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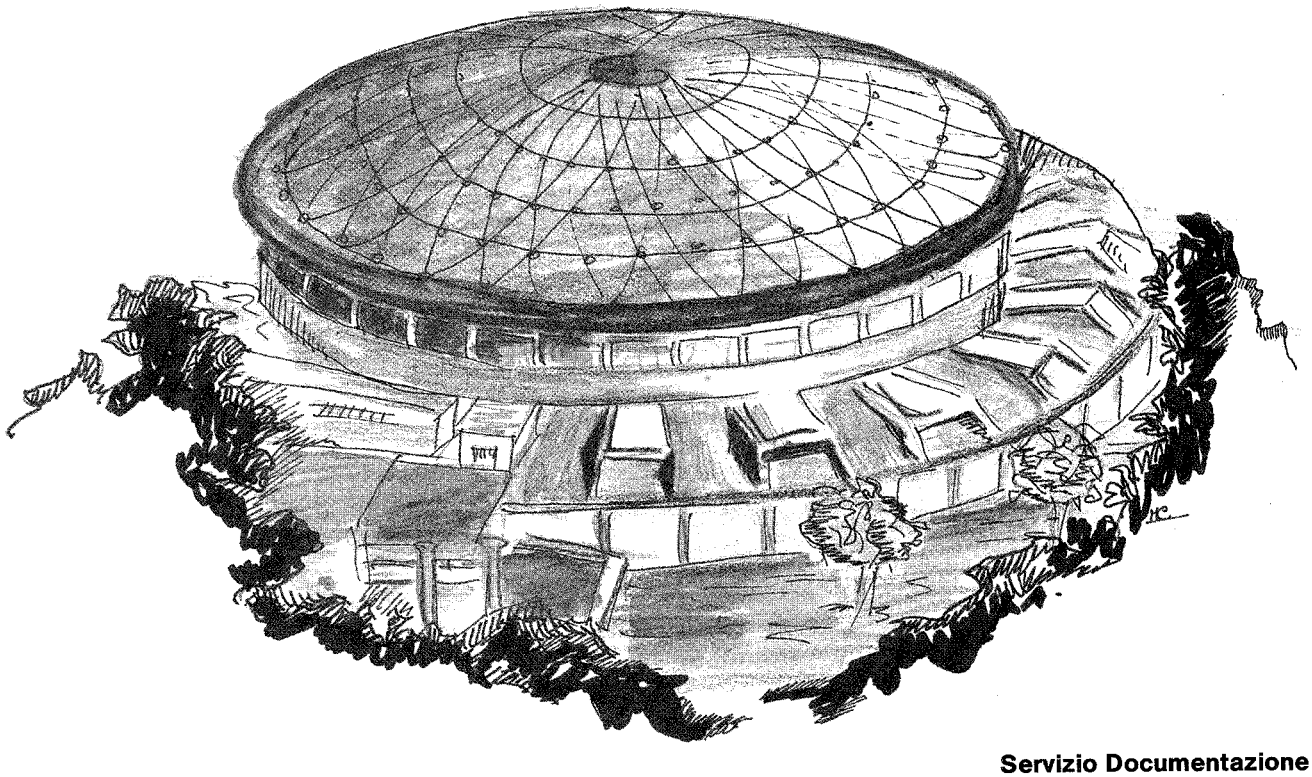

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NON COMPACT FORMULATION OF YANG-MILLS THEORIES ON A LATTICE

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ABSTRACT

We formulate Yang-Mills theories on a lattice by using a polar representation which separates the fields into pure gauge and gauge-invariant degrees of freedom. We define all the variables on the sites and get a noncompact theory.

If we do not make any requirement on the states we get no confinement in strong coupling. If, however, we require that the eigenstates of the Hamiltonian expressed in terms of polar fields be also eigenstates of the Hamiltonian expressed in terms of cartesian fields we get superconfinement, namely quarks cannot be separated even by one lattice spacing. A strong coupling expansion shows that at finite but large values of the coupling constant the quark-quark wave-function decays not faster than exponentially at large distance, a behaviour incompatible with a linear effective potential.

