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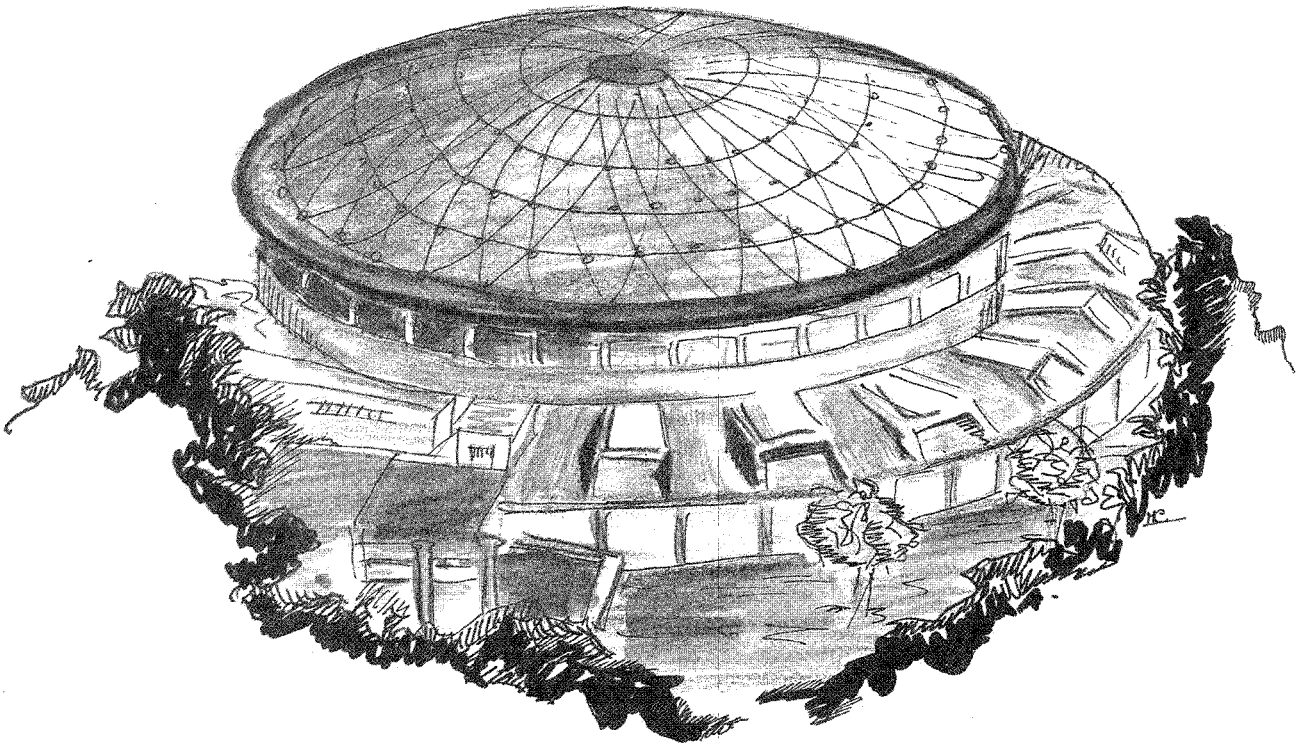
CONSISTENT CHIRAL BOSONIZATION WITH ABELIAN AND NON-ABELIAN GAUGE SYMMETRIES

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Abstract

An $O(N)$ non-abelian generalization of Siegel's action for chiral bosons coupled to gauge fields is presented. We show that the only consistent gauging of the $O(N)$ symmetry corresponds to that of N free Majorana-Weyl fermions. The equivalence between the quantized $O(N)$ chiral model without gauge fields and the free fermion theory is then established. The theory is treated as a constrained hamiltonian system in which zeromodes are carefully taken into account. Specializing to the gauged abelian case, the solution to the chiral Schwinger model is discussed in the framework of chiral bosonization.

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1. Introduction

The covariant Siegel action for describing a chiral boson [1] is particularly suited as a starting point for defining a theory in which the chiral boson is coupled to other fields. In a previous paper [2] in which the boson was coupled to two-dimensional gravity, we have shown that this action, when treated according to Dirac's quantization procedure for hamiltonian systems in the presence of constraints, leads to a theory which has no anomaly associated with Siegel symmetry [1], and therefore leads to a consistent bosonization of chiral fermions coupled to gravity. In the present paper we concentrate on the theory of $O(N)$ non-abelian chiral bosons coupled to gauge fields. To that end we present rather complete results. We first show that the chiral constraints allow coupling to only $O(N)_L$ or $O(N)_R$ gauge fields but not both. Then we show that the anomaly structure of the theory is precisely what one would expect from a theory of left-handed or right-handed fermions coupled to gauge fields. (The anomaly appears already in the classical boson theory as opposed to the fermion theory in which it only appears when quantum effects are considered.) Next we treat the quantum theory where we first derive the quantum hamiltonian explicitly, and then show that in the absence of gauge fields the physical currents of the theory satisfy the same Kac-Moody algebra as those for the corresponding free fermion theory. An essential aspect of this result is that when all constraints are resolved (including the first class constraints which vanish on physical states of the theory), the resulting hamiltonian is just what one would obtain from a gauged version of the action of Floreanini and Jackiw [3]. Thus a quantum theory defined through this latter action is equivalent to the theory which follows from the gauged Siegel action. Of course the advantage in Siegel's approach is that Lorentz covariance is maintained from the beginning, in contrast to the latter where only a restricted Lorentz invariance exists.

An important aspect of our presentation is that throughout the discussion, the presence of zero modes is allowed for explicitly, so that the derivation is of quite general validity. The significance of this is that although the chiral constraints arising in the theory are second class in general, if zero modes exist as part of the set of the constraints, these are first class. Hence the latter cannot be imposed in the classical theory, but must be kept as constraints to be imposed on the Hilbert space of the quantum theory. This derivation generalizes Witten's prescription for non-abelian bosonization [4] to cases in which zero modes are present, and carries over without modification to the gauged chiral theory. Finally, to complete the discussion of the non-abelian theory, we examine the renormalization structure. A one loop calculation leads us to the conclusion that for values of the coupling f^2 other than $\pm 4\pi/n$, in the absence of gauge fields, the model is not free, but nevertheless the chiral structure of the theory is preserved under renormalization.

Next we specialize to the abelian theory which we solve explicitly by a diagonalization of the lagrangian. This result makes contact with an earlier solution [5,6] to the chiral Schwinger model [7] in which bosonization was also employed. The difference is that in the latter case, in order to use the standard bosonization procedure, a free right-handed Weyl fermion was added to the theory, whereas using chiral bosonization this extra field is avoided. Otherwise the results coincide.

Some other work that overlaps to some extent with the present paper should be mentioned. For the free theory, the non-abelian Kac-Moody algebra has been obtained by Salomonson et. al. [8] using a non-abelian generalization [9] of the action of Floreanini and Jackiw, and independently by Bastianelli and van Nieuwenhuizen [10] using a first order formalism. These results are consistent with ours, but neither takes into account the presence of zero modes in the quantization procedure, the coupling to gauge fields, or the relation to Siegel's approach. In a previous attempt to incorporate gauge fields in the classical theory, Frishman and Sonnenschein [11] considered both a vector and a left-right coupling of gauge fields to chiral bosons, and found the anomaly structure to be that of the fermion theory. However, the inconsistency of both an $O(N)_L$ and an $O(N)_R$ coupling simultaneously renders some of their results superfluous. Another attempt to couple gauge fields to chiral bosons in the abelian case has been presented by Labastida and Ramallo [12]. Under the misconception that one must add further fields to the theory to cancel a "Siegel" anomaly [13], these authors work in an unnecessarily complicated theory and restrict themselves to theories with an even number of chiral bosons.

