemann integrals.

familiar, they occur in a wide variety of interesting physical problems and it is easier, with them, to check, for the validity of analytic continuation.

We review, in Sect. 2, a few problems in high energy physics where the Hankel transform occurs prominently. We show there that this transform is a representation of the conformal inversion operator $R^{(*)}$ in Hilbert space. In Sect. 3 we restrict ourselves to the class of Hankel transforms of powers i.e. to eigenfunctions of the dilatation operator. The general case requires the function to be transformed to admit an expansion in the basis of these eigenfunctions. In other words their Mellin transforms (in particular, their Taylor series expansions) are assumed to exist. The method of Cesàro summability is described and then applied to the integrals in question. Sect. 4 concludes the paper and provides further theoretical insights to the general procedure of Cesàro summability.

2. - THE INVERSION TRANSFORM AND ITS HANKEL REPRESENTATION

Eq. (1) is the main formula in Glauber's theory of diffraction scattering^(4, 5). This theory gives the scattering amplitude $T(s, t)$, in momentum space as the Hankel transform

$$T(s, t) = \int_0^\infty db \, b \, J_0(b \sqrt{|t|}) \, T(s, b) \quad (2)$$

of a matter distribution $T(s, b)$ of 2-dimensional disc^(°), s is the square of the CM energy of the colliding particles, t the square of the momentum transfer and b the impact parameter. $J_0(z)$ is the Bessel function of zeroth order. The origin of eq. (2) in scattering problems is much more general. It is to be traced to the partial wave expansion or Sonine's formula⁽⁶⁾.

$$e^{i\vec{q} \cdot \vec{x}} = \Gamma(\lambda) \left(\frac{2}{qr} \right)^\lambda \cdot \sum_{n=0}^{\infty} i^n (n + \lambda) J_{n+\lambda}(qr) C_n^\lambda(u) \quad (3)$$

where $\vec{q} \cdot \vec{x} = qr \cos \theta$ is the scalar product of the momentum vector q_i ($i = 1, 2, \dots, D$) and the position vector x_i ($i = 1, 2, \dots, D$) in a D -dimensional space. $C_n^\lambda(u)$ is the Gegenbauer polynomial, $u = \cos \theta$, where θ is the angle between the vectors q_i and x_i . The index λ is related to the dimension D of the space by $\lambda = (D - 2)/2$. In three dimensions $\lambda = 1/2$ and $C_n^{1/2}(u) = P_n(u)$ are the Legendre polynomials corresponding to orbital angular momentum quantum number $n = 0, 1, 2, \dots$. The problem of regularisation is not encountered in the Glauber

(*) R is the transformation by reciprocal radii: $z \rightarrow 1/z^*$, where z^* is the complex conjugate of z .

(°) Lorentz contraction along the direction of motion leads to a model of the colliding particles as 2-dimensional discs extended along their transverse dimensions.

theory⁽⁺⁾ because the function $T(s,b)$ is usually approximated by a Gaussian^(4, 5) in b , i.e. $T(s,b)=T(s) e^{-b^2/b_0^2}$. The rapid fall-off of $T(s, b)$ in this case strongly damps the oscillations of the Bessel function $J_0(b\sqrt{|t|})$. The support of the integrand in eq. (2) is thus effectively bounded. The Gaussian approximation for $T(s, b)$ acts effectively as regularisation, although this fact is never explicitly stated.

In conformal invariant quantum field theories, R can be related to the non- positive-definite metric in the Hilbert space of solutions of the Klein-Gordon equation. Let

$$\varphi(x) := \frac{1}{(2\pi)^{D/2}} \int \frac{d^D p}{2p_0} \hat{\varphi}(p) e^{ipx} \quad (4)$$

be a solution of the massless Klein-Gordon equation

$$\square\varphi(x) := \frac{\partial^2 \varphi(x)}{\partial x_0^2} - \sum_{i=1}^D \frac{\partial^2 \varphi(x)}{\partial x_i^2} = 0 \quad (5)$$

in a $(D + 1)$ dimensional space-time. Our metric convention is also defined in eq. (5); thus, the scalar product $p \cdot x = p_\mu x_\mu = p_0 x_0 - \vec{p} \cdot \vec{x}$ where $\mu = 0, 1, 2, \dots, D$, x_0 the time, \vec{p}, \vec{x} D -dimensional vectors.

The conformal inversion operator acts on x_μ as

$$R : x_\mu \rightarrow x'_\mu = x_\mu/x^2 \quad (6)$$

and on the scalar field $\varphi(x)$ as

$$R : \varphi(x) \rightarrow \varphi_R(x) := \left(\frac{1}{x^2}\right)^{\frac{D-1}{2}} \varphi(x') = \left(\frac{1}{x^2}\right)^{\lambda+\frac{1}{2}} \varphi\left(\frac{x}{x^2}\right) \quad (7)$$

where, as before, $\lambda = (D - 2)/2$. The invariance of eq. (5) under R implies that $\varphi_R(x)$ is also a solution of this equation. More generally the function $\psi(x)$ with the expansion⁽⁷⁾

$$\psi(x) = \frac{1}{(2\pi)^{D/2}} \left(\frac{1}{x^2}\right)^{\lambda+\frac{1}{2}} \int \frac{d^D k}{2k_0} \hat{\psi}(k) \exp(ikx/x^2) \quad (8)$$

(+) more generally in the potential scattering theory.

in the basis of eigenfuntions $\left(\frac{1}{x^2}\right)^{\lambda+\frac{1}{2}} \exp(ikx/x^2)$ is a solution of eq. (5). Note that the $(D+1)$ -vector k_μ ($\mu = 0, 1, 2, \dots, D$) has the scale dimension of inverse momentum. As usual, one defines the scalar product of the solutions $\varphi(x)$ and $\psi(x)$ of eq. (3) by^(*)

$$\langle \varphi | \psi \rangle := i \int d^D x \left(\varphi^*(x) \frac{\overleftrightarrow{\partial}}{\partial x_0} \psi(x) \right)_{x_0=0} \quad (9)$$

It is well known that this scalar product is not a positive-definite metric. This fact can be expressed explicitly in terms of properties of R . In fact substituting from eqs. (4) and (8) into (9) one gets

$$\langle \varphi | \psi \rangle = \langle \widehat{\varphi} | R | \widehat{\psi} \rangle \quad (10)$$

where

$$\langle \widehat{\varphi} | R | \widehat{\psi} \rangle = \int \frac{d^D p}{2p_0} \frac{d^D k}{2k_0} \widehat{\varphi}^*(p) R(p, k) \widehat{\psi}(k) \quad (11)$$

and

$$R(p, k) := \frac{1}{(2\pi)^D} \int d^D x \left[\frac{p_0}{(x^2)^{\lambda+\frac{1}{2}}} + \frac{k_0}{(x^2)^{\lambda+\frac{3}{2}}} \right] \exp(i\vec{p} \cdot \vec{x} - i\vec{k} \cdot \vec{x} / x^2) \quad (12)$$

is the representation of the inversion operator R in Hilbert space. It is easy to check that the kernel $R(p, k)$ satisfies all the properties of R : first of all it is unitary, that is

$$\int \frac{d^D k}{2k_0} R(p, k) R(p', k) = 2p_0 \delta^D(\vec{p} - \vec{p}') \quad (13)$$

The unitarity of $R(p, k)$ is with respect to the positive-definite scalar product

(*) $A \frac{\overleftrightarrow{\partial}}{\partial x_0} B := A \left(\frac{\partial B}{\partial x_0} \right) - \left(\frac{\partial A}{\partial x_0} \right) B.$

$$\langle \hat{\varphi} | \hat{\psi} \rangle := \int \frac{d^D p}{2p_0} \hat{\varphi}^*(p) \hat{\psi}(p) \quad (14)$$

of momentum space wave functions. Because of (13) we have

$$\langle R \hat{\varphi} | R \hat{\psi} \rangle = \langle \hat{\varphi} | \hat{\psi} \rangle \quad (15)$$

as to be expected of a unitary operator. Secondly, $R(p, k)$ is also Hermitian (i.e. self-adjoint) with respect to the scalar product in eq. (14), that is

$$R^\dagger(p, k) := R(k, p)^* = R(p, k) \quad (16)$$

or equivalently

$$\langle R \hat{\varphi} | \hat{\psi} \rangle = \langle \hat{\varphi} | R^\dagger \hat{\psi} \rangle = \langle \hat{\varphi} | R \hat{\psi} \rangle = \langle \hat{\varphi} | R | \hat{\psi} \rangle \quad (17)$$

The eigenvalues of a unitary, self adjoint operator are ± 1 . Consequently, we re-obtain the known result that the scalar product defined by [cf eqs (10) and (17)]

$$\langle \hat{\varphi} | \hat{\psi} \rangle := \langle \hat{\varphi} | R | \hat{\psi} \rangle = \langle \varphi | \psi \rangle \quad (18)$$

is not positive-definite^(*) on the basis of properties of the inversion operator. The eigenfunctions $\varphi_n(k)$ ($n = 0, 1, 2, \dots$) of $R(p, k)$ therefore satisfy the integral equation

$$\int \frac{d^D k}{2k_0} R(p, k) \hat{\varphi}_n(k) = (-)^n \hat{\varphi}_n(p) \quad (19)$$

and the orthonormality condition

$$\langle \hat{\varphi}_m | \hat{\varphi}_n \rangle := (-)^n \delta_{mn} \quad (20)$$

Eq. (20) expresses explicitly and clearly the non-positive definiteness of the metric in eq. (18). To find the explicit form of $R(p, k)$ one expands the exponentials $e^{i\vec{k} \cdot \vec{x}}$ and $e^{-i\vec{k} \cdot \vec{x}/x^2}$ in partial waves (cf. eq. (3))⁽⁶⁾

(*) Equivalently, it is an indefinite metric in Hilbert space.

$$e^{i\vec{p}\vec{x}} = \Gamma(\lambda) \left(\frac{2}{p_0 r}\right)^\lambda \sum_{n=0}^{\infty} i^n (n+\lambda) J_{n+\lambda}(p_0 r) C_n^\lambda(\cos \theta_p) \quad (21.a)$$

$$e^{-i\vec{k}\vec{x}} = \Gamma(\lambda) \left(\frac{2r}{k_0}\right)^\lambda \sum_{n=0}^{\infty} (-i)^n (n+\lambda) J_{n+\lambda}\left(\frac{k_0}{r}\right) C_n^\lambda(\cos \theta_k) \quad (21.b)$$

where $p_0 = |\vec{p}|$ and $k_0 = |\vec{k}|$. $C_n^\lambda(\cos \theta)$ is the Gegenbauer polynomial and θ_p, θ_k the angles between \vec{x}, \vec{p} and \vec{x}, \vec{k} respectively. Before substituting (21) in (12), one first expresses θ_k in terms of θ_p and the angle θ between \vec{p} and \vec{k} . This leads to the addition formula for Gegenbauer polynomials⁽⁸⁾.

$$C_n^\lambda(\cos \theta_k) = \sum_{n'=0}^n 2^{n'} \frac{(2\lambda + 2n' - 1)(n - n')(\Gamma(\lambda + n'))^2 \Gamma(2\lambda - 1)}{(\Gamma(\lambda))^2 \Gamma(2\lambda + n + n')} \cdot (\sin \theta_p)^n (\sin \theta)^{n'} C_{n-n'}^{n+\lambda}(\cos \theta_p) C_{n-n'}^{n+\lambda}(\cos \theta) C_{n'}^{n-\frac{1}{2}}(\cos \varphi). \quad (22)$$