1. - INTRODUCTION

Compact gauge theories[1] have the properties of i) giving rise to charge quantization in the abelian case and ii) confining charges by a linear potential in the strong coupling limit both in the abelian and non-abelian case.

Since a non-compact version of non-abelian theories is assumed to be impossible[2], confinement is considered a specific feature of non-abelian theories. It is not specific of abelian

theories because for the latter a non-compact formulation is known, and moreover a phase transition deconfines the electric charge in the continuum limit.

The compact version is however preferred also for QED, on the assumption that in the context of a noncompact theory the quantization of the electric charge would be an unexplained mistery [2]. Moreover if QED should arise from some non-Abelian theory it should necessarily be compact.

As it will be shown in Sect. 2, we have found that actually in strong coupling confinement is not an intrinsic property of compact gauge theories. Such theories in fact confine or do not confine according to the choice of boundary conditions (b.c.) for the states. We have proved this only in the abelian case, but we presume that it should be a general result for the reason that we are going to explain.

In compact theories the Gauss constraint is enforced by writing the states in the form[3]

$$\Psi = \psi^{+}(x)\, e^{i\int_y^x dx_K A_K}\, \psi(y)\, |0\rangle . \tag{1.1}$$

Now the gauge potential can be separated into its transverse and longitudinal parts

$$A_K = A_{TK} + A_{LK} . \tag{1.2}$$

The transverse part being gauge-invariant, a gauge-invariant state can also be written inserting only A_{LK} in the phase[4]

$$\Psi = \psi^{+}(x)\, e^{i\int_y^x dx_K A_{LK}}\, \psi(y)\, |0\rangle \tag{1.3}$$

In such a state the interaction is Coulombic, the linear part in the state (1.1) being due to the unnecessary transverse field. Since a corresponding separation into pure-gauge and gauge-invariant degrees of freedom can be performed also in the non-abelian case[5] (and it will be discussed in this paper) we expect that nonconfining gauge-invariant states can be written also in compact non-abelian theories.

It should be noted that the state (1.1) is a periodic function of the angles A_K, while, as it will be shown in Sect. 2, the state (1.3) obeys generalized periodic b.c. Linear confinement for strong coupling is therefore not an intrinsic property of the compact formulation but depends on the choice of b.c.. Moreover the b.c. giving rise to confinement select states of higher energy.

We think that a complete understanding of gauge theories an a lattice requires a thorough investigation of the relevance of b.c. to the hypotesis of universality.

The present paper, however, does not pursue such on objective, but is devoted to a development which is suggested by the above analysis, namely the formulation of noncompact gauge theories on a lattice. Such a suggestion comes from the expression (1.3) for the state. The same expression can be used in the Hamiltonian to define products of matter fields at different points. It accounts for the interaction of matter fields with the longitudinal part of the gauge field.

The interaction with the transverse part, which remains noncompact, must be added separately.

Since a longitudinal field is a gradient

$$A_{LK} = \partial_K \varphi, \tag{1.4}$$

the expression (1.3) can be rewritten

$$\Psi = \chi^+ (x)\, \chi\, (y)\ |0>, \tag{1.5}$$

in terms of the gauge-invariant field

$$\chi = e^{\,i\varphi}\, \psi\,. \tag{1.6}$$

The fields χ and A_{TK}, being gauge-invariant, can be safely defined on the sites and discrete derivatives can be applied to them.

Such a formulation for the abelian case will be presented in a separate paper, where it will be shown that the variable φ appearing in Eq. (1.4) must be compact for consistency of the Gauss law (only A_{TK} are not compact). As a consequence the electric charge is quantized also in non compact QED, and there is, in principle, the possibility that QED come from a non-abelian theory.