In the next section we present the classical theory of chiral bosons coupled to gauge fields, derive the consistency conditions, and present the Noether current which has divergence equal to that of a left-handed chiral fermion current. The third section is taken up with quantizing the theory according to the Dirac procedure. This section contains a careful treatment of the zero modes of the chiral constraints. The following section presents the arguments concerning the renormalization structure of the theory, and the

calculation of the one-loop β-function. The fifth section contains the solution of the abelian model, and we close with some concluding remarks. An appendix shows that no operator ordering ambiguities arise in the quantization procedure.

2. Gauged Non-abelian Chiral Bosons

Defining the light-cone coordinates

$$x^+ = \frac{1}{\sqrt{2}}(t + x),$$

$$x^- = \frac{1}{\sqrt{2}}(t - x).$$

(2.1)

the lagrangian for the bosonization of a two dimensional theory of N free Majorana fermions with the non-abelian symmetry group $O(N)$ is given by the action [4]

$$S = S_0^{(0)} + nS_{wz}^{(0)},$$

(2.2)

where

$$S_0^{(0)} = -\frac{1}{2f^2} \int d^2x \, \text{tr}[\partial_- gg^{-1}\partial_+ gg^{-1}],$$

(2.3)

and

$$S_{wz} = \frac{1}{8\pi} \int d^2x \int_0^1 dy \, \text{tr}[\partial_y \overline{gg}^{-1}(\partial_- \overline{gg}^{-1}\partial_+ \overline{gg}^{-1} - \partial_+ \overline{gg}^{-1}\partial_- \overline{gg}^{-1})]$$

(2.4)

is the Wess-Zumino term. The coupling n in (2.2) can only take integer values [14]. The field $g(x)$ is an element of $O(N)$ and $\overline{g}(x,y)$ is an extension of $g(x)$ such that $\overline{g}(x,0) = 1$ and $\overline{g}(x,1) = g(x)$. This action posseses the global $O(N)_L \times O(N)_R$ symmetry

$$g \to h_L^{-1}gh_R,$$

which can be gauged by introducing left and right handed gauge fields L_μ and R_μ. For the gauged action, the derivatives in $S_0^{(0)}$ are replaced by covariant derivatives:

$$S_0 = -\frac{1}{2f^2} \int d^2x \, \text{tr}[D_- gg^{-1}D_+ gg^{-1}],$$

(2.5)

where

$$D_\mu g = \partial_\mu g + L_\mu g - gR_\mu,$$

(2.6)

and to the Wess-Zumino term a second term depending on the gauge fields is added [15]:

$$S_{wz} = S_{wz}^{(0)} + S_{wz}^{(1)},$$

$$S_{wz}^{(0)} = \frac{1}{8\pi} \int d^2x \int_0^1 dy \text{tr}[\partial_y \overline{gg}^{-1}(\partial_- \overline{gg}^{-1}\partial_+ \overline{gg}^{-1} - \partial_+ \overline{gg}^{-1}\partial_- \overline{gg}^{-1})],$$

$$S_{wz}^{(1)} = \frac{1}{8\pi} \int d^2x \, \text{tr}[\partial_- gg^{-1}L_+ - \partial_+ gg^{-1}L_-$$

$$+ g^{-1}\partial_- gR_+ - g^{-1}\partial_+ gR_- + g^{-1}L_- gR_+ - g^{-1}L_+ gR_-].$$

(2.7)

Under an infinitesimal gauge transformation

$$\delta g = g\xi_R - \xi_L g,$$

$$\delta L_\mu = \partial_\mu \xi_L + [L_\mu, \xi_L],$$

$$\delta R_\mu = \partial_\mu \xi_R + [R_\mu, \xi_R]$$

(2.8)

the term S_0 is invariant, and the Wess-Zumino term gives the standard left-right anomaly:

$$\delta S_{wz} = \frac{1}{8\pi} \int d^2x \, \text{tr}[\xi_L(\partial_- L_+ - \partial_+ L_-) - \xi_R(\partial_- R_+ - \partial_+ R_-)].$$ (2.9)

We may also add a kinetic term for the gauge fields so the final Lagrangian is given by

$$S = S_0 + n S_{wz} + S_{YM}$$ (2.10)

with

$$S_{YM} = \frac{1}{8e^2} \int d^2x \, \text{tr}[F^L_{\mu\nu} F^{L\mu\nu} + F^R_{\mu\nu} F^{R\mu\nu}],$$ (2.11)

where

$$F^L_{\mu\nu} = \partial_\mu L_\nu - \partial_\nu L_\mu + [L_\mu, L_\nu],$$

and likewise for F^R. The action (2.10) leads to a quantum theory equivalent to a theory of fermions coupled to non-abelian gauge fields. Varying with respect to the field g, we find the equation of motion

$$\frac{1}{f^2} D_-(D_+ gg^{-1}) + D_+(D_- gg^{-1}) + \frac{n}{4\pi}([D_- gg^{-1}, D_+ gg^{-1}] - F^L_{-+} - gF^R_{-+}g^{-1}) = 0.$$ (2.12)

When the gauge fields are set to zero, this equation becomes

$$(\frac{1}{f^2} + \frac{n}{4\pi})\partial_-(\partial_+ gg^{-1}) + (\frac{1}{f^2} - \frac{n}{4\pi})\partial_+(\partial_- gg^{-1}) = 0.$$ (2.13)

For equivalence to the free fermion theory we need $f^2 = 4\pi$ and $n = \pm 1$ [4].

In order to describe a theory of chiral bosons, following Siegel [1], we add a term

$$-\frac{1}{2f^2} \int d^2x \, \lambda \, \text{tr}[D_- gg^{-1} D_- gg^{-1}],$$ (2.14)

to the action where λ is a Lagrange multiplier to enforce the constraint

$$\text{tr}[D_- gg^{-1} D_- gg^{-1}] = 0$$

Because $D_- gg^{-1}$ is a positive definite matrix, this can only be satisfied if

$$D_- gg^{-1} = \partial_- gg^{-1} + L_- - gR_- g^{-1} = 0.$$ (2.15)

When the chiral constraint (2.15) is imposed, equation (2.12) becomes

$$(\frac{1}{f^2} - \frac{n}{4\pi})F^L_{-+} + (\frac{1}{f^2} + \frac{n}{4\pi})gF^R_{-+}g^{-1} = 0.$$ (2.16)

Thus in the chiral theory, the consistency condition tells us that for $f^2 = \pm \frac{4\pi}{n}$, a chiral boson can only couple to either L or R, but not both. This consistency condition can be understood as a consequence of the fact that the left handed part of a Dirac fermion couples only to the $V - A$ gauge combination. This is unlike the case of an ordinary scalar where both left-handed and right-handed currents can be coupled to gauge fields. We should also remark at this point that although we have added a kinetic term for the gauge fields (2.11) which breaks conformal symmetry, this term in no way contributes to equation (2.16). Thus (2.16) is also valid in a case such as in applications to string theory for which no kinetic term arises.