φ is here the azimuthal angle between the directions defined by θ and θ_p . Substituting now from eq. (22) into (21.b) and then (21.a) and (21.b) into (12), one gets

$$R(p, k) = \sum_{n=0}^{\infty} R_n(p, k) \quad (23.a)$$

$$R_n(p, k) := \frac{2^{n-2\lambda}}{\pi} \left(\frac{1}{p_0 k_0}\right)^\lambda (n+\lambda) J_{2n+2\lambda} \left(2(p_0 k_0)^{\frac{1}{2}}\right) C_n^\lambda \cos \theta \quad (23.b)$$

The Bessel function $J_{2n+2\lambda}(2(p_0 k_0)^{1/2})$ in eq. (23.b) arises from the radial integral in eq. (12)⁽⁹⁾, i. e.

$$\begin{aligned} \int_0^\infty dr r^{2\lambda+1} \left(\frac{p_0}{r^{2\lambda+1}} + \frac{k_0}{r^{2\lambda+3}}\right) J_{n+\lambda}(p_0 r) J_{n+\lambda}\left(\frac{k_0}{r}\right) = \\ = 2 \int_0^\infty dt J_{n+\lambda}(t) J_{n+\lambda}\left(\frac{p_0 k_0}{t}\right) = 2 J_{2n+2\lambda} \left(2(p_0 k_0)^{\frac{1}{2}}\right) \end{aligned} \quad (24)$$

In eq. (24), one is once more faced with a Hankel transform. Similarly when one makes use of eq. (23) in eqs. (18) and (19). From eq. (23) in fact one sees that the Hankel transform is a representation of the action of the inversion operator R . Indeed, the reduced kernel $R_n(p,R)$ in the Hilbert sub-space with orbital angular momentum n is proportional to $J_{2n+2\lambda}(2(p_o k_o)^{1/2})$. Consequently substituting (23) in eq. (18) gives rise to integrals of the form

$$x^{2\lambda+1} \widehat{\phi}_\nu(x) := \int_0^\infty dy y^{2\lambda+1} J_\nu(xy) \widehat{\psi}(y^2) \quad (25)$$

where $\nu = 2(n+1)$ ($n = 0, 1, 2, \dots$). Viceversa given any Hankel transform one may associate with it a suitable representation of the inversion operator. Now in a conformal invariant field theory (7,10), the Green's functions in coordinate space are power distributions, the powers depending on the scale dimensions of the operators involved. The propagator is a particular such Green's function. It is proportional to $(x^2)^{-d}$ for a field of scale dimension d . Its Fourier transform is also a power distribution $(p^2)^{d-\lambda-3/2}$. The inversion transform of these power distributions in momentum space are of the form

$$A_\nu(\mu;x) = \int_0^\infty dy y^\mu J_\nu(xy) \quad (26)$$

For given ν , the integral in eq. (26) does not exist for all values of μ . The restriction on μ may be removed by regularisation⁽⁷⁾. Basically this means analytically continuing the integral to those values of μ for which it was not originally defined. Consequently, any consistent method of carrying out this regularisation must be regular since it must give the correct result for the integral in the domain of μ where it is originally defined. We shall deal exclusively with eq. (26) in this paper. However, note that eq. (1) reduces to this case if the function $f(y)$ to be transformed admits an expansion in terms of a Mellin transform, i.e.

$$f(y) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\mu \widehat{f}(\mu) y^\mu \quad (27)$$

To close this section, let us recall another instance of the problem of regularisation of the Hankel transform. This arises in the framework of the QCD sum rules⁽¹¹⁾. It was first suggested by Shifman, Vainshtein and Zakharov⁽¹¹⁾ that QCD asymptotic expansions for flavour current propagators are Borel summable⁽¹²⁾. Let $\Pi(q^2)$ be any of the invariant functions of such a propagator, with $\text{Im } \Pi(q^2)$ its imaginary part. The Borel transform of $\Pi(q^2)$ reduces to the so-called exponential moment⁽¹¹⁾

$$Z_0(t) := \frac{t^2}{\pi} \int_0^\infty ds \operatorname{Im} \Pi(s) e^{-st} \quad (28)$$

The first Borel improvement^(*) of $Z_0(t)$ is the Hankel transform⁽¹¹⁾

$$Z_1(t) := \frac{1}{\pi} \int_0^\infty ds \left(\frac{t}{s}\right)^{1/2} J_1(2(st)^{1/2}) \operatorname{Im} \Pi(s) e^{-st} \quad (29)$$

while the second Borel improvement is the Hankel transform⁽¹¹⁾

$$Z_2(t) = \frac{t}{\pi} \int_0^\infty ds J_0(2(st)^{1/2}) \operatorname{Im} \Pi(s) e^{-st} \quad (30)$$

The weight function $\operatorname{Im} \Pi(s)$ in eq. (28) is positive definite whereas those in the integrals of eqs. (29) and (30) are not, due to the oscillations of the Bessel functions. This behaviour of the integrands of eqs. (29) and (30) constitutes the main difficulty in using these transforms in the QCD sum rules. This difficulty is overcome in the method of summability proposed in this paper. This suggests to apply Cesàro summability to the integral in eq. (28) or, even more directly, to the QCD asymptotic expansion of $\Pi(q^2)$ itself.

3. - CESÀRO SUMMABILITY OF HANKEL TRANSFORM INTEGRALS

We return in this section to the Hankel transform in the general form of eq. (1) without any specific physical meaning to the function $f(y)$. We require however that $f(y)$ have a convergent expansion in terms of a Mellin transform as in eq. (27). Under this assumption, the Hankel transform to be regularised is given by eq. (26). The integral on the right hand side of eq. (26) exists for $-\operatorname{Re}(\nu) - 1 < \operatorname{Re}(\mu) < -1/2$ ^(°) and is given by⁽¹²⁾

$$\int_0^\infty dy y^\mu J_\nu(xy) = 2^\mu \frac{\Gamma\left(\frac{1+\nu+\mu}{2}\right)}{\Gamma\left(\frac{1+\nu-\mu}{2}\right)} x^{-\mu-1} \quad (31)$$

(*) i. e. the Borel transform of the Borel transform.

(°) $\operatorname{Re}(z)$ stands for the real part of z .

For fixed ν , the gamma function $\Gamma\left(\frac{1+\nu+\mu}{2}\right)$ is a meromorphic function of μ with simple poles at

$$\mu = -2n - (\nu + 1); \quad n = 0, 1, 2, \dots \quad (32)$$

and residues $(-)^n/n!$. The lower limit $\text{Re}(\mu) > -\text{Re}(\nu) - 1$ of the domain of convergence of the integral in the μ -plane is determined by the first singularity $n = 0$ in eq. (32). This limit may also be established from the behaviour of the integrand $y^\mu J_\nu(xy)$ for $y \rightarrow 0$ i.e.

$$\left| y^\mu J_\nu(xy) \right| \xrightarrow{y \rightarrow 0} y^{\text{Re}(\mu) + \text{Re}(\nu)} \quad (33.a)$$

whence, upon integration, $\text{Re}(\mu)$ must be greater than $-\text{Re}(\nu) - 1$ for convergence of the integral at $y \rightarrow 0$. Analogously, the upper limit $\text{Re}(\mu) < -1/2$ arises from the asymptotic behaviour

$$\left| y^\mu J_\nu(xy) \right| \xrightarrow{y \rightarrow \infty} y^\mu \left(\frac{\pi y}{2}\right)^{-\frac{1}{2}} \cos\left(y - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) \sim y^{\text{Re}(\mu) - \frac{1}{2}} \quad (33.b)$$

Upon integrating, one sees that $\text{Re}(\mu)$ must be less than $-1/2$ for the integral to converge at the upper limit $y \rightarrow \infty$. Note that the convergence restriction $-\text{Re}(\nu) - 1 < \text{Re}(\mu) < -1/2$ also constrains ν , namely $\text{Re}(\nu) > -1/2$. For μ and ν outside of these ranges, the integral in eq. (31) is defined indirectly by assigning to it the corresponding values of the analytic function of μ and ν on the right hand side of (31). For the function $A_\nu(\mu; x)$ in eq. (26) one therefore sets, for all values of μ and ν

$$A_\nu(\mu; x) := 2^\mu \frac{\Gamma\left(\frac{1+\nu+\mu}{2}\right)}{\Gamma\left(\frac{1+\nu-\mu}{2}\right)} x^{-\mu-1} \quad (34)$$

It is useful to consider ν fixed and to study the behaviour of $A_\nu(\mu; x)$ as an analytic function of μ . As an analytic function of μ , $A_\nu(\mu; x)$ has simple poles at the values of μ given in eq. (32) and zeroes at

$$\mu = 2n + (\nu + 1); \quad n = 0, 1, 2, \dots \quad (35)$$

The zeroes arise from the poles of the gamma function $\Gamma\left(\frac{1+v-\mu}{2}\right)$ in the denominator of the right hand side of eq. (34). Eq. (34) may now be substituted into eq. (27) and the contour integral evaluated to yield the regulated form of the Hankel transform of $f(y)$.

Our purpose in this section is to show that one may arrive at eq. (34) also much more directly by an explicit summability procedure applied to the Hankel transform integral itself. The oscillatory behaviour of the integrand $y^\mu J_\nu(x y)$ is not a problem, as far as this procedure is concerned. On the contrary, the procedure may be defined, in general, with respect to such oscillatory behaviour. Non-oscillating integrands then constitute a special and much simpler case. The behaviour of the integrand in eq. (26) is shown in figs(1) as a function of y for four values of ν ($\nu = 0, 1, 3/2$ and 4) and for μ taken for the two intervals^(*) $\nu < \mu < 1/2$ and $1/2 < \mu < \infty$. For simplicity we have also set $x = 1$ in eq. (26) so that the integral to be regularised becomes

$$A_\nu(\mu) := A_\nu(\mu; x = 1) = \int_0^\infty dy y^\mu J_\nu(y) \quad (36)$$

with analytic continuation

$$A_\nu(\mu) = 2^\mu \frac{\Gamma\left(\frac{1+\nu+\mu}{2}\right)}{\Gamma\left(\frac{1+\nu-\mu}{2}\right)} \quad (37)$$

There is no loss of generality in restricting ourselves to eq. (36). By a change of variables, eq. (26) can be brought into the form

$$A_\nu(\mu, x) = x^{-\mu-1} A_\nu(\mu) \quad (38)$$

Now with every oscillating function, e.g. those in Figs. (1), we associate a property, the quasi-period T , which generalises that of a period for periodic functions. By definition a quasi-period T is the distance between any two maxima or minima of the function or between any two zeroes separated by one maximum and one minimum. Unlike for a periodic function, T is not a constant so that the oscillating function has not got one but many (may be even an infinite number of) quasi-periods. The usefulness of this concept emerges if it becomes convenient to measure distances in terms of the number of oscillations of the function. This is certainly the case if we evaluate the integral by successive approximations generated by varying the cut-offs

(*) We take both μ and ν to be real. For $\mu < 0$ see eq. (50).