In the present paper we present the noncompact formulation of non-abelian gauge theories with SU(2) color group following the above line.

In Sect. 3 we perform in the continuum the manipulations necessary to separate the gauge-invariant variables from the gauge degrees freedom. This allows us to discretize the theory by defining all the variables on the sites.

In Sect. 4 we discuss the Gauss low. In Sect. 5 we derive the strong coupling limit of the Hamiltonian. Here, as in the compact case, we get or we do not get confinement according to the conditions imposed on the states. The Hamiltonian is expressed in terms of polar fields. We get confinement if we require that the states expressed in terms of polar fields be also eigenstates of the Hamiltonian expressed in terms of cartesian fields. This is a rather natural requirement analogous to the one adopted in quantum mechanics. Moreover, such a requirement does not select the states higher in energy. In the strong coupling limit we get what we call superconfinement, namely quarks cannot be separated even by one lattice spacing. We then perform a strong coupling expansion showing that at finite but large values of the coupling constant the quark-quark wave function decays not faster than exponentially at large distance, a behaviour incompatible with a linear potential.

Some of the results of this paper have been anticipated in Refs. (6).

2. - LINEAR POTENTIAL AND BOUNDARY CONDITIONS

The gauge field Hamiltonian density of the abelian theory can be written

$$\mathcal{H} = \frac{1}{a} \sum_{r,k} \left\{ \frac{1}{2} g^2 E_k^2(\vec{r}) + \frac{1}{2g^2} \sum_h \left[1 - \cos\left(\Delta_k A_h(\vec{r}) - \Delta_h A_k(\vec{r}) \right) \right] \right\} \tag{2.1}$$

where a is the lattice spacing, $\vec{r}$ a lattice vector, and Δ_k the discrete derivative

$$\Delta_k f(\vec{r}) = f(\vec{r} + \hat{k}) - f(\vec{r}) . \tag{2.2}$$

The space of states must satisfy the Gauss constraint

$$\left[\Delta_k E_k(\vec{r}) + \psi^*(\vec{r}) \, \psi(\vec{r}) \right] \Psi = 0 \tag{2.3}$$

where ψ is the spinor matter field.

Both Hamiltonian and constraint are periodic with respect to the gauge field. The most general b.c. are therefore

$$\Psi\left(A_k(\vec{r}) = 0\right) = e^{-i\alpha_k(\vec{r})} \Psi\left(A_k(\vec{r}) = 2\pi\right), \tag{2.4}$$

often referred to as generalized or quasi-periodic b.c..

When it is stated that in strong coupling the potential is linear it is tacitly assumed that b.c. are strictly periodic, namely $\alpha_k(r) = 0$. In such a case the solution to Eq. (2.3) for two charges localized at $\vec{r}_1$, $\vec{r}_2$ is

$$\Psi = \exp\left\{ i \sum_{\vec{r} = \vec{r}_1}^{\vec{r}_2}{}_{\Gamma} A_k(\vec{r}) \right\}, \tag{2.5}$$

where Γ is any path leading from $\vec{r}_1$ to $\vec{r}_2$, and for a link in the k direction we must insert A_k in the sum.

For large g we can neglect the magnetic term in the Hamiltonian and find that the energy

$$E = \frac{1}{2} g^2 N_\Gamma, \tag{2.6}$$

where N_Γ is the number of links along the path Γ.