In order to recover the correct Kac-Moody algebra for the case in which no gauge fields are present, we will eventually need $n = 1$ and $f^2 = 4\pi$. From now on let us make that assumption, and also set $R_\mu = 0$ to satisfy the consistency condition. Substituting these conditions back into the action leads to

$$S = \int d^2x\ \mathcal{L}$$

$$\mathcal{L} = -\frac{1}{8\pi}\mathrm{tr}[\partial_-gg^{-1}\partial_+gg^{-1} + \lambda\partial_-gg^{-1}\partial_-gg^{-1}]$$

$$+ \frac{1}{8\pi}\int_0^1 dy\,\mathrm{tr}[\partial_y\overline{gg}^{-1}(\partial_-\overline{gg}^{-1}\partial_+\overline{gg}^{-1} - \partial_+\overline{gg}^{-1}\partial_-\overline{gg}^{-1})]$$

$$- \frac{1}{4\pi}\mathrm{tr}[L_-(\partial_+gg^{-1} + \lambda\partial_-gg^{-1})] - \frac{1}{8\pi}\mathrm{tr}[L_-(L_+ + \lambda L_-)]$$

$$+ \frac{1}{8e^2}\mathrm{tr}[F_{\mu\nu}F^{\mu\nu}] - \frac{1}{2}\frac{m^2}{e^2}\mathrm{tr}[L_+L_-], \tag{2.17}$$

where we have renamed F^L to F, and a mass term for the gauge fields has been added. The addition of a mass term is probably necessary in general, as it has been shown to be essential in the abelian case [7]. Note that only the L_- component of the gauge field couples to the matter fields. This is as expected, reflecting the correspondence with the fermion theory in which this is the case.

The Noether current associated with a global gauge transformation can be calculated by varying the matter part of the action with respect to the gauge field. The variation of the full action is

$$\delta S = \int d^2x\ \mathrm{tr}[\delta L_-(\frac{1}{2e^2}D_+F_{+-} - \frac{1}{4\pi}(\partial_+gg^{-1} + \lambda\partial_-gg^{-1} + \lambda L_-), \tag{2.18}$$

$$- (\frac{1}{8\pi} + \frac{1}{2}\frac{m^2}{e^2})L_+)]$$

$$+ \int d^2x\ \mathrm{tr}[\delta L_+(\frac{1}{2e^2}D_-F_{-+} - (\frac{1}{8\pi} + \frac{1}{2}\frac{m^2}{e^2})L_-)],$$

so with the convention

$$D^\mu F_{\mu\nu} = e^2 J_\nu,$$

we have

$$J_+ = -\frac{1}{2\pi}(\partial_+gg^{-1} + \lambda\partial_-gg^{-1} + \lambda L_-) - (\frac{1}{4\pi} + \frac{m^2}{e^2})L_+, \tag{2.19}$$

$$J_- = -(\frac{1}{4\pi} + \frac{m^2}{e^2})L_-.$$

This current has divergence

$$D^\mu J_\mu = -\frac{1}{2\pi}D_-(\partial_+gg^{-1}) - (\frac{1}{4\pi} + \frac{m^2}{e^2})(\partial_-L_+ + \partial_+L_-) - \frac{1}{4\pi}D_-[\lambda(\partial_-gg^{-1} + L_-)]. \tag{2.20}$$

When the chiral constraint (2.15) is imposed (and $m = 0$) the divergence of the current (2.20) becomes

$$D^\mu J_\mu = \frac{1}{4\pi}(\partial_+L_- - \partial_-L_+). \tag{2.21}$$

Thus J_μ has precisely the same anomalous divergence as the fermion current in a quantum theory of chiral fermions coupled to gauge fields.

To close this section we would like to make a few comments about related work. In ref. 16, "Siegel symmetry"[1] was employed to construct the interaction of the chiral boson ϕ with an external field L_- (eq. (2.5) of ref. 16). Their lagrangian $\mathcal{L}^*$ can be obtained from the abelian version of (2.7) by subtracting the quadratic term L_-L_+ which is independent of ϕ and λ:

$$\mathcal{L}^* = \mathcal{L} - L_-L_+.$$

Thus $\mathcal{L}^*$ also describes a consistent model in the sense that the consistency condition (2.16) between the λ and the ϕ equations of motion is satisfied. This lagrangian corresponds to employing a different regularization scheme in the equivalent fermionic theory.

From our analysis it also follows that the action for chiral bosons coupled to abelian gauge fields in a "vector conserving" regularization scheme as discussed in ref. 11 is inconsistent. There the action in the "left-right symmetric scheme" was analyzed as well, where terms quadratic in the gauge vector field coupled to λ were introduced in order to recover a correct (i.e. λ independent) expression for the divergence of the left handed current. Such terms are present automatically in our analysis.

3. The Quantization of Non-abelian Chiral Bosons

The quantization is best performed by introducing local coordinates $q^i(x)$ on the group manifold. To this end we define

$$g = g(q^i(x)),$$

and introduce the generators T_a of the group G satisfying

$$\text{tr}[T_a, T_b] = 2\delta_{ab}, \tag{3.1}$$

$$[T_a, T_b] = f_{abc}T_c.$$

The vielbein e_i^a is then defined through the equation

$$\partial_\mu g g^{-1} = i T_a e_i^a \partial_\mu q^i. \tag{3.2}$$

Defining the interpolating field $\bar{g} = g(yq^i(x))$ and the gauge fields $L_\mu = i L_\mu^a T_a$ the lagrangian density in (2.17) becomes

$$
\begin{aligned}
\mathcal{L} =& \frac{1}{8\pi} g_{ij}\big(\partial_t q^i \partial_t q^j (1+\lambda) - 2\lambda \partial_t q^i \partial_x q^j - \partial_x q^i \partial_x q^j (1-\lambda)\big) \\
&+ A_{ij}(q)\partial_t q^i \partial_x q^j \\
&+ \frac{1}{4\pi}\Big((L_0{}^a - L_1{}^a)e_i^a\big(\partial_t q^i(1+\lambda) + \partial_x q^i(1-\lambda)\big)\Big) \\
&+ \frac{1}{8\pi}\Big(L_0{}^a L_0{}^a(1+\lambda) - 2\lambda L_0{}^a L_1{}^a - L_1{}^a L_1{}^a(1-\lambda)\Big) \\
&+ \frac{1}{2e^2}F_{01}^a F_{01}^a + \frac{1}{2}\frac{m^2}{e^2}(L_0{}^a L_0{}^a - L_1{}^a L_1{}^a),
\end{aligned}
\tag{3.3}
$$

where

$$g_{ij} = e_i^a e_j^a, \tag{3.4}$$

and

$$A_{ij} = -i\frac{1}{4\pi}\int_0^1 dy\, y^2 f_{abc}\bar{e}_i^a \bar{e}_j^b \bar{e}_k^c q^k, \tag{3.5}$$

in which we have used

$$\partial_\mu \bar{g}\bar{g}^{-1} = i T_a \bar{e}_i^a \partial_\mu \bar{q}^i,$$

with

$$\bar{q}(x, y) = y q(x).$$

From this lagrangian density we find that the momenta conjugate to the fields q^i, L_1^a, L_0^a and λ are

$$p_i = \frac{1}{4\pi}g_{ij}\big(\partial_t q^j(1+\lambda) - \lambda\partial_x q^j\big) + A_{ij}\partial_x q^j + \frac{1}{4\pi}e_i^a(1+\lambda)(L_0{}^a - L_1{}^a), \tag{3.6}$$

$$\pi_1^a = \frac{1}{e^2}F_{01}^a,$$

$$\pi_0^a = 0,$$

$$\pi_\lambda = 0.$$

respectively. The last two must be treated as constraints so we introduce two Lagrange multipliers v_0^a and v_λ to impose them [17]. This leads to the hamiltonian

$$H = H' + H_q = \int dx (\mathcal{H}' + \mathcal{H}_q), \tag{3.7}$$

$$\mathcal{H}' = \frac{e^2}{2}\pi_1^a \pi_1^a + \pi_1^a \partial_x L_0{}^a - i f_{abc}\pi_1^a L_0{}^b L_1{}^c - \frac{1}{2}\frac{m^2}{e^2}(L_0{}^a L_0{}^a - L_1{}^a L_1{}^a)$$

$$+ \frac{1}{4\pi}(-L_0{}^a L_1{}^a + L_1{}^a L_1{}^a) + \pi_0^a v_0^a + \pi_\lambda v_\lambda,$$

$$\mathcal{H}_q = \frac{2\pi}{1+\lambda}(e_a^i p_i - e_a^i A_{ij}\partial_x q^j - \frac{1}{4\pi}e_i^a \partial_x q^i)^2 + p_i \partial_x q^i$$

$$+ (L_0{}^a - L_1{}^a)(-e_a^i p_i + e_a^i A_{ij}\partial_x q^j - \frac{1}{4\pi}e_i^a \partial_x q^i).$$