on the upper limit of integration. The convenience consists in taking these cutt-offs to be related to properties of the integrand, e.g. its zeroes, its maxima and its minima. Let us call the zeroes of the oscillating function and its maxima and minima, the fixed points of the function. We order them in increasing order of their distance from the origin and denote them by y_N ($N = 0, 1, 2, \dots$). Note that y_0 need not coincide with the origin $y = 0^{(\circ)}$. Between any two consecutive fixed points, the oscillating function is monotonic, either increasing or decreasing. By definition, this distance is a quarter of a quasi-period. We denote the interval of monotonicity between the N -th and the $(N + 1)$ -th fixed point by $T_N/4$ ($N = 0, 1, 2, \dots$)

The integral of the oscillating function in this interval we denote by $a(N/4)$ ($N = 0, 1, 2, \dots$). We now define, iteratively, the sum of the $a(N/4)$ by

$$A\left(-\frac{N}{4}\right) := 0 \quad ; \quad N \geq 1 \quad (39.a)$$

$$A\left(\frac{N}{4}\right) := A\left(\frac{N-1}{4}\right) + a\left(\frac{N}{4}\right) \quad ; \quad N \geq 0 \quad (39.b)$$

The $A(N/4)$ ($N = 0, 1, 2, \dots$) form a sequence of approximations for the integral in eq. (31). As explained earlier, they are obtained by varying the cut-off introduced as the upper limit of integration. Next we define the following hierarchy of arithmetic means of the $A(N/4)$:

$$C^{(0)}\left(\frac{N}{4}\right) := A\left(\frac{N}{4}\right) \quad ; \quad N = 0, 1, 2, \dots \quad (40.a)$$

$$C^{(k)}\left(\frac{N}{4}\right) := \frac{1}{N} \sum_{n=0}^{N-1} C^{(k-1)}\left(\frac{n}{4}\right) \quad ; \quad k = 1, 2, \dots \quad N = 1, 2, \dots \quad (40.b)$$

The $C^{(k)}(N/4)$ ($k = 1, 2, \dots$) are known as the Cesàro means of the sequence $A(N/4)$ ($N = 0, 1, 2, \dots$). Now, if the limit

$$C^{(k)} := \lim_{N \rightarrow \infty} C^{(k)}\left(\frac{N}{4}\right) \quad (41)$$

exists, it follows from eq. (40.b) that the limits

^(\circ) If $y_0 \neq 0$, the integral of the integrand between $y = 0$ and $y = y_0$ can always be added at the end to the result we shall obtain later starting from y_0 .

$$C^{(k+k')} := \lim_{N \rightarrow \infty} C^{(k+k')} \left(\frac{N}{4} \right) ; k' = 0, 1, 2, \dots \quad (42)$$

all exist and we have the equality

$$C^{(k+k')} = C^{(k)} ; k' = 0, 1, 2, \dots \quad (43)$$

It does not however mean that the limits

$$C^{(k-k')} := \lim_{N \rightarrow \infty} C^{(k-k')} \left(\frac{N}{4} \right) ; k' = 1, 2, \dots, k \quad (44)$$

exist. Under these circumstances one defines the Cesàro summable value of the integral of the function to be $C^{(k)}$. Applying this definition to the Hankel transform integral in eq. (36) we have

$$A_v(\mu) = \int_0^\infty dy y^\mu J_v(y) := C_v^{(k)}(\mu) \quad (45)$$

as the Cesàro summable value of $A_v(\mu)$. We have restored the dependence of the Cesàro mean $C^{(k)}$ on μ and v . It turns out that for the integral in eq. (45), the limit

$$C_v^{(1)}(\mu) := \lim_{N \rightarrow \infty} C_v^{(1)} \left(\mu ; \frac{N}{4} \right) \quad (46)$$

of the first Cesàro mean exists. More importantly it gives a value for $A_v(\mu)$ according to the definition in eq. (45) which agrees with eq. (37), i.e.

$$C_v^{(1)}(\mu) = 2^\mu \frac{\Gamma\left(\frac{1+v+\mu}{2}\right)}{\Gamma\left(\frac{1+v-\mu}{2}\right)} \quad (47)$$

In other words, Cesàro summability of the integral in eq. (45) coincides with analytic continuation. It is interesting to see how the limit $C_v^{(1)}(\mu)$ is attained starting from the sequence

of approximations $A(N/4)$ of the integral^(*). Putting $k = 1$ in eq. (40.b) and making use of (40.a) one gets

$$C^{(1)}\left(\frac{N}{4}\right) = \frac{1}{N} \sum_{n=0}^{\left[\frac{N-1}{2}\right]} A\left(\frac{n}{2}\right) + \frac{1}{N} \sum_{n=0}^{\left[\frac{N-1}{2}\right]} A\left(\frac{2n-1}{4}\right) \quad (48)$$

By definition $[N/2]$ is the integer closest to $N/2$. Taking the limit $N \rightarrow \infty$ in (48) yields

$$C^{(1)} = \lim_{N \rightarrow \infty} \frac{1}{2} \left[A\left(\frac{N}{2}\right) + A\left(\frac{2N-1}{4}\right) \right] \quad (49)$$

The approach to the limit defined in eq. (49) and in higher Cesàro means is illustrated graphically in Figs (2) for four values of ν ($\nu = 0, 1, 3/2$ and 4) and fixed $\mu > 0$. The evaluations of the integrals $A(N/4)$ ($N = 0, 1, 2, \dots$), their sum $A(N/4)$ ($N = 0, 1, 2, \dots$) and the Cesàro means $C^{(k)}(N/4)$ were all carried out numerically. This was done only for $\mu > 0$. For $\mu < 0$ we make use of the relation (cf. eq. (37)).

$$A_{\nu}(\mu) A_{\nu}(-\mu) = 1 \quad (50)$$

In Figs. (2), for each pair of values of μ and ν , we first give the plot of the Hankel transform integral as a function of the cut-off. The next plots give the first few successive Cesàro means of the sequence of cut-off integrals. These plots allow to see the rapid convergence of these means.

In Figs. (3), on the other hand, we give the behaviour, of $A_{\nu}(\mu)$ for $\nu = 0, 1, 3/2$ and 4 , as a function of μ .

4. - CONCLUSIONS

Quite generally integrals of oscillating functions over an infinite interval tend to have bad convergence properties unless the oscillations are suitably damped. An acceptable level of accuracy in their numerical computations requires some labour and particular care. Such integrals e.g. the Hankel transform, occur in many physical applications. Their analytic expressions are often not available and often not easy to find. Worse still these integrals are often required in domains where they must be defined by some procedure of regularisation. We

(*) For ease of notation, we again suppress the dependence of $A(N/4)$, $C^{(1)}(N/4)$ and $C^{(1)}$ on μ and ν .

have discussed in Sect. 3 a simple method of summability based on the convergence of a sequence of arithmetical means. The simplicity of the method resides in this fact but also in the fact that the method attacks the problem of regularisation directly: it is applied to a sequence of approximations of the integral. This sequence of approximations may be generated by any suitable infinite set of cut-offs. The hierarchy of arithmetical means, which may be constructed from the cut-off integrals, allows to extend the definition of convergence of the sequence and hence to arrive at a new definition for the value of the integral. The hierarchy of arithmetical means is nothing but a set of linear transformations which operate on the sequence of cut-off integrals. For instance from eqs. (40.a) and (40.b) one has $\left(C^{(0)}\left(\frac{N}{4}\right) := A\left(\frac{N}{4}\right) \right)$

$$C^{(1)}\left(\frac{N}{4}\right) = \frac{1}{N} \sum_{N'=0}^{N-1} C^{(0)}\left(\frac{N'}{4}\right) \quad (51.a)$$

$$C^{(2)}\left(\frac{N}{4}\right) = \frac{1}{N} \sum_{N'=0}^{N-1} L_{NN'} C^{(0)}\left(\frac{N'}{4}\right) \quad (51.b)$$

where the matrix elements $L_{NN'}$ are defined by

$$L_{NN'} := \sum_{n=1}^{N-N'-1} \frac{1}{N(N'+n)} \quad (52)$$

From eq. (40.b) one gets, on generalising (51.b),

$$C^{(k)}\left(\frac{N}{4}\right) = \frac{1}{N} \sum_{N'=0}^{N-1} L_{NN'} C^{(k-2)}\left(\frac{N'}{4}\right) \quad ; \quad k \geq 2 \quad (53)$$

Iterating eq. (53), one sees clearly that the $C^{(k)}(N/4)$ are obtained from $C^{(0)}(N/4) = A(N/4)$ by linear transformations. If in a domain D of the dependent variable μ , the limit $A(\mu)$ of the sequence $A(N/4)$ exists then it follows, as pointed in eq. (43), that the limits $C^{(k)}(\mu)$ of Cesàro means $C^{(k)}(N/4)$ ($k \geq 1$) also exist in D , so that we have

$$A(\mu) = C^{(k)}(\mu) \quad ; \quad \mu \in D \quad ; \quad k \geq 0 \quad (54.a)$$

On the other hand, if in the complementary domain \bar{D} , or in a part of it, the $C^{(k)}(\mu)$ exist for some $k = \bar{k} \geq 1$, it does not follow that the limits $C^{(k)}(\mu)$ ($k = 0, 1, \dots, \bar{k} - 1$) exist in \bar{D} , Often they do not. In this, case, one uses the limit $C^{(\bar{k})}(\mu)$ to define the value of the integral $A(\mu)$ for $\mu \in \bar{D}$, i.e.