Let us separate the gauge field into its transverse and longitudinal components

$$A_{Th}(\vec{r}) = \sum_{\vec{r}'} \left[\delta_{hk} \delta_{\vec{r},\vec{r}'} - \Delta_h(\vec{r}) \Delta_k(\vec{r}') \Delta^{-1}(\vec{r} - \vec{r}') \right] A_k(\vec{r}')$$

$$A_{Lh}(\vec{r}) = \sum_{\vec{r}'} \Delta_h(\vec{r}) \Delta_k(\vec{r}') \Delta^{-1}(\vec{r} - \vec{r}') A_k(\vec{r}'). \tag{2.7}$$

In the above equations

$$\Delta^{-1}(\vec{r}) = \sum_{\vec{r}'} G(r') e^{i \frac{2\pi}{N} \vec{r}' \cdot \vec{r}}, \tag{2.8}$$

N being the number of links along one edge of the cubic lattice, and

$$G(\vec{r}) = \left\{ \sum_h \left[\cos \frac{2\pi}{N} r_h - 1 \right] \right\}^{-1}. \tag{2.9}$$

Since

$$[A_{Tk}, \Delta_h E_h] = 0, \tag{2.10}$$

we can omit A_T in Eq. (2.5), getting

$$\Psi = \exp \left\{ i = \sum_{\vec{r} = \vec{r}_1}^{\vec{r}_2} \Gamma A_{Lk} \right\}. \tag{2.11}$$

The sum does not depend on the path and Ψ can be rewritten

$$\Psi = \exp \left\{ i \sum_{\vec{r}} \Delta_h \left(\Delta^{-1}(\vec{r}_2 - \vec{r}) - \Delta^{-1}(\vec{r}_1 - \vec{r}) \right) A_h(\vec{r}) \right\}. \tag{2.12}$$

Such a state, however, is no longer periodic with respect to $A_h(\vec{r})$, but it satisfies generalized periodic b.c. with

$$\alpha_h(\vec{r}) = 2\pi \left\{ \Delta_h(\vec{r}) \left(\Delta^{-1}(\vec{r}_2 - \vec{r}) - \Delta^{-1}(\vec{r}_1 - \vec{r}) \right) - \left[\Delta_h(\vec{r}) \left(\Delta^{-1}(\vec{r}_2 - \vec{r}) - \Delta^{-1}(\vec{r}_1 - \vec{r}) \right) \right] \right\}. \tag{2.13}$$

The square brackets in the second term on the r.h.s. mean integral part. Different phases $\alpha_h(\vec{r})$ must be chosen for each charge configuration.

It is very easy to check that in the state (2.12) the energy is Coulombic. The linear term in Eq. (2.6) comes from the transverse photons that we require to accompany the longitudinal photons by our choice of strictly periodic b.c.. This means that the number of physical photons is not arbitrary with such b.c.. Neither is it arbitrary with the generalized ones. In fact the most general solution to the Gauss constraint with the b.c. (2.4) is

$$\Psi = \exp\left\{ i \sum_{r} \Delta_h(\vec{r}) \left[\Delta^{-1}(\vec{r}_2 - \vec{r}) - \Delta^{-1}(\vec{r}_1 - \vec{r}) \right] A_h(\vec{r}) \right\} . \Phi(A_T), \tag{2.14}$$

where Φ is an arbitrary function of $A_T(\vec{r})$ periodic with respect to $A_h(\vec{r})$. An arbitrary function of A_T can be written as

$$\Phi(A_T) = \sum_{k,r} \sum_{\mu_k(\vec{r})} C(\mu_k(\vec{r})) \exp\left\{ i \mu_k(\vec{r}) A_{Tk}(\vec{r}) \right\} . \tag{2.15}$$

The condition of periodicity requires that, for every $\vec{r}$

$$\sum_{r} \mu_k(\vec{r}) \Delta_k(\vec{r}) \Delta_h(\vec{r}') \Delta_k^{-1}(\vec{r} - \vec{r}') = \text{integer}, \tag{2.16}$$

which is a set of constraints on the integers $\mu_k(\vec{r})$.