By demanding that the constraints are stationary (their Poisson bracket with the hamiltonian density vanishes) we arrive at two further constraints on the system

$$\chi_a = e_a^i p_i - e_a^i A_{ij}\partial_x q^j - \frac{1}{4\pi}e_i^a \partial_x q^i = 0 \tag{3.8}$$

and

$$\omega_a = e_a^i p_i - e_a^i A_{ij}\partial_x q^j + \frac{1}{4\pi}e_i^a \partial_x q^i + \frac{1}{4\pi}L_1{}^a + \partial_x \pi_1^a - i f_{abc}\pi_1^b L_1{}^c + \frac{m^2}{e^2}L_0{}^a = 0. \tag{3.9}$$

The Poisson brackets of the constraints (3.8) with H vanish when the constraints are subsequently imposed and the Poisson bracket of the constraints (3.9) with H only lead to the determination of the Lagrange multipliers v_0^a which are irrelevant for our purposes, since they multiply the constraints π_0^a which are second class as long as $m^2 \neq 0$. Thus no further constraints need be introduced. The next step is to determine the algebra structure of these constraints in terms of their Poisson brackets so as to isolate the second class constraints which are to be eliminated.

First it is obvious that π_λ is first class: its Poisson bracket with all others vanishes. Next, both ω_a and π_0^a are second class, as

$$\{\omega_a(x), \pi_0^b(y)\} \neq 0, \quad m^2 \neq 0.$$

Further

$$\{\omega_a(x), \chi_b(y)\} = 0,$$

so we are left to consider the Poisson bracket $\{\chi_a(x), \chi_b(y)\}$. Here we must be more careful. First note that one can write

$$\delta S_{wz}^{(0)} = \int d^2 x F_{ijk}\partial_t q^i \partial_x q^j \delta q^k, \tag{3.10}$$

where

$$F_{ijk} = \frac{\partial A_{ij}}{\partial q^k} + \text{terms cyclic in } i, j, k$$

$$= -\frac{i}{4\pi}f_{abc}e_i^a e_j^b e_k^c.$$

Then using

$$\frac{\partial e_a^i}{\partial q^k} = -e_b^i \frac{\partial e_j^b}{\partial q^k}e_a^j, \tag{3.11}$$

the Maurer-Cartan equation

$$\frac{\partial e_j^a}{\partial q^i} - \frac{\partial e_i^a}{\partial q^j} = i f_{abc}e_i^b e_j^c, \tag{3.12}$$

and the basic Poisson bracket relation

$$\{q^i(x), p_j(y)\} = \delta_j^i \delta(x - y), \tag{3.13}$$

we find

$$\{\chi_a(x), \chi_b(y)\} = -\frac{1}{2\pi}\left(\delta_{ab}\partial_x + if_{abc}e_i^c\partial_x q^i(x) - 2\pi if_{abc}\chi_c(x)\right)\delta(x - y). \tag{3.14}$$

The matrix $\{\chi_a(x), \chi_b(y)\}$ is not invertible when the constraints are imposed, which means there are still first class constraints in the game [17]. To isolate the first class constraints let us define

$$X = iT^a\chi_a \tag{3.15}$$

and introduce the field

$$\Phi = iT^a\phi_a.$$

Then (3.14) when acting on the field Φ can be rewritten

$$\int d^2y\, \{X(x), \mathrm{tr}\,(X(y)\Phi(y))\} = -\frac{1}{\pi}\left(\partial_x\Phi - [\partial_x gg^{-1}, \Phi] + 2\pi[X, \Phi]\right). \tag{3.16}$$

Aside from the last term (which vanishes when the constraints are imposed) the right hand side is just the covariant derivative of Φ in a background 'pure gauge' field $\partial_x gg^{-1}$. The zero modes of this operator are the fields which satisfy $g\Phi g^{-1} = $ constant, which means there are as many zero modes as there are generators of the group. Since under global gauge transformations

$$\partial_\mu gg^{-1} \rightarrow h_L^{-1}\partial_\mu gg^{-1}h_L$$

and the gauged version of this, $D_\mu gg^{-1}$, has the covariant derivative

$$D_\mu(D_\nu gg^{-1}) = \partial_\mu(D_\nu gg^{-1}) + [L_\mu, D_\nu gg^{-1}].$$

we can refer to these quantities as 'right invariant'. We could have equally started with the left invariant expressions $g^{-1}\partial_\mu g$ and $g^{-1}D_\mu g$ which have the global and local properties respectively

$$g^{-1}\partial_\mu g \rightarrow h_R^{-1}g^{-1}\partial_\mu gh_R$$
$$D_\mu(g^{-1}D_\nu g) = \partial_\mu(g^{-1}D_\nu g) + [R_\mu, g^{-1}D_\nu g].$$

Because $R_\mu = 0$ in our case, these turn out to be more convenient in isolating the zero modes. First let us rename X to X_L. Because $g^{-1}D_\mu g = g^{-1}(D_\mu gg^{-1})g$, the constraint we would have derived using the right-handed formalism is

$$X_R = g^{-1}X_L g = 0. \tag{3.17}$$

Thus defining χ_{Ra} through

$$X_R = iT^a\chi_{Ra}$$

we have

$$\{\chi_{Ra}(x), \chi_{Rb}(y)\} = -\frac{1}{2\pi}\delta_{ab}\partial_x\delta(x - y) - if_{abc}\chi_{Rc}\delta(x - y). \tag{3.18}$$

Now we can isolate the zero modes by defining

$$\Xi_R = i\xi_{Ra}T_a = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\chi_R(x) \tag{3.19}$$

(we have introduced a finite spatial volume L) from which it follows that

$$\{\xi_{Ra}, \xi_{Rb}\} = if_{abc}\xi_{Rc}. \tag{3.20}$$

Hence with

$$\overline{X}_R = X_R - \Xi_R = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\overline{\chi}_R(x), \tag{3.21}$$

we have

$$\{\overline{\chi}_{Ra}(x), \overline{\chi}_{Rb}(y)\} = -\frac{1}{2\pi}\delta_{ab}\partial_x\delta(x-y) + if_{abc}\overline{\chi}_{Rc}\delta(x-y) \tag{3.22}$$

Since the right hand side of (3.20) only involves constraints, we have isolated the first class constraints: $\xi_{Ra} = 0$. The remaining constraints

$$\overline{\chi}_{Ra} = \chi_{Ra} - \xi_{Ra} = 0 \tag{3.23}$$

are second class. For convenience let us also define

$$Q_R = iQ_{Ra}T_a = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, g^{-1}\partial_x g, \tag{3.24}$$

$$\overline{q} = g^{-1}\partial_x g - Q_R.$$

In terms of these variables, if we impose the second class constraints $\overline{\chi}_{Ra} = 0$ in the hamiltonian (3.7), we obtain

$$H_q = -\left(\int dx\, \frac{\pi}{1+\lambda}\right)\mathrm{tr}[\Xi_R\Xi_R] - \frac{1}{2}L\mathrm{tr}[\Xi_R Q_R] - \frac{1}{8\pi}L\mathrm{tr}[Q_R Q_R] - \frac{1}{8\pi}\int \mathrm{tr}[\overline{q}_R\overline{q}_R] \tag{3.25}$$

$$+ \frac{1}{2}\mathrm{tr}\left[(\int dx\, g^{-1}(L_0 - L_1)g)(\Xi_R + \frac{1}{2\pi}Q_R)\right] + \frac{1}{4\pi}\int dx\mathrm{tr}\left[g^{-1}(L_0 - L_1)g\overline{q}_R\right],$$

with H' unchanged.