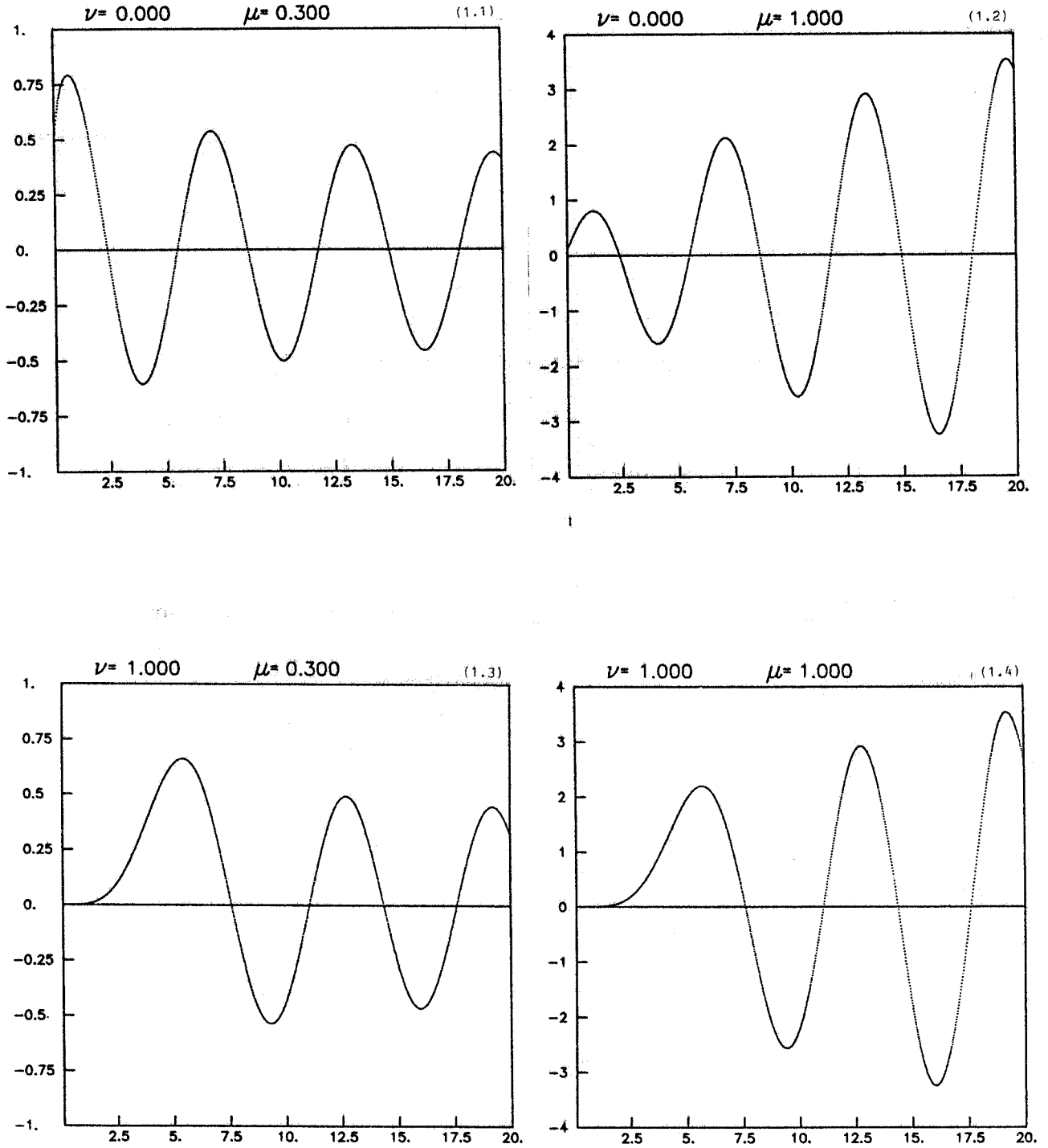
$$A(\mu) := C^{(\bar{k})}(\mu) \quad ; \quad \mu \in \bar{D} \quad ; \quad \bar{k} \geq 1 \quad (54.b)$$

It is in the sense of eqs. (54.a) and (54.b) that $C^{(\bar{k})}(\mu)$ constitutes an analytic regularisation of $A(\mu)$ for $\mu \in \bar{D}$ or, equivalently, the analytic continuation of $A(\mu)$ from D to \bar{D} . Under these circumstances one reads eq. (54.b) to mean that the integral $A(\mu)$ is Cesàro summable to the value $C^{(\bar{k})}(\mu)$. Note that it is $C^{(\bar{k})}(\mu)$ which defines $A(\mu)$ for $\mu \in \bar{D}$ and not vice versa. On the other hand for $\mu \in D$, eq. (54.a) is a set of identities, once $A(\mu)$ exists. In regularising the Hankel transform integral we have done no more than apply eqs. (54.a) and (54.b). The procedure is therefore very direct.

In Sect. 2 we reviewed some important problems in high energy physics in which the Hankel transform figures prominently. We showed there that this transform is related to representations of the conformal inversion operator R in Hilbert space. It would be interesting to see how this connection could be exploited in the problem of analytic continuation involved in the QCD sum rules, quite apart from any particular method of summability. In any case, the most immediate and promising application of the summability method discussed here is to the investigation of the QCD sum rules. Comparison with and a check of the results obtained by Borel summability are both necessary and important.

Lastly, to avoid any misunderstandings, we stress that the method of Cesàro summability is not applicable only to integrals of oscillating functions. The sequence of approximations $A(N/4)$ for the integral is available in all cases. For oscillating functions, the cut-offs giving rise to the approximations have been conveniently chosen to coincide with its fixed points (i.e. the zeroes, maxima and minima). They could have been chosen differently without any change in the procedure of summability. However, divergences of integrals caused by amplified oscillations may be compared to and even used to simulate the ultraviolet divergences caused by the large fluctuations of quantum fields at short distances. Therein lies their special importance and of the methods of regularisation which can eliminate the associated divergences. The idea is useful also in the study of the dynamics of random systems. The direct approach to regularisation offered by Cesàro summability is interesting in view of its possible application to these problems.

FIG. 1 - Plots of the integrand $y^\mu J_\nu(y)$ as a function of y for two values of μ ($\mu = 0, 3$ and $1, 0$) and four values of ν ($\nu = 0, 1, 3/2$ and 4).