3. - GAUGE-INVARIANT VARIABLES ON THE SITES

The Hamiltonian density and the Gauss constraint read

$$\mathcal{H} = \frac{1}{2} E_{ka}^2 + \frac{1}{4} E_{hka}^2 + i\bar{\psi} \gamma_k (\partial_k - ig A_{ka} \frac{1}{2} \tau_a) \psi + m \bar{\psi} \psi , \tag{3.1}$$

$$(\partial_k E_{ka} - g \varepsilon_{abc} A_k^c E_k^b - g j_{oa}) \Psi = 0, \tag{3.2}$$

where

$$F_{hka} = \partial_h A_{ka} - \partial_k A_{ha} + g \varepsilon_{abc} A_{hb} A_{kc}, \tag{3.3}$$

$$j_{oa} = \frac{1}{2} \psi^+ \tau_a \psi, \tag{3.4}$$

τ_a being the Pauli matrices. E_{ka} are the momenta canonically conjugate to A_{ka}. We adopt the convention of summation over repeated indices, although in some cases the sum will be explicitely indicated.

We parametrize the gauge field by separating the gauge-invariant and the gauge degrees of freedom

$$A_k = u^+ \overline{A}_k u + i/g\, u^+ \partial_k u, \tag{3.5}$$

or in color components

$$A_{ka} = \overline{A}_{kb}\, h_{ba} - 1/(2g)\, \varepsilon_{abc}\, h_{db}\, \partial_k\, h_{dc}. \tag{3.6}$$

The gauge degrees of freedom are the three angles φ_a which parametrize the orthogonal matrix h_{ba}. The gauge-invariant degrees of freedon are 6 variables, let us say Q_r necessary to parametrize $\overline{A}_{ka}$. These are 9 quantities subject to the three constraints

$$\sum_k \overline{A}_{ka}\, \overline{A}_{kb} = 0,\ a \neq b.$$

These constraints are a local gauge-fixing, which is singular when the above matrix has degenerate eigenvalues[7]. As observed by Goldstone and Jackiw[5] there are terms in the Hamiltonian which serve as a "centrifigual barrier" which should prevent the system from achieving degenerate configurations. We will explicitely see such terms in the strong coupling limit.

The spinor field ψ is replaced by the gauge-invariant spinor

$$\chi = u\psi \tag{3.7}$$

The fermionic part of the Hamiltonian density in the new variables reads

$$\mathcal{H}_F = i\, \overline{\chi}\, \gamma_k (\partial_k - ig\, 1/2\, \tau_a\, \overline{A}_{ka})\, \chi + m\, \overline{\chi}\chi. \tag{3.8}$$

No difficulty comes from discretization of the Gauss law. Using the angles φ_a the spatial derivatives disappear and it takes the local form

$$(L_a(\varphi) + j_{oa})\, \Psi = 0, \tag{3.9}$$

where L_a are the cartesian components of the orbital angular momentum of a rigid rotator.

The magnetic strength components can be written

$$F_{hka} = \mathcal{F}_{hkb}(Q)\, h_{ba}, \tag{3.10}$$

so that

$$F_{hka}\, F_{hka} = \mathcal{F}_{hkb}\, \mathcal{F}_{hkb}. \tag{3.11}$$

The magnetic energy density depends only on the gauge-invariant variables Q_r.

The electric strength components can be expressed in terms of the momenta P_r conjugate to Q_r and of the angular momenta L_a

$$E_{ka}(\vec{x}) = -i\,\frac{\delta}{\delta A_{ka}(\vec{x})} = \int d\vec{y}\left[\frac{\delta Q_r(\vec{y})}{\delta A_{ka}(\vec{x})}\,P_r(\vec{y}) - \frac{\delta \varphi_c(\vec{y})}{\delta A_{ka}(\vec{x})}\,\eta_{cb}(\vec{y})\,L_b(\vec{y})\right], \tag{3.12}$$

where

$$\eta_{cb} = \frac{1}{2}\,\varepsilon_{bae}\,h_{da}\frac{\partial}{\partial \varphi_c}\,h_{de}, \tag{3.13}$$