In order to impose the first class constraints $\Xi_R = 0$ in the quantum theory, we must construct a Hilbert space for H such that on physical states $|\mathrm{phys}\rangle$ we have

$$\Xi_R|\mathrm{phys}\rangle = 0.$$

On such states

$$H_q|\mathrm{phys}\rangle = \frac{1}{8\pi}\int dx\, \mathrm{tr}[(-\partial_x gg^{-1} + 2L_0 - 2L_1)\partial_x gg^{-1}]|\mathrm{phys}\rangle. \tag{3.26}$$

The hamiltonian (3.26) is precisely that which would follow from the action (2.17) if we set $\lambda = -1$. This hamiltonian has been shown to lead to a consistent quantization in the abelian case without gauge fields [3,9,18]. Note that if we would have set $\lambda = -1$ from the beginning, then the chiral constraint (3.28) would have been a primary constraint. This value of λ can be found by requiring terms quadratic in the time derivative of g to be absent. The same holds for the case of chiral bosons coupled to gravity, although in that case the value obtained for λ is more complicated [2].

The commutators for the quantum theory are now obtained from the Dirac bracket

$$\{A(x), B(y)\}_D = \{A(x), B(y)\} \tag{3.27}$$

$$- \int dz_1 \int dz_2 \sum_{ab}\{A(x), \overline{\chi}_{Ra}(z_1)\}\{\overline{\chi}_{Ra}(z_1), \overline{\chi}_{Rb}(z_2)\}^{-1}\{\overline{\chi}_{Rb}(z_2), B(y)\}$$

through the correspondence principle. Thus to complete the demonstration of the equivalence of the present theory with a quantized theory of N Majorana-Weyl fermions coupled to gauge fields, we must show that the Dirac brackets of the physical currents lead to the expected Kac-Moody algebra. Although we cannot solve the general non-abelian theory, we will give an explicit demonstration of the equivalence of the abelian theory with the corresponding chiral Schwinger model in section five. Here we proceed by setting the gauge fields L_μ to zero and we show that the Kac-Moody algebra for the theory of N free Majorana-Weyl fermions is reproduced by our non-abelian chiral boson model.

When $L_\mu = 0$ the Noether current given in (2.19) becomes

$$J_+ = \frac{1}{4\pi}(\partial_t gg^{-1}(1+\lambda) + \partial_x gg^{-1}(1-\lambda)), \tag{3.28}$$

$$= iT_a(e_a^i p_i - e_a^i A_{ij}\partial_x q^j + \frac{1}{4\pi}e_i^a\partial_x q^i),$$

$$= iT_a j_a,$$

$$J_- = 0.$$

When the second class constraints are imposed, this current becomes

$$j_a = \frac{1}{2\pi} e^a_i \partial_x q^i + \xi_{La},$$

which on physical states satisfies

$$j_a|\text{phys}\rangle = \frac{1}{2\pi} e^a_i \partial_x q^i |\text{phys}\rangle. \tag{3.29}$$

To calculate the Dirac bracket of the currents, first we note that the Poisson bracket $\{j_a(x), j_b(y)\}$ vanishes, so the Dirac bracket is given by

$$\{j_a(x), j_b(y)\}_D = -\int dz_1 \int dz_2 \sum_{cd} \{j_a(x), \overline{X}_{Rc}(z_1)\}\{\overline{X}_{Rc}(z_1), \overline{X}_{Rd}(z_2)\}^{-1}\{\overline{X}_{Rd}(z_2), j_b(y)\}. \tag{3.30}$$

Now from equation (3.22) we have

$$\{\overline{X}_{Ra}(x), \overline{X}_{Rb}(y)\}^{-1} = -\pi \delta_{ab} \epsilon(x-y). \tag{3.31}$$

since the zero modes have been removed. We also find

$$\{j_a(x), \overline{X}_{Rb}(y)\} = \frac{1}{4\pi} \text{tr}[T_a g^{-1}(x) T_b g(x)] \partial_x \delta(x-y) \tag{3.32}$$

and similarly

$$\{\overline{X}_{Ra}(x), j_b(y)\} = \frac{1}{4\pi} \text{tr}[T_a g^{-1}(x) T_c g(x)] \tag{3.33}$$
$$\times \left(\delta_{cb} \partial_x \delta(x-y) + i f_{cbd} e^d_{Li} \partial_x q^i \delta(x-y) \right).$$

Combining (3.32) and (3.33) leads to

$$\{j_a(x), j_b(y)\}_D = \frac{1}{8\pi} \text{tr}[T_a g(x) T_c g^{-1}(x)] \text{tr}[T_c g^{-1}(x) T_d g(x)]$$
$$\times \left(\delta_{db} \partial_x \delta(x-y) + i f_{dbe} e^e_j(x) \partial_x q^j(x) \delta(x-y) \right)$$

and using the identity

$$\sum_c (T_c)_{\alpha\beta}(T_c)_{\gamma\delta} = \delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta},$$

the result

$$\{j_a(x), j_b(y)\}_D = \frac{1}{2\pi} \delta_{ab} \partial_x \delta(x-y) + i f_{abc} j_c(x) \delta(x-y), \tag{3.34}$$

follows. Choosing periodic boundary conditions for the physical currents (3.29), the Fourier transform of this expression leads to the required Kac-Moody algebra [19]. One may wonder if some operator ordering ambiguities may arise in the above derivation. Although we have not discussed operator ordering here, in fact no problem exists. This is explained in an appendix.

In order to complete the demonstration that the theory is fully equivalent to the theory of chiral fermions, the Virasoro algebra must also be checked. This insures that the energy momentum tensors are also equivalent. In a left-right theory, at this stage one arrives at two commuting Kac-Moody algebras, for left-handed and right-handed modes respectively, and the Sugawara construction leads to the Virasoro algebras associated with them. In our construction, although we have only one of the two Kac-Moody algebras present, we expect the construction of the correct Virasoro algebra to proceed as usual [2]. Thus the present theory is fully equivalent to the theory of N free Majorana-Weyl fermions.

It is worth noting that our formulation here, if one neglects the zero modes, is equivalent to that of Witten as found in ref. 4. We start by noticing that after setting $\lambda = -1$, we are left with an action which

is linear in time derivatives (see discussion after eqn. (3.26)). Such a structure is exactly what leads to the conjugate momentum to act as a constraint. That is, starting with an action of the form

$$S = \int dt \left[A_i(\phi)\dot{\phi} + V(\phi) \right],$$ (3.35)

the canonical momentum is

$$p_i = A_i(\phi).$$

Since the right side is not a function of $\dot{\phi}$ this amounts to a constraint

$$p_i - A_i(\phi) = 0.$$ (3.36)

Then computing the Poisson bracket with itself, we obtain

$$[p_i - A_i, p_j - A_j] = -\frac{\partial A_i}{\partial \phi_j} + \frac{\partial A_j}{\partial \phi_i} = F_{ij},$$ (3.37)

where F_{ij} is that given by eq. (25) of ref. 4. Thus the Dirac bracket (3.27) in this case becomes

$$\{X(\phi), Y(\phi)\}_D = \frac{\delta X}{\delta \phi^i} \frac{\delta Y}{\delta \phi^j} F^{ij},$$ (3.38)

where F^{ij} is the inverse of F_{ij}. This last equation is identical to Witten's generalized Poisson bracket. Of course, in the presence of zero modes, the modifications explained earlier in this section must be taken into account.