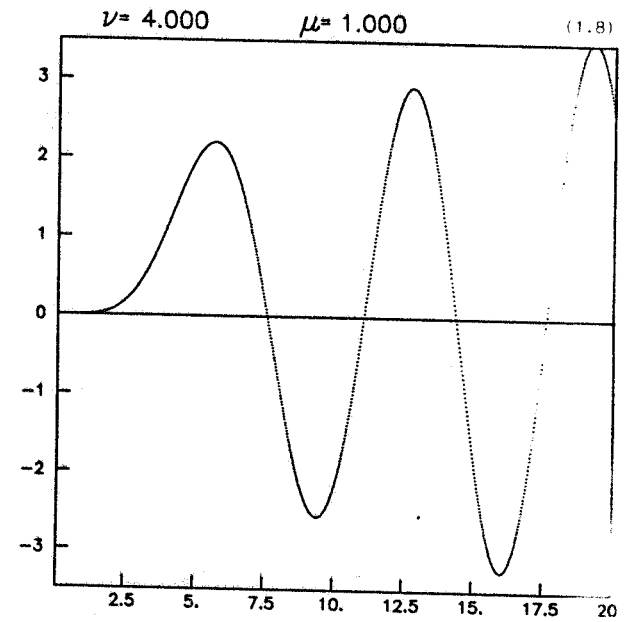
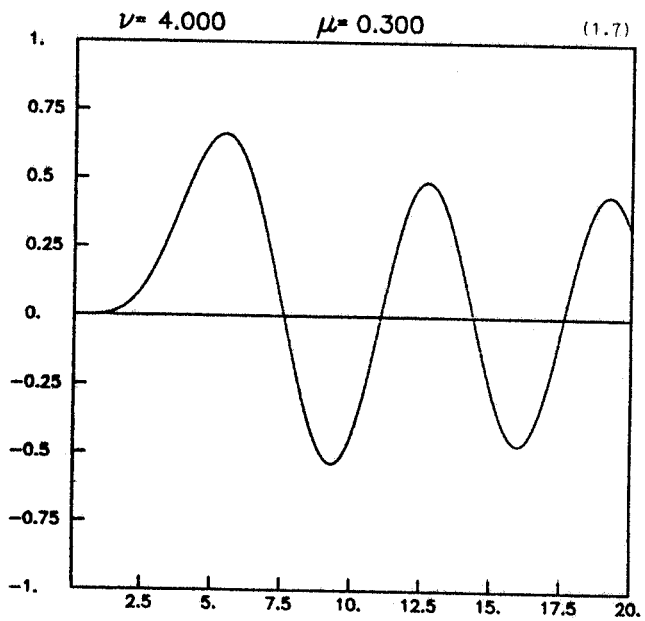
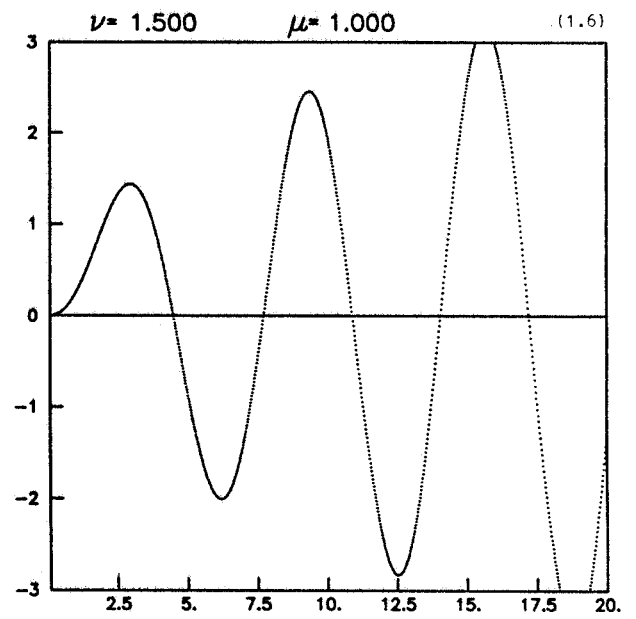
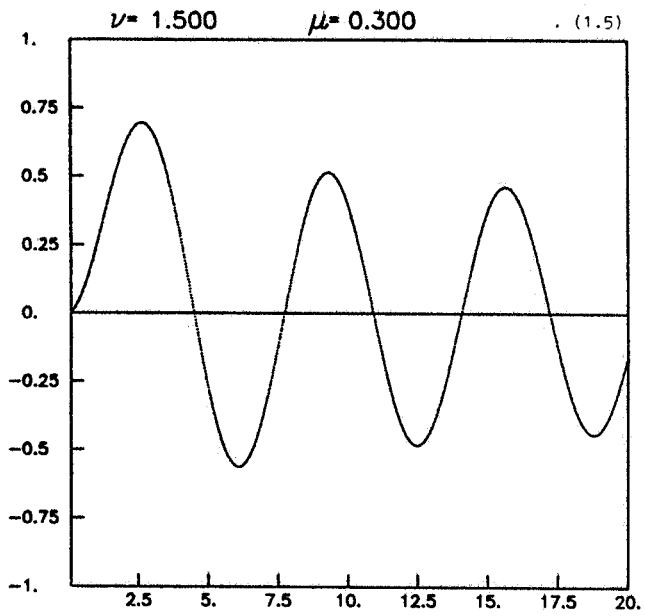
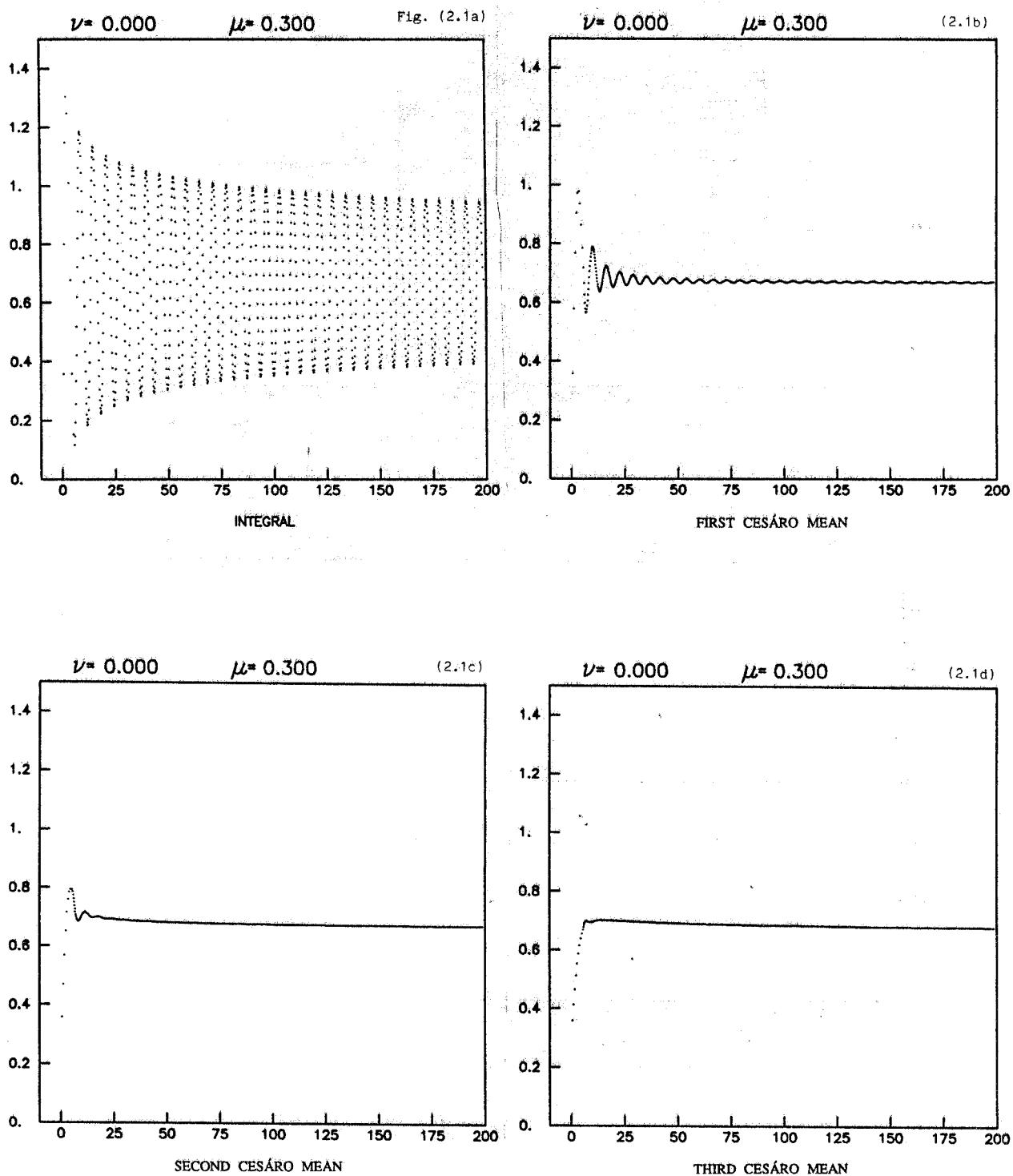
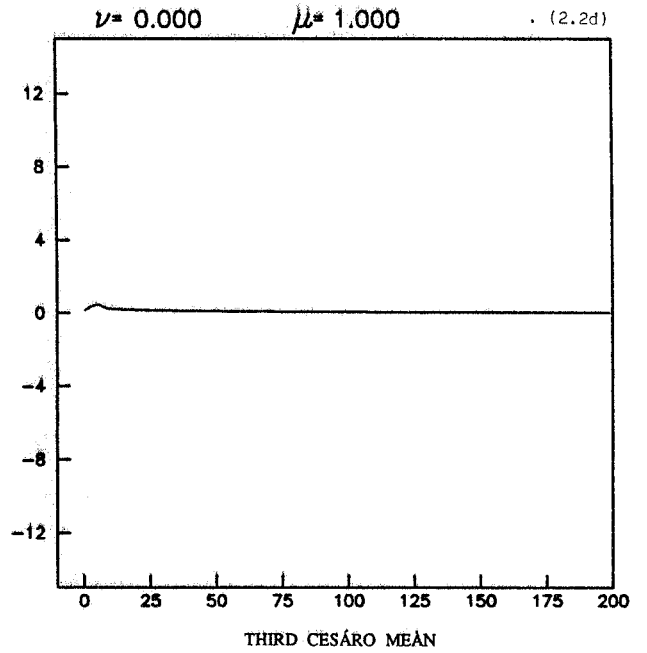
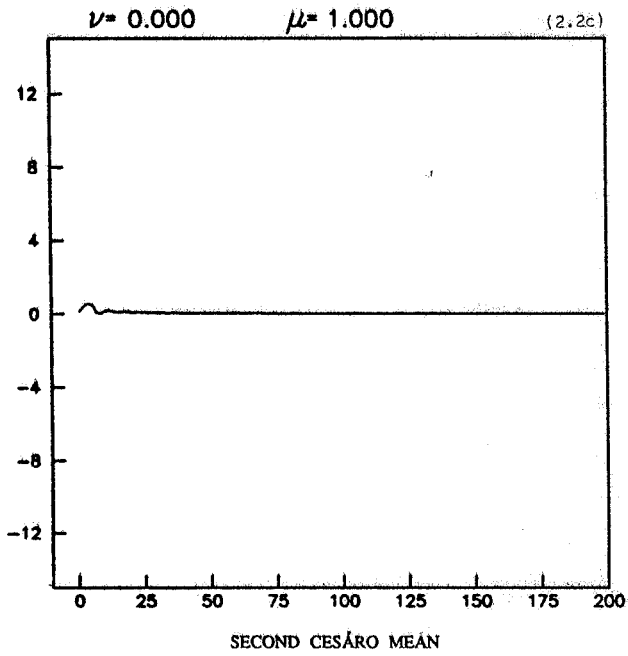
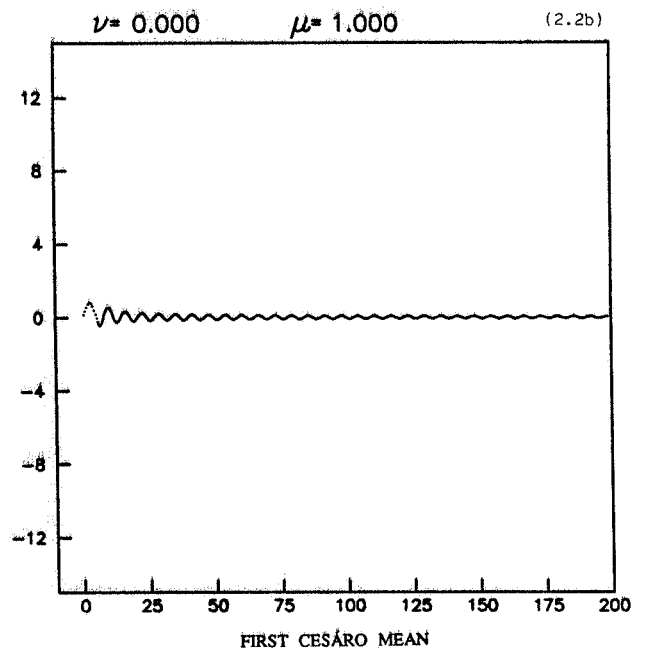
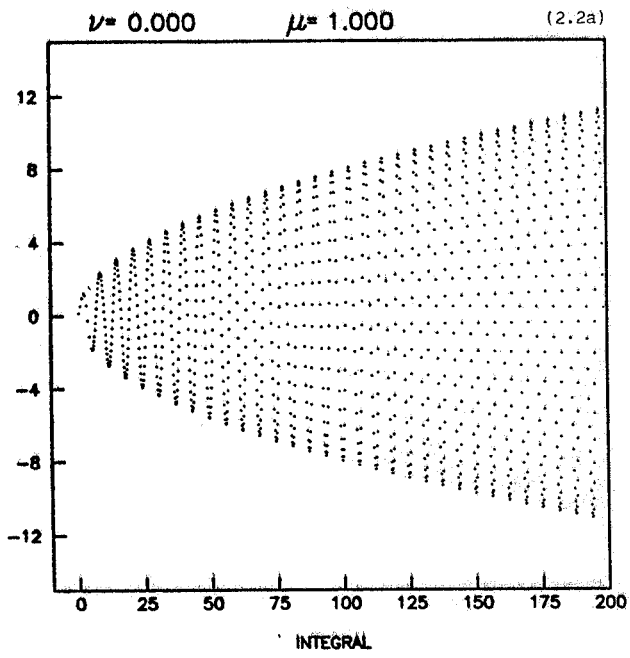
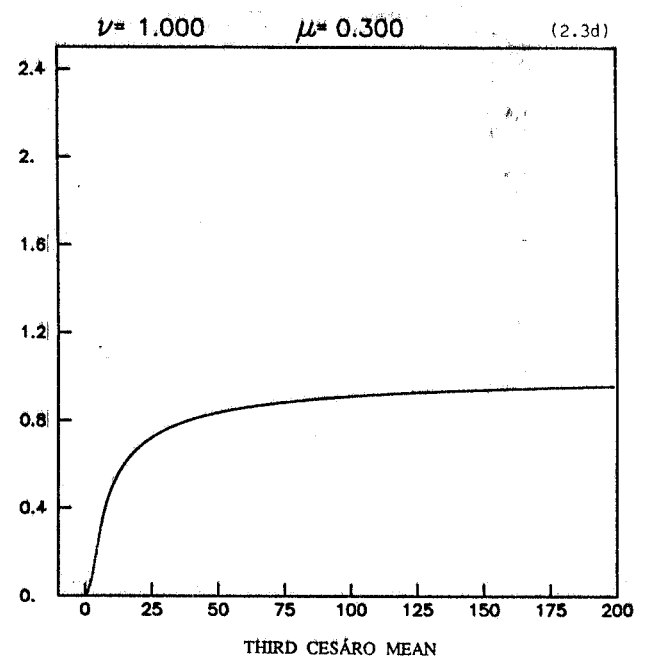
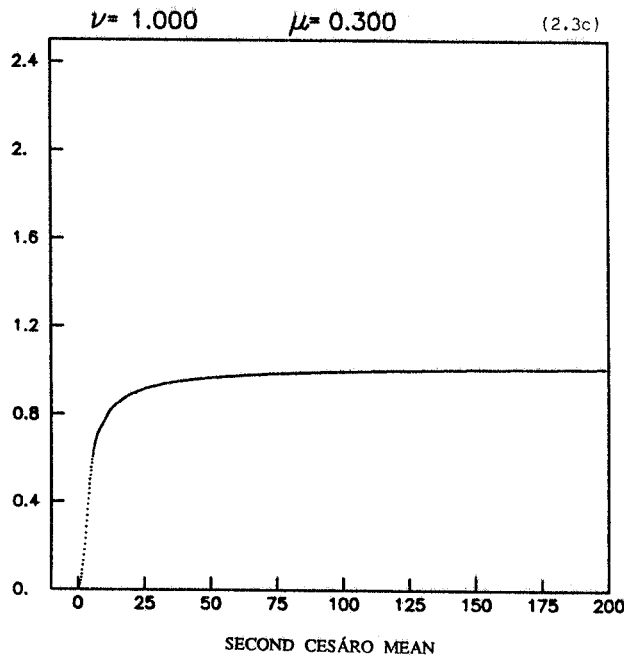
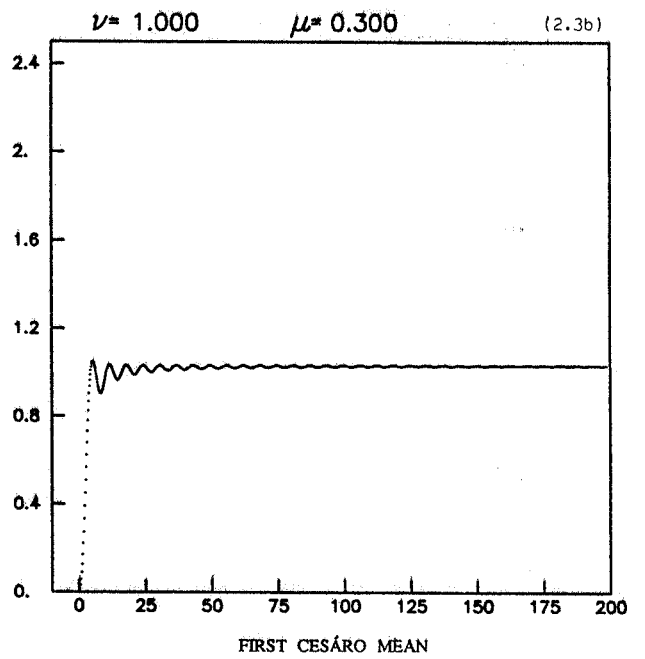
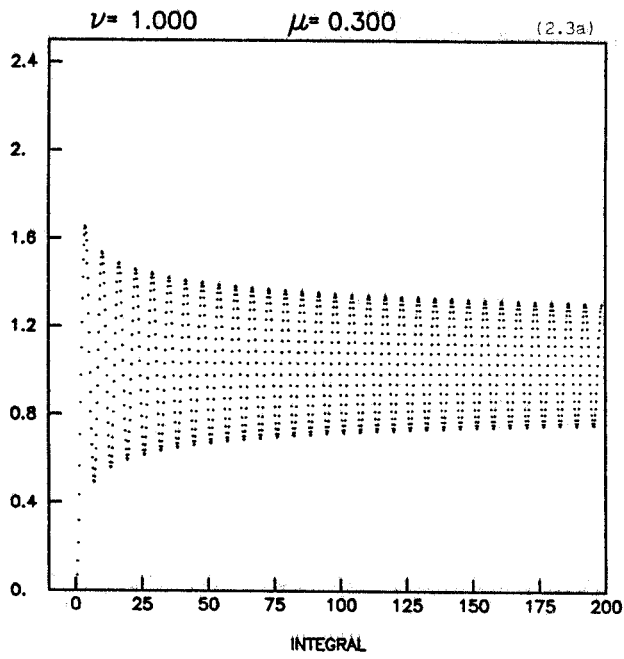
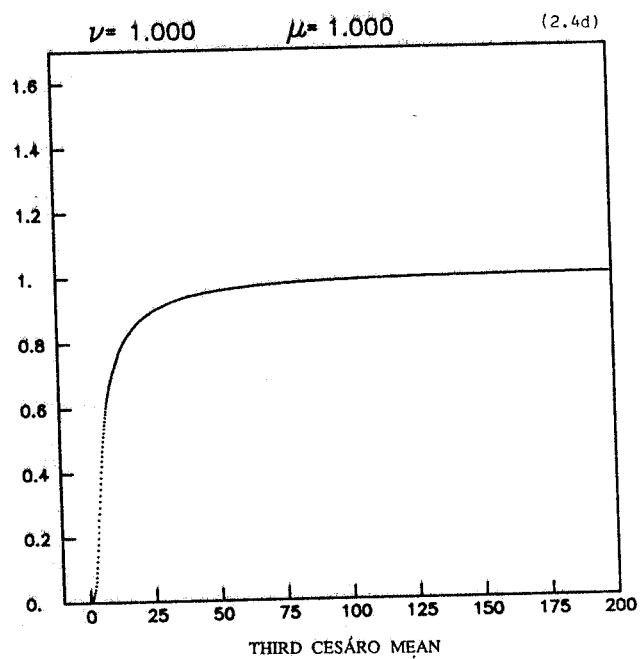
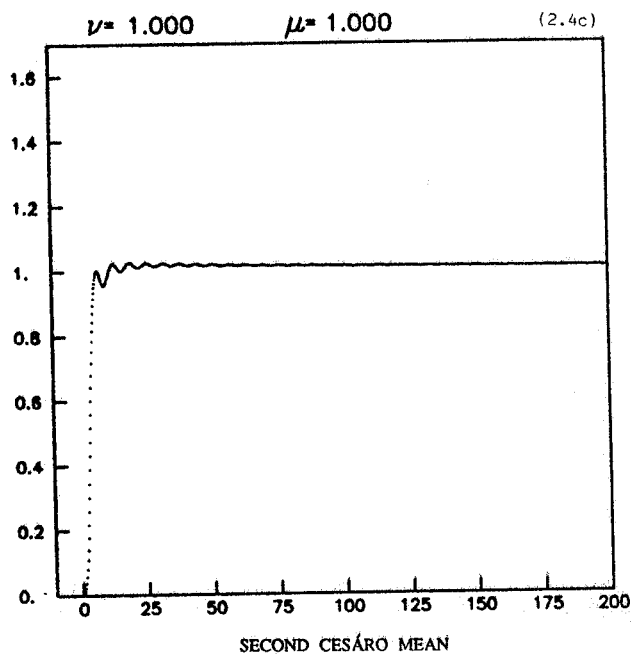
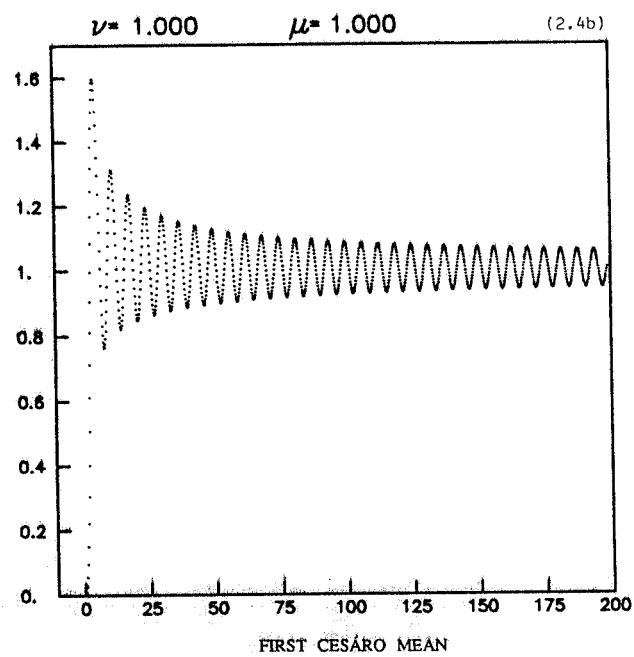
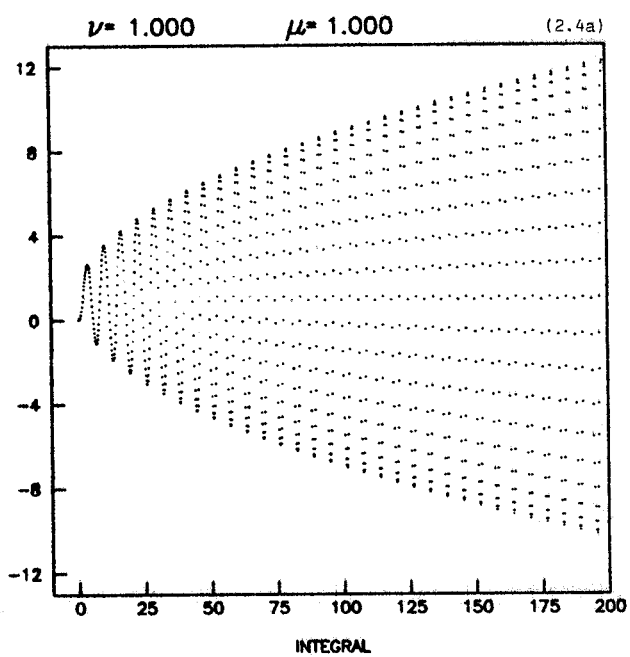