and we have used the relation

$$L_a = i\,\eta^{-1}_{ab}\frac{\delta}{\delta \varphi_b}. \tag{3.14}$$

Since under gauge transformations E_{ka} transforms according to $E_{ka} \to E_{kb}\, h_{ba}$, it is convenient to put

$$E_{ka} = \mathcal{E}_{kb}\, h_{ba}, \tag{3.15}$$

where

$$\mathcal{E}_{ka}(\vec{x}) = \int d\vec{y}\; h_{bc}(\vec{x})\left[\frac{\delta Q_r(\vec{y})}{\delta A_{kc}(\vec{x})}\,P_r(\vec{y}) - \frac{\delta \varphi_c(\vec{y})}{\delta A_{ka}(\vec{x})}\,\eta_{db}(\vec{y})\,h_{eb}(\vec{y})\;L'_e(\vec{y})\right] \tag{3.16}$$

In the above equation

$$L'_e = h_{ea}\,L_a \tag{3.17}$$

are the color angular momentum components in the "proper frame". They commute with L_a and are therefore gauge-invariant. As a consequence the functions multiplying P_r and L'_a must also be gauge-invariant, so that can depend only on the variables Q_r.

In conclusion the gauge field Hamiltonian density can be written

$$\mathcal{H}_G = 1/2\, \mathcal{E}_{ka}\, \mathcal{E}_{ka} + 1/4\, \mathcal{F}_{hka}\, \mathcal{F}_{hka} \tag{3.18}$$

We have performed the transformation to curvilinear coordinates at the classical level for simplicity, in order to avoid the quantum additions due to non-commuting terms.

We can now adopt the standard prescription for discretization

$$\begin{aligned}
&\varphi_b(\vec{x}) \to \varphi_b(\vec{r}) \\
&L_b(\vec{x}) \to a^{-3} L_b(\vec{r}) \\
&\psi(\vec{x}) \to a^{-3/2} \psi(\vec{r}) \\
&\overline{A}_{kb}(\vec{x}) \to a^{-1} \overline{A}_{kb}(\vec{r}) \\
&\mathcal{E}_{ka}(\vec{x}) \to a^{-2} \mathcal{E}_{ka}(\vec{r})
\end{aligned} \tag{3.19}$$

The discretized version of H_G is obvious

$$H_G = \frac{1}{a} \sum_{\vec{r}} \frac{1}{2} \mathcal{E}_{ka}(\vec{r})\, \mathcal{E}_{ka}(\vec{r}) + \frac{1}{4} \mathcal{F}_{hka}(\vec{r})\, \mathcal{F}_{hka}(\vec{r}) \tag{3.20}$$

The spatial derivatives appearing in $\mathcal{F}_{hka}$ do not spoil the gauge invariance on the lattice, because they act on the gauge-invariant variables Q (r).

For the fermionic Hamiltonian we must distinguish according to the way we define the fermion field on the lattice.

If we adopt Wilson's prescription $\psi(r)$ is a four-component spinor and H_F takes the form

$$\begin{aligned}
H_F = {} & \sum_{\vec{r}} i\, \overline{\chi}(\vec{r}) \sum_h \gamma_h \frac{1}{2} \left[\chi(\vec{r}+h) - \chi(\vec{r}-h)\right] + i\, \overline{\chi}(\vec{r}) \frac{1}{2} \tau_a \gamma_h \chi(\vec{r})\, \overline{A}_{ka}(\vec{r}) \\
& + am\, \overline{\chi}(\vec{r}) \chi(\vec{r}) + \sum_h K\, \overline{\chi}(\vec{r}) \left[\chi(\vec{r}) - \chi(\vec{r}+h) - \chi(\vec{r}-h)\right],
\end{aligned} \tag{3.21}$$

where K is the Wilson gap parameter.