4. Renormalization Structure

In this section we continue our analysis of the theory of non-abelian chiral bosons without gauge fields. First, we show that a non-conventional Lorentz invariance [3] restricts the counterterms in such a way that the chiral structure of the theory is preserved. Then a one-loop computation of the β-function tells us that as in the non-chiral case [4], the equivalence with the theory of N left- (or right-) handed free fermions only holds for the special values of the coupling $f^2 = \pm 4\pi/n$ which was considered in the preceding sections. Otherwise the β-function is non-zero so the coupling has a non-trivial renormalization.

We first write the boson field as

$$g = g_0 \exp(if\Pi)$$ (4.1),

where g_0 is an arbitrary solution of the classical equations of motion and $\Pi = T_a \pi^a$ represents quantum fluctuations around this field. Then to second order in the π fields, the action (with $\lambda = -1$) becomes

$$S = -\frac{1}{2f^2} \int d^2x \; \text{tr} \left(\partial_- g_0 g_0^{-1} \partial_+ g_0 g_0^{-1} - \partial_- g_0 g_0^{-1} \partial_- g_0 g_0^{-1} \right)$$ (4.2)

$$+ \int d^2x \left(\partial_+ \pi^a \partial_- \pi^a - \partial_- \pi^a \partial_- \pi^a \right)$$

$$+ \frac{1}{4} \int d^2x \; \text{tr} \left((g_0^{-1}\partial_+ g_0 - g_0^{-1}\partial_- g_0)[\Pi, \partial_-\Pi] \right)$$

$$+ \frac{1}{4} \int d^2x \; \text{tr} \left(g_0^{-1}\partial_- g_0[\Pi, \partial_+\Pi - \partial_-\Pi] \right)$$

$$- \frac{1}{4} f^2 \frac{n}{4\pi} \int d^2x \; \text{tr} \left(g_0^{-1}\partial_+ g_0[\Pi, \partial_-\Pi] - g_0^{-1}\partial_- g_0[\Pi, \partial_+\Pi] \right).$$

Because λ is fixed to -1, the action is no longer invariant under the full Lorentz group, but only a subgroup thereof [3]. The transformation law which leaves the action invariant is

$$\delta g = \epsilon x_+ (\partial_+ - \partial_-)g.$$ (4.3)

This invariance limits the possible counterterms that can arise in a one loop calculation. First of all, the Wess-Zumino term doesn't renormalize because it is a topological invariant [4,14]. Thus the only possible counterterms that can arise are given by

$$S_{\text{counterterm}} = a \int d^2x \, (g_0^{-1}\partial_- g_0)^2 \tag{4.4}$$

$$+ b \int d^2x \, g_0^{-1}\partial_+ g_0 g_0^{-1}\partial_- g_0 + c \int d^2x \, (g_0^{-1}\partial_+ g_0)^2$$

where a, b and c are arbitrary coefficients. It is then straightforward to show that under the transformation (4.3), the variation of (4.4) is

$$\delta S_{\text{counterterm}} = -\epsilon(a+b) \int d^2x \, (g_0^{-1}\partial_- g_0)^2 \tag{4.5}$$

$$+ \epsilon c \int d^2x \, g_0^{-1}\partial_+ g_0 (g_0^{-1}\partial_+ g_0 - 2g_0^{-1}\partial_- g_0).$$

From this it follows that in order for the symmetry to be maintained we must have

$$a = -b, \quad c = 0. \tag{4.6}$$

Thus the only possible counterterm which respects the symmetry (4.3) has the form

$$S_{\text{counterterm}} = a \int d^2x \, \left((g_0^{-1}\partial_- g_0)^2 - g_0^{-1}\partial_+ g_0 \, g_0^{-1}\partial_- g_0 \right). \tag{4.7}$$

This indicates that only a coupling renormalization is needed and the chiral structure is preserved.

The divergent integral at the one-loop level which gives rise to a term of the form of the second term in (4.7) can easily be shown to have the form (in momentum space)

$$I = (1 - f^2 \frac{n}{4\pi}) \int \frac{d^2k}{(2\pi)^2} \frac{(k_- + p_-)[(1 + f^2 \frac{n}{4\pi})k_+ - 2k_-]}{k_-(k_+ - k_-)(k_- + p_-)(k_+ + p_+ - k_- - p_-)}. \tag{4.8}$$

With the change of variables,

$$l_- = k_-, \quad l_+ = k_+ - k_- \tag{4.9}$$

and using dimensional regularization on the infinite part, one can show

$$I \approx \frac{1}{D-2}(1 - (f^2 \frac{n}{4\pi})^2) + \text{finite terms}. \tag{4.10}$$

The only other divergent integral at the one-loop level contributing to the coupling constant renormalization is of the form of the first term in (4.7). As required by the symmetry arguments, the divergent part of this term has exactly the same value as I but with opposite sign. Thus the β-function can be written

$$\beta(f, n) \sim f^2(N-2)(1 - (f^2 \frac{n}{4\pi})^2)$$

to one loop order, where N comes from the group manifold $O(N)$. Hence we conclude that the β-function vanishes only when $f^2 \frac{n}{4\pi} = \pm 1$, and for other values of the coupling, the theory is not free.

5. Solution to the Abelian Theory

In this section we present the solution to the theory of chiral bosons coupled to abelian gauge fields. We start with the lagrangian (3.3) with $g = \exp(i\phi)$. Then multiplying by a factor of 4π, rescaling the constants m and e accordingly, and rescaling L_μ by a factor e we arrive at

$$\mathcal{L} = (\partial_+\phi + eL_+)(\partial_-\phi + eL_-) + \lambda(\partial_-\phi + eL_-)^2 \qquad (5.1)$$
$$- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + m^2 L_+ L_- - e(L_+\partial_-\phi - L_-\partial_+\phi).$$

We could of course proceed along the lines of section 3, but in this case a more direct solution is available along the lines of refs. 5 and 6. We start
by decomposing the gauge field into longitudinal and transverse parts:

$$L_\mu = \partial_\mu\alpha + \epsilon_{\mu\nu}\partial^\nu\beta, \qquad (5.2)$$

which when substituted into (5.1) yields

$$\mathcal{L} = \frac{1}{2}(\Box\beta)^2 + \frac{1}{2}(m^2 + e^2)(-\alpha\Box\alpha + \beta\Box\beta) - e^2(\partial_-(\alpha - \beta))^2$$
$$+ \partial_+\phi\partial_-\phi + \lambda(\partial_-\phi)^2 + 2e(\partial_+ + \lambda\partial_-)\phi\partial_-(\alpha - \beta). \qquad (5.3)$$

To disentangle the double pole present in the propagator of β, we introduce the auxiliary field F and replace the higher derivative kinetic term $\frac{1}{2}(\Box\beta)^2$ with

$$-\frac{1}{2}F^2 + F\Box\beta. \qquad (5.4)$$

The equation of motion for F then leads back to the original form of the lagrangian (5.3). With this replacement, the lagrangian can be (partially) diagonalized by making the change of field variables

$$\phi = \phi' - \frac{e}{m}(\eta - \beta_2 + \beta_1), \qquad (5.5a)$$

$$\alpha = \frac{1}{m}(\eta - \frac{e^2}{e^2 + m^2}(\beta_2 - \beta_1)), \qquad (5.5b)$$

$$F = m'\beta_1, \quad m' = \frac{m^2 + e^2}{m}, \qquad (5.5c)$$

$$\beta = \frac{1}{m'}(\beta_2 - \beta_1), \qquad (5.5d)$$

leading to

$$\mathcal{L} = \frac{1}{2}\partial_+\beta_1\partial_-\beta_1 - \frac{1}{2}m'^2\beta_1^2 - \frac{1}{2}\partial_+\beta_2\partial_-\beta_2 + \frac{1}{2}\partial_+\eta\partial_-\eta + \partial_+\phi'\partial_-\phi' + \lambda(\partial_-\phi')^2. \qquad (5.6)$$