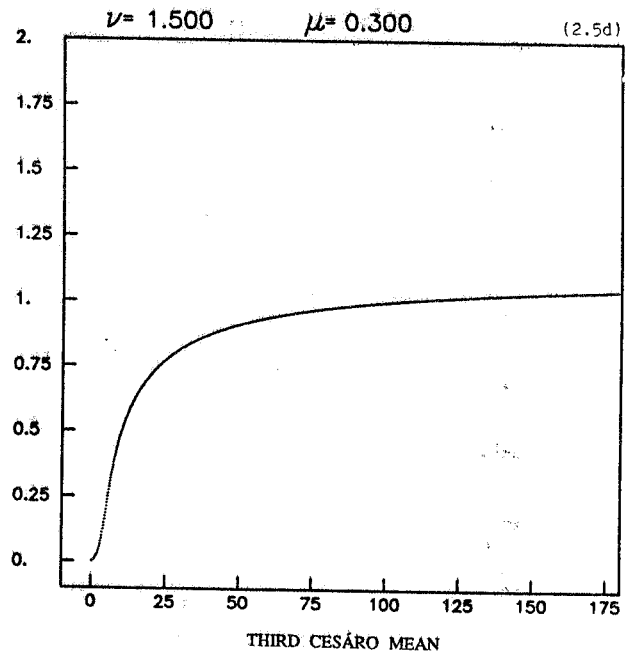
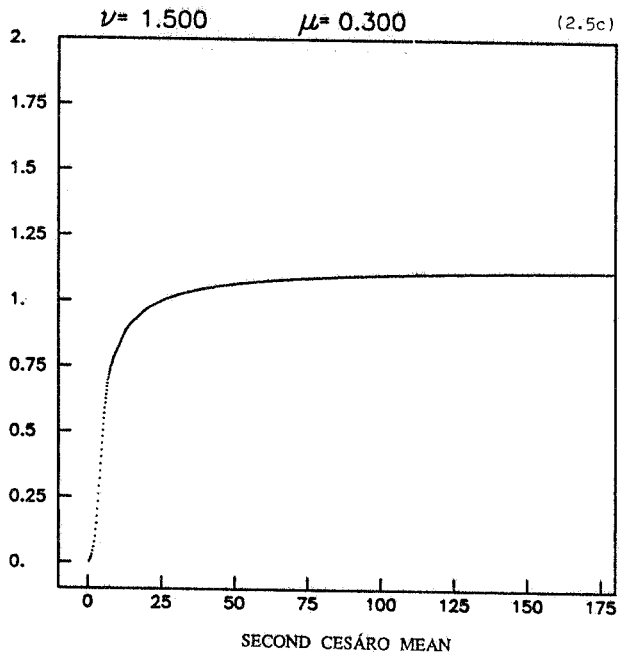
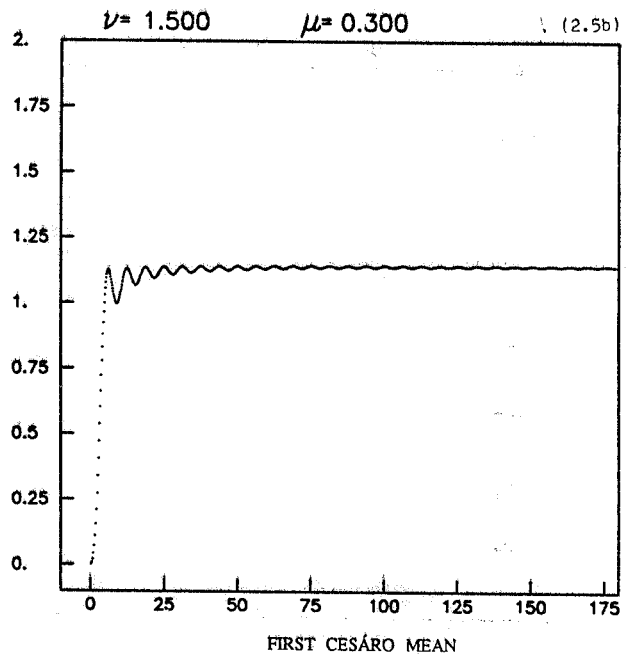
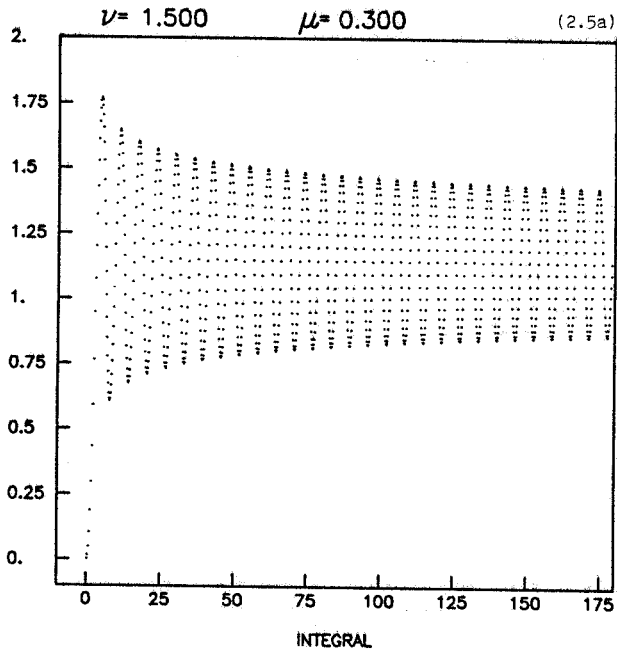
FIG. 2 - Plots of the Hankel transform integral of eq. (36) as a function of cut-off and the first three Cesàro means of the integral for $\nu = 0, 1, 3/2$ and 4.