If we adopt the Casher-Susskind method ψ (r) is a two-component spinor and H_F takes the form

$$\begin{aligned}
H_F = {} & \sum_{\vec{r}} \sum_h i\, \chi^{+}(\vec{r}) \frac{\vec{\sigma} \cdot h}{2} \chi(\vec{r}+h) + g\, \chi^{+}(\vec{r})\, \sigma_a \frac{1}{2} \tau_a \chi(\vec{r}+h)\, \overline{A}_{ha}(\vec{r}) \\
& + am\, (-1)^{r_1 + r_2 + r_3}\, \chi^{+}(\vec{r}) \chi(\vec{r})
\end{aligned} \tag{3.22}$$

We have reported explicitely H_F in the two cases because they have a different behaviour for

$g \to \infty$, as it will be shown in Sec. 5.

Finally the discretized version of the Gauss law reads

$$\left[L_a(\vec{r}) + h_{ba} \chi^+(\vec{r}) \frac{1}{2} \tau_b \chi(\vec{r}) \right] \Psi = 0 \tag{3.23}$$

$\chi(r)$ being a two-component or a four-component spinor according to the choice for the Hamiltonian.

We conclude this Section by noting that although $\bar{A}_{ka}$ and the gauge angles φ_a have been defined on the sites, the cartesian components A_{ka} do not result to be defined on the sites because of the term $1/2g\, \varepsilon_{abc}\, h_{db}\, \partial_k\, h_{dc}$. This fact has an important consequence which will be discussed in the next Section.

4. - THE GAUSS LAW

The Gauss operator is the sum of a spin and an orbital angular momentum in color space. The spin takes integral or half-integral values at each site according to the occurrence of an even or odd number of quarks at that site. As for the orbital angular momentum one would expect naively that it should take integral values at each site. If this were the case, the Gauss law would require an even number of quarks at each site, and a single quark could not be separated by another one even by one lattice spacing, what we call superconfinement.

In order to determine the content of the Gauss law it is necessary to understand the role of the curvilinear coordinates which give the form of an angular momentum to the Gauss operator.

The expression of the cartesian components of the gauge field in terms of gauge angles plus other variables bears some resemblance to the transformation from cartesian to polar coordinates in quantum mechanics. In order to analyze in detail such an analogy, let us consider the problem of a particle moving in a plane and interacting with a fermion. Let the Hamiltonian be

$$H = -1/2\ p^2 + \psi^* \psi + V(r, \Sigma_3, L_3), \tag{4.1}$$

where

$$r = \sqrt{x^2 + y^2}$$

$$L_3 = x p_y - y p_x$$

$$\Sigma_3 = \psi^+ \frac{1}{2} \tau_3 \psi. \tag{4.2}$$

The above Hamiltonian acts in the configuration space of the particle and in the Fock space of the Fermion, ψ, ψ^+ being destruction and creation operators resp.

Let us consider the motion of the particle subject to the constraint

$$(L_3 + \Sigma_3)\,\Psi = 0, \tag{4.3}$$

which obviously commutes with the Hamiltonian. Introducing polar coordinates H becomes

$$H = -\frac{1}{2}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{2}\frac{L^2_3}{r^2} + V(r, \Sigma_3, L_3). \tag{4.4}$$

As far as this form of the Hamiltonian is concerned, any number of fermions is allowed. We have the additional requirement, however, that the eigenfunctions of the Hamiltonian in polar coordinates be also eigenfunctions of the Hamiltonian in cartesian coordinates. This requirement rules out the states with an odd number of fermions which, due to the constraint, belong to a half-integral eigenvalue of L_3 and are therefore multivalued.

Here is the limit of the analogy between this quantum mechanical model and the present formulation of gauge theories on a lattice. In the latter case only the Hamiltonian in curvilinear coordinates is defined on the sites, and there is no corresponding cartesian form. As a consequence multivalued functions of the gauge angles cannot be ruled out, and are actually required by the structure of the Hamiltonian. Suppose in fact that we start with a wave function

$$\Psi = \sum_s (-1)^{\frac{1}{2}+s}\ \psi^+_s(\vec{r})\,\psi^+_{-s}(\vec{r})\,|\,0>$$

(4.5)

where s is the third component of color spin.