This lagrangian appears to describe several decoupled free particles: a massive field β_1 with mass m', a massless field β_2 with negative metric (a ghost), a massless field η and the chiral field ϕ'. However, since the change of variables (5.5) does not constitute a canonical transformation, to insure consistency we compare the equations of motion before and after the transformation. This leads to a set of constraints among the fields which must be resolved.
The equations of motion for the original lagrangian are

$$\partial_-\phi + eL_- = 0, \qquad (5.7a)$$

$$\partial_-(\partial_-L_+ - \partial_+L_-) = (m^2 + e^2)L_-, \qquad (5.7b)$$

$$\partial_+(\partial_-L_+ - \partial_+L_-) + (m^2 + e^2)L_+ + 2e\partial_+\phi = 0. \qquad (5.7c)$$

Rewriting these in terms of the new fields ϕ', η, β_1 and β_2 and demanding consistency with the equation of motion for β_1 that follows from the transformed lagrangian (5.6):

$$\left(\Box + m'^2\right)\beta_1 = 0$$

we arrive at the consistency conditions

$$\partial_-(\beta_2 - \eta) = 0, \tag{5.8a}$$

$$2e\partial_+\phi' + \overline{m}\partial_+\eta + m'\partial_+\beta_2 = 0, \tag{5.8b}$$

$$\overline{m} = \frac{m^2 - e^2}{m}. \tag{5.8c}$$

From the decomposition of L_μ given in (5.2) and the chiral condition (5.7a) we also find

$$\partial_-\phi' = 0, \tag{5.9}$$

which is also obtained from the equation of motion for λ following from (5.6).

Since constraints (5.8) and (5.9) do not involve the field β_1, this field indeed describes a massive excitation with mass m' as in refs. 5 and 7. For the other (massless) degrees of freedom, we go on to resolve the constraints explicitly. We start by putting the system in a finite box in the spatial direction with volume L, $-L/2 \le x \le L/2$. With periodic boundary conditions on the derivatives of the fields [6], a mode expansion for the fields ϕ', η and β_2 gives

$$\phi' = \frac{\phi'_0}{\sqrt{\pi}} + \frac{\sqrt{\pi}}{L}(Qx + Pt) + \sum_{n>0} \frac{1}{\sqrt{2L\omega_n}}[\phi_n e^{-ik_n(x+t)} + \phi_n'^\dagger e^{ik_n(x+t)}], \tag{5.10a}$$

$$\eta = \frac{\eta_0}{\sqrt{\pi}} + \frac{\sqrt{\pi}}{L}(Q_\eta x + P_\eta t) + \sum_n \frac{1}{\sqrt{2L\omega_n}}[\eta_n e^{-i\omega_n t + ik_n x} + \eta_n^\dagger e^{i\omega_n t - ik_n x}], \tag{5.10b}$$

$$\beta_2 = \frac{\beta_0}{\sqrt{\pi}} + \frac{\sqrt{\pi}}{L}(Q_\beta x + P_\beta t) + \sum_n \frac{1}{\sqrt{2L\omega_n}}[\beta_n e^{-i\omega_n t + ik_n x} + \beta_n^\dagger e^{i\omega_n t - ik_n x}], \tag{5.10c}$$

in which

$$\omega_n = |k_n|, \quad k_n = \frac{2\pi n}{L}. \tag{5.11}$$

Note here that we have already incorporated the second class chiral constraint

$$\overline{\chi} = p - \partial_x\phi' - \Xi = 0, \tag{5.12}$$

$$\Xi = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx(p - \partial_x\phi'),$$

corresponding to (3.23) of the non-abelian case into the mode expansion where p is the momentum conjugate to ϕ'. This accounts for the fact that the non-zero frequency modes of ϕ' depend only on the combination $x+t$. The first class chiral constraint Ξ remains to be imposed on the Hilbert space. Using the correspondence principle, the commutators

$$[\phi'_0, P] = i, \tag{5.13a}$$

$$[\eta_0, P_\eta] = i, \quad [\beta_0, P_\beta] = -i, \tag{5.13b}$$

$$[\phi_n, \phi_m'^\dagger] = \delta_{nm}, \tag{5.13c}$$

$$[\eta_n, \eta_m^\dagger] = \delta_{nm}, \quad [\beta_n, \beta_m^\dagger] = -\delta_{nm}. \tag{5.13d}$$

follow from the Dirac brackets of the classical theory. The negative sign in the last commutator arises because β_2 is a field with negative metric (a ghost).

Using the field definitions above, the hamiltonian for the massless sector can now be written

$$H_{\text{massless}} = H_{n>0} + H_{n<0} + H_{\text{zeromodes}} \tag{5.14}$$

$$= \sum_{n>0} |k_n| [\phi_n'^{\dagger} \phi_n + \eta_n^{\dagger} \eta_n - \beta_n^{\dagger} \beta_n]$$

$$+ \sum_{n<0} |k_n| [\eta_n^{\dagger} \eta_n - \beta_n^{\dagger} \beta_n]$$

$$+ \frac{\pi}{L} [\frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{1}{2} \frac{1}{1+\lambda} \Xi^2 + Q^2 + Q\Xi]$$

$$+ \frac{\pi}{L} [\frac{1}{2}(P_\eta^2 + Q_\eta^2) - \frac{1}{2}(P_\beta^2 + Q_\beta^2)],$$

and, since the constraints (5.8) are first class, the physical subspace of Hilbert space is defined as those states which are annihilated by these constraints:

$$(\beta_n - \eta_n)|\text{phys}\rangle = 0, \quad n < 0, \tag{5.15a}$$

$$(2e\phi_n' + \overline{m}\eta_n + m'\beta_n)|\text{phys}\rangle = 0, \quad n > 0, \tag{5.15b}$$

$$(P_\beta - Q_\beta - P_\eta + Q_\eta)|\text{phys}\rangle = 0, \tag{5.15c}$$

$$(2e(P+Q) + \overline{m}(P_\eta + Q_\eta) + m'(P_\beta + Q_\beta))|\text{phys}\rangle = 0, \tag{5.15d}$$

$$\Xi|\text{phys}\rangle = 0. \tag{5.15e}$$

This hamiltonian in the presence of the constraints (5.8) can be understood by making the change of variables

$$h = \phi' - \frac{e}{m}(\eta - \beta_2) \tag{5.16}$$

which is a canonical transformation:

$$[h_n, h_m^{\dagger}] = \delta_{nm}, \quad [h_0, P_h] = i. \tag{5.17}$$

Further, by virtue of the constraint (5.8a), the combination $\eta - \beta_2$ has an expansion which on physical states will only depend on the combination $x + t$, similar to the expansion of ϕ'. Thus the field h on physical states is also a left-mover. With (5.16) it follows that the hamiltonian takes precisely the form of the hamiltonian for a free chiral boson [3] on physical states. That is,

$$\langle \text{phys}'|H_{\text{massless}}|\text{phys}\rangle = \langle \text{phys}'| \int_{-\frac{L}{2}}^{\frac{L}{2}} dx (\partial_x h)^2 |\text{phys}\rangle \tag{5.18}$$

$$= \langle \text{phys}'| \frac{\pi}{L} Q_h^2 + \sum_{n>0} |k_n| h_n^{\dagger} h_n |\text{phys}\rangle,$$

with

$$Q_h = Q_\phi - \frac{e}{m}(Q_\eta - Q_\beta),$$

and

$$h_n = \phi_n - \frac{e}{m}(\eta_n - \beta_n), \quad n > 0,$$

$$h_n = 0, \quad n < 0.$$

Of course, on physical states, $P_h = Q_h$.