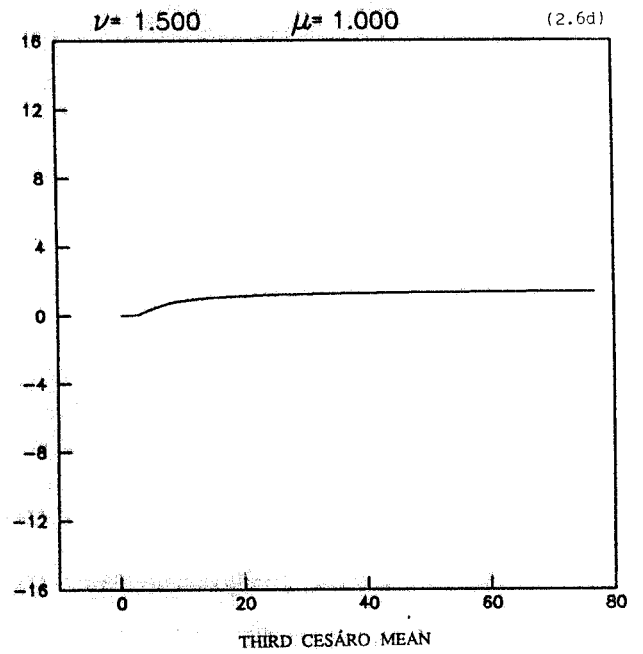
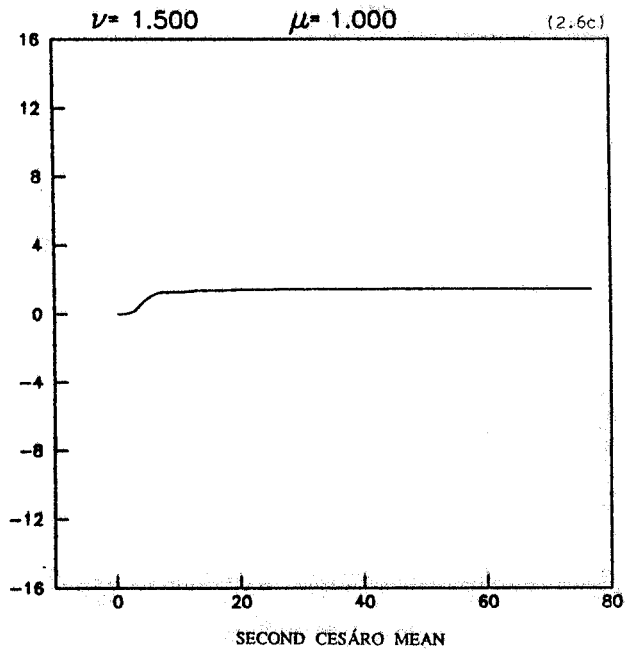
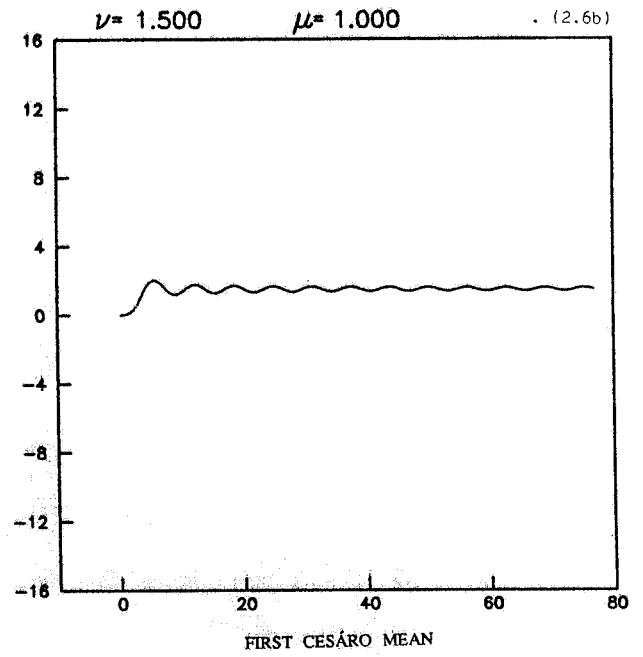
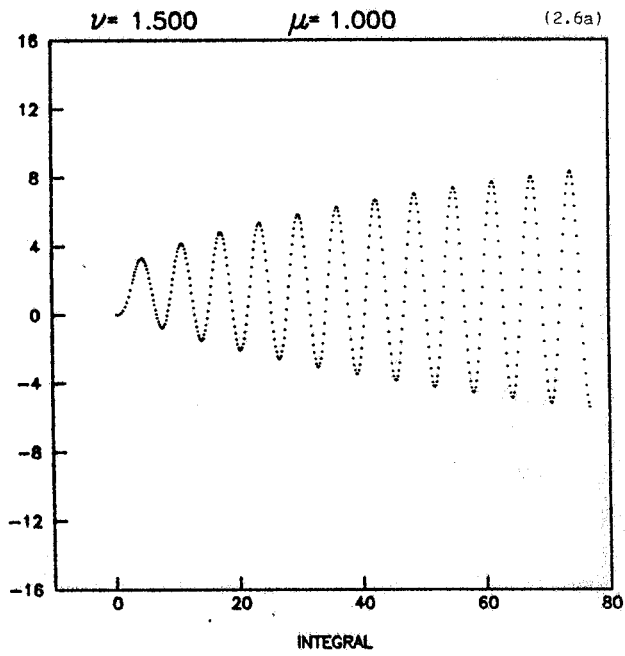


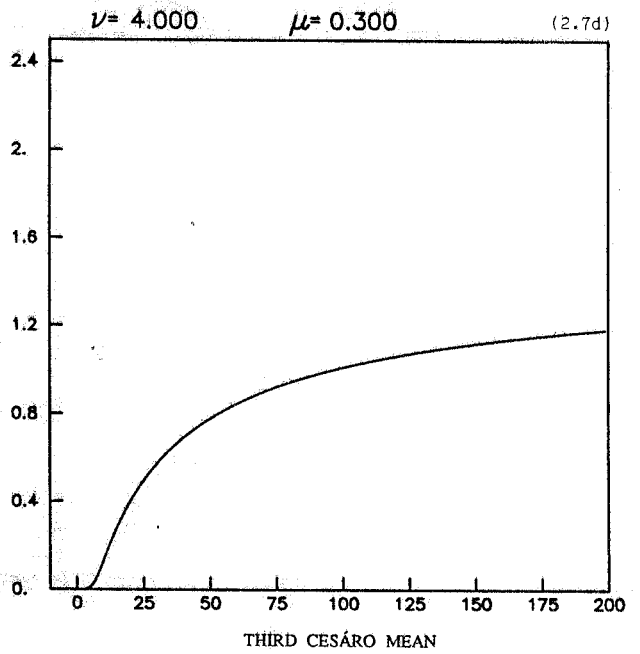
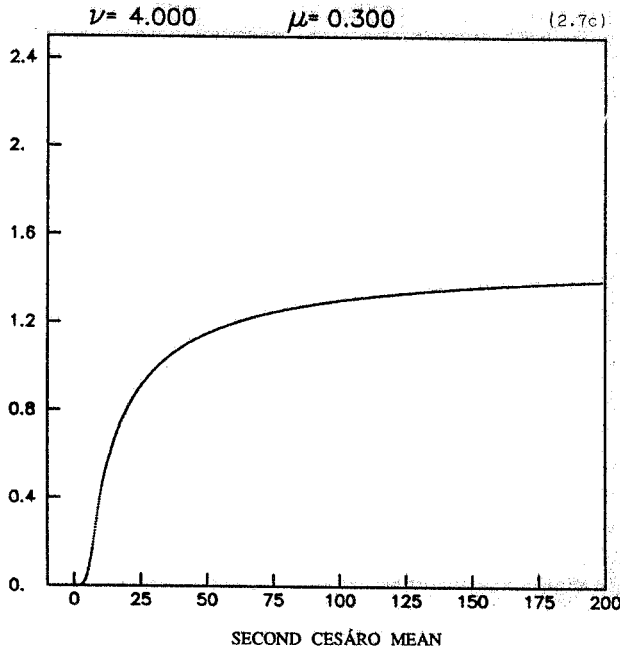
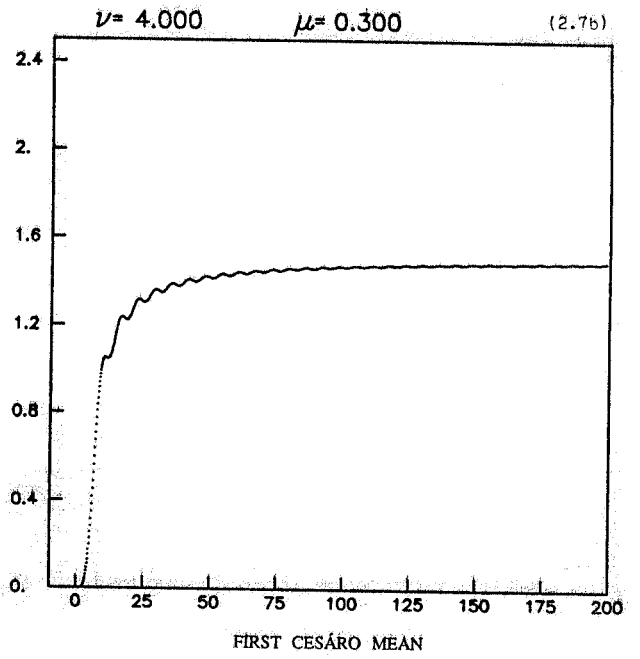
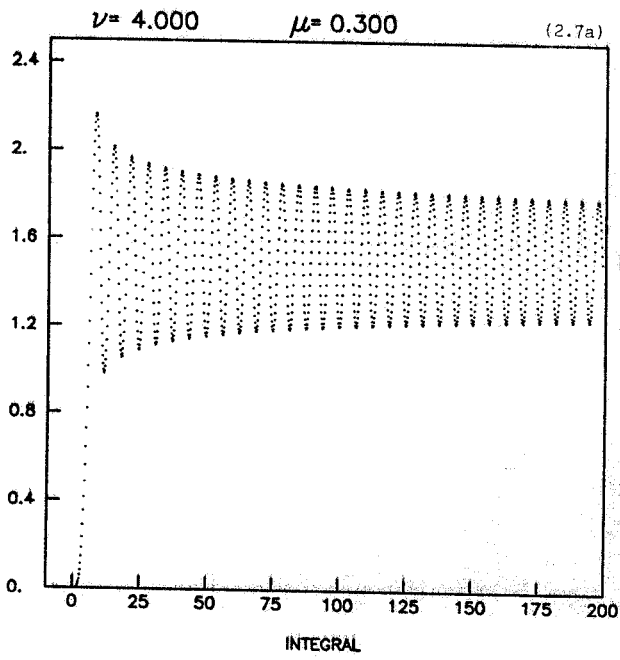












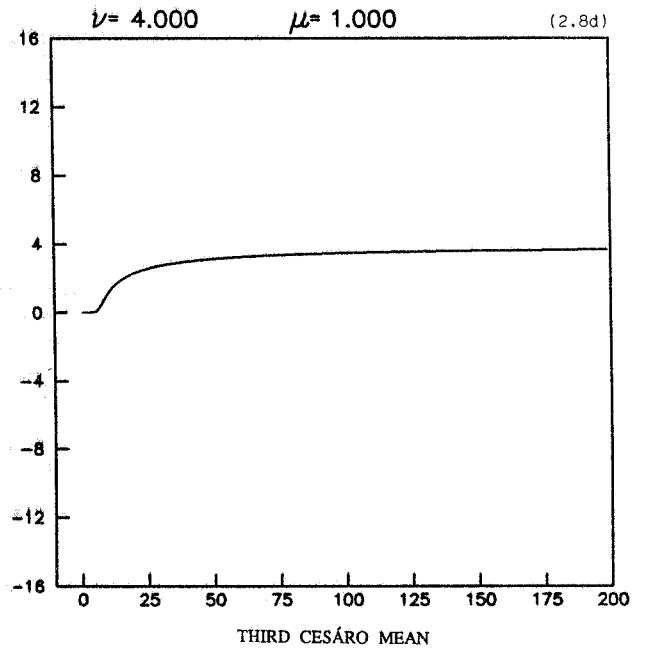
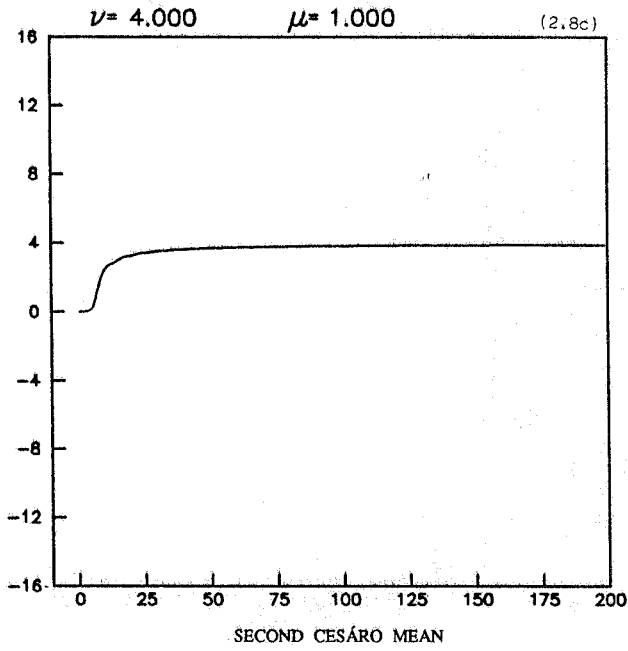
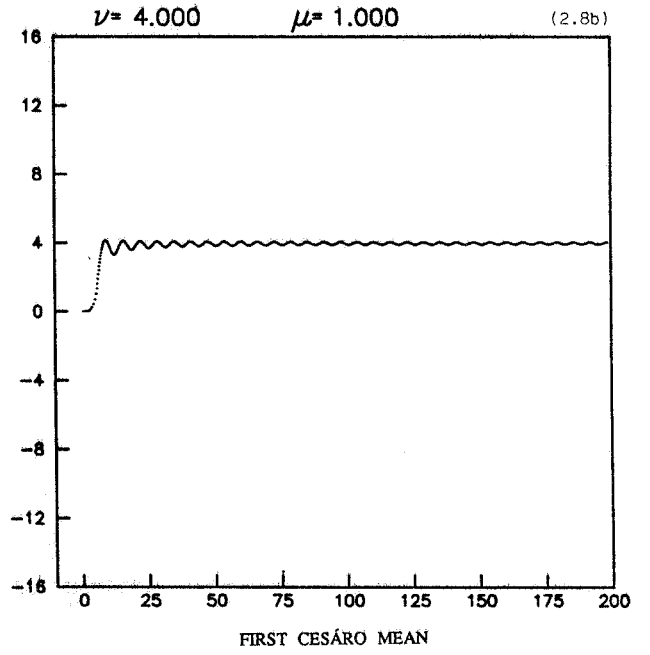
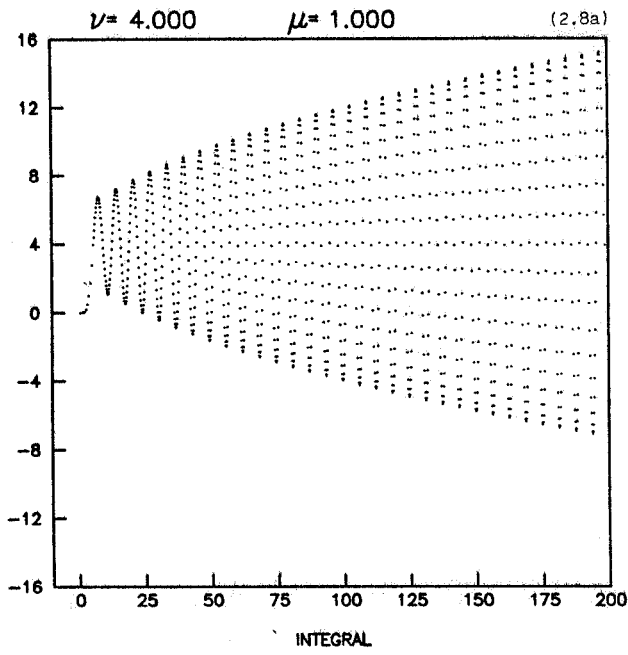
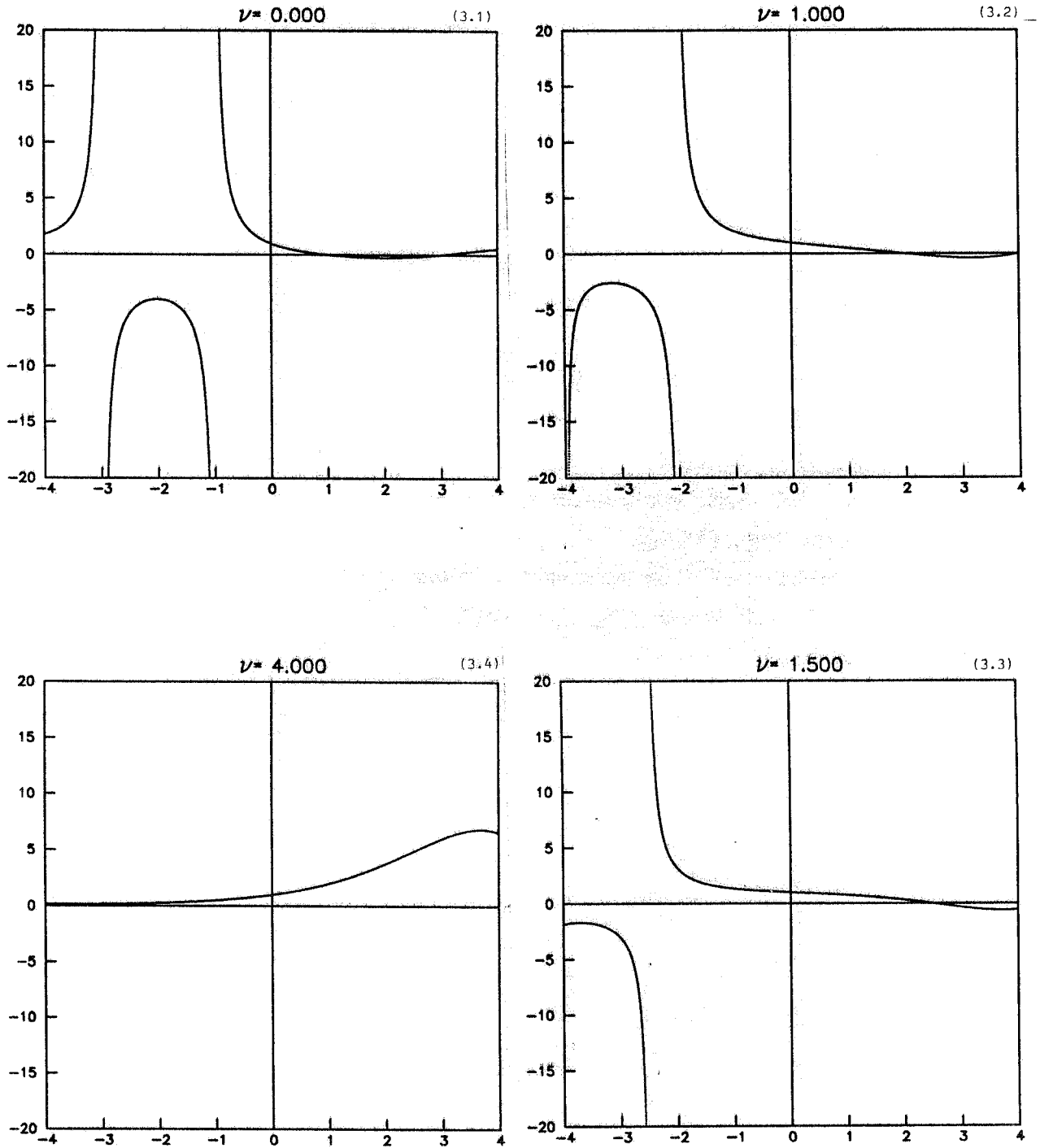


FIG. 3 - Plots of the completely regularised integral $A_\nu(\mu)$ of eq. (36) as a function of μ for $\nu = 0, 1, 3/2$ and 4.



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