Thus we have shown the physical spectrum of the abelian model to be that of a (massless) chiral boson h and a free massive scalar field β_1. This is similar to the spectrum found in ref. 6, the main difference lying in the inclusion of a free right handed Weyl fermion in the latter case, in order to employ conventional bosonization techniques. Of course, this 'trivial' massless mode is then also present in the spectrum obtained in that paper. By contrast, in the present construction no such field exists.

6. Conclusion

Starting with a non-abelian generalization of Siegel's action [1] for chiral bosons, we have first shown that although the original action posesses a global $O(N)_L \times O(N)_R$ symmetry, the constraints which follow from the action allow only one or the other of these symmetries to be consistenly gauged, but not both. Thus the theory passes the first test for comparing it with chiral fermions. Setting the boson coupling f^2 equal to $4\pi/n$ and n equals ± 1 as is required to reproduce the free theory of chiral fermions [4], we next examined the Noether current of the gauged $O(N)$ symmetry. Here we found that the divergence of the current has precisely the anomaly of the corresponding fermion theory. Thus a second requirement is satisfied. Next we turned to the quantization of the theory for which we followed Dirac's procedure for the quantization of a constrained hamiltonian system [17].

An important aspect of our quantization of the model was to take into account zero modes of the constraints explicitly, in order to separate first class constraints from second class. Our treatment is a generalization of Witten's non-abelian bosonization, and carries over to the gauged theory. In this analysis, we were led to the hamiltonian for quantization (3.25) which involves first class constraints. When these constraints were imposed on physical states in the Hilbert space, the remaining non-vanishing or 'physical' hamiltonian turned out to be precisely what one would obtain from the original lagrangian, but with the lagrange multiplier λ set to -1. This is consistent with the abelian free case [2] showing that the constraining of λ is not affected by the presence of gauge fields. So once again we are led to the conclusion that the non-abelian generalization of Siegel's action leads to a quantum theory identical to that which follows from a non-abelian generalization of the action of Floreanini and Jackiw [3]. Further, because Siegel's action is fully Lorentz covariant, and therefore easily gauged, this equivalence between lagrangians tells us how to couple gauge fields consistently to Floreanini and Jackiw's action in the quantum theory.

After presenting the fully gauged hamiltonian and Poisson bracket algebra for quantization purposes, we dropped the gauge fields and examined the free theory. Here we have shown that the Kac-Moody algebra for N Majorana-Weyl fermions is reproduced by the boson theory, and because the Virasoro algebra follows from the Sugawara construction, the identification of the two quantum theories is established. Next we relaxed the condition that f^2 equal to $\pm 4\pi/n$, and studied the renormalization structure of the model. A consideration of one-loop perturbation theory leads us to two conclusions. First, a remnant of Lorentz symmetry in the chiral model [3] is enough to show that the chiral structure of the theory is preserved under renormalization, and that only the coupling f^2 is renormalized. Second, the explicit one loop calculation shows that for values of f^2 not equal to $\pm 4\pi/n$, the coupling is indeed renormalized, so that it is only when f^2 equals $\pm 4\pi/n$ that the theory can possibly be equivalent to a free fermion theory. For this value the first term of the β-function vanishes. This is exactly as in the case of Witten's non-abelian bosonization.

In the last section, we finally specialized to the abelian gauge theory, or chiral Schwinger model [7], for which an explicit solution is known. In this section we gave the complete solution of the theory in the context of chiral bosonization, in order to clarify those parts of the derivation which are affected by the chiral structure of the boson sector. In particular, we found that because of the chiral constraints, certain of the fields are left-moving (or right-moving), and that in the final result the spectrum only differs in the sector where chiral bosons replace the normal bosons used in a previous solution [5,6].

Finally we end with some comments about the implications of our results. First, we have shown that both in the presence of gravity [2] and in the presence of gauge fields, that the approach of Siegel [1] is identical to that of Floreanini and Jackiw [3] on the quantum level. An alternate method for quantizing chiral bosons exists, which also starts from Siegel's action [20]. This method is based on the erroneous assumption that further fields and further constraints must be added to the theory in order to cancel a local 'Siegel anomaly'. Hence, 'no-movers' are added, and an analysis is performed on the resulting lagrangian. On the basis of our study, this alternative approach is unnecessarily complicated and can be avoided. Because Dirac's procedure for isolating the canonical degrees of freedom on a manifold defined by certain constraints within a larger space is obtained from an analysis of the underlying geometry of the manifold, we are quite confident that when the procedure is applied to Hull's approach, exactly the same quantum hamiltonian we have found here will be reproduced.

The first order formalism of [10] on the other hand is very close to a hamiltonian formulation of the theory. The key to its simplicity is that the 'square root' constraints (see ref. 2) are assumed from the beginning, and therefore when quantized along the lines presented here, the resulting theory should be the

same. Thus a justification of this very nice 'quick and dirty' method for obtaining the 'physical' hamiltonian would be obtained. Of course, in the proof, the presence of zero modes must be carefully taken into account.

In summary, we have analyzed the quantum theory of non-abelian chiral bosons coupled to gauge fields, and shown that whenever the theory can be checked against the theory of gauged chiral fermions, the two are in agreement.

Appendix

The careful reader may wonder whether some of our conclusions concerning zero modes may be affected by ambiguities in operator ordering when defining the quantum theory. In this appendix we show that in the places where one might suspect this to be a problem, no difficulty arises.

The first potential problem occurs in the term $\text{tr}[\Xi_R Q_R]$ of the hamiltonian (3.25). However because the operators ξ_{Ra} and Q_{Rb} commute when a equals b, and since in the hamiltonian they occur inside a trace, no ambiguity arises here.

The second potential problem is a little more subtle. In the the derivation (3.29) one encounters terms of the form

$$\text{tr}(T_a g(x) P_{Rb} T_b g^{-1}(x)))|phys>, \qquad (A.1)$$

where P_{Rb} is defined through

$$P_R = i P_{Ra} T_a = \Xi_R + \frac{1}{4\pi} Q_R, \qquad (A.2)$$

and of course g depends on q. Now in order to use the vanishing of Ξ_R on physical states, which in terms of P_R means

$$P_{Rb}|phys> = \frac{1}{4\pi} Q_{Rb}|phys>, \qquad (A.3)$$

we have to check that P_{Rb} commutes with $g^{-1}(x)$ inside expression (A.1). A short computation gives the desired result. Introducing the commutator

$$[q^i(y), p_j(x)] = i\delta^i_j \delta(x-y), \qquad (A.4)$$

where $p_i(x)$ is defined through

$$P_R = \frac{1}{L} \int dx\, g^{-1}(x) e^i(x) g(x)(p_i(x) - A_{ij}(x)\partial_x q^j(x)), \qquad (A.5)$$

we have

$$\text{tr}(T_a g(x)[P_R, g^{-1}(x)]) = \text{tr}(T_a g(x)[\frac{1}{L}\int dy\, g^{-1}(y) e^i_b(y) T_b g(y) p_i(y), g^{-1}(x)]) \qquad (A.6)$$

$$= i\frac{1}{L} e^i_b \text{tr}(T_a T_b \frac{\partial g}{\partial q^i} g^{-1})$$

$$= -\frac{1}{L} e^i_b e^c_i \text{tr}(T_a T_b T_c)$$

$$= -\frac{1}{L}\text{tr}(T_a T_b T_b)$$

$$= 0,$$

where the last line follows because $T_b T_b$ is proportional to the unit matrix, and $\text{tr}(T_a) = 0$. Thus we conclude that no operator ordering ambiguities occur that affect the derivations in the main text.

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