By applying the Hamiltonian to such a state we get states of the type

$$\Psi = \sum_s (-1)^{\frac{1}{2}+s} \sum_h \left[\psi^+(\vec{r}+\hat{h})\,u^+(\vec{r}+\hat{h})\,\gamma_0\gamma_h\,u(\vec{r})\right]_s\ \psi^+_{-s}(\vec{r}),$$

(4.6)

which are multivalued functions of the gauge angles, because u belongs to the spinorial representation of 0(3).

As already said the Hamiltonian cannot be defined on the sites in terms of the cartesian components of the gauge fields because these components are not defined on the sites due to the term $i/g\ u^+\partial_k u$ in Eq. (3.5). In the limit of large g, however, such a term disappears and also the cartesian components of A_{ka} result to be defined on the sites. The large g limit will be studied in the next Section.

Before ending this Section we want to show a practical way to enforce the Gauss law.

Let us define the unitary operator

$$S = \prod_{\vec{r}} e^{-i\varphi_1 j_{03}}\ e^{-i\varphi_2 j_{02}}\ e^{-i\varphi_3 j_{03}}.$$

(4.7)

We can easily check that the state

$$\Psi = S\Lambda \tag{4.8}$$

satisfies the Gauss law for Λ arbitrary but independent of φ_a. We can therefore work with the states Λ and the Hamiltonian S^+HS. In order to find its expression we need the transformation of the spinor field

$$S^+\psi S = u^+\psi \tag{4.9}$$

according to which

$$\begin{aligned} &S^+\chi S = \psi \\ &S^+\chi^+ S = \psi^+ \\ &S^+ j_{oa} S = h_{ba}\, j_{ob} \\ &S^+ L'_a S = S^+ h_{ab}\, L_a S = -S^+ h_{ab}\, j_{ob}\, S = j_{oa}. \end{aligned} \tag{4.10}$$

In conclusion S^+H_GS is obtained from H_G by replacing L'_a by $-j_{oa}$ in the electric strength components, while the transformed fermionic Hamiltonian density is obtained from Eqs. (3.21) or (3.22) by replacing χ by ψ.

5. - THE STRONG COUPLING EXPANSION

In order to study the large g limit it is convenient to specify the variables Q_r. We chose the polar representation[5] for the gauge field

$$A_{ka} = g^{-\frac{1}{3}}\left[f_{kh}\,\lambda_n\, h_{ha} - g^{-\frac{2}{3}}\,\frac{1}{2}\,\varepsilon_{abc}\, h_{hb}\,\partial_k\, h_{hc}\right] \tag{5.1}$$

so that

$$\overline{A}_{ka} = f_{ka}\,\lambda_a. \tag{5.2}$$

The variables Q_r are the three angles θ_i which parametrize the orthogonal matrix f_{kh} and the three λ_h. The factor $g^{-1/3}$ has been introduced for convencience.

The large g limit of the gauge field Hamiltonian has been derived by Simonov[5] in the continuum

$$H_{G\infty}=\frac{g^{\frac{2}{3}}}{2}\int d^3x\left\{-\sum_n\frac{\delta^2}{\delta\lambda_n^2}-2\delta(0)\sum_{n>m}\frac{1}{\lambda_n^2-\lambda_m^2}\left(\lambda_n\frac{\delta}{\delta\lambda_n}-\lambda_m\frac{\delta}{\delta\lambda_m}\right)\right.$$

$$+\frac{1}{2}\sum_{nm}\varepsilon_{mnk}^2\frac{1}{\left(\lambda_n^2-\lambda_m^2\right)^2}\left[\left(\lambda_m^2+\lambda_n^2\right)\left(L'^2_k+l'^2_k\right)+4\lambda_m\lambda_n L'_k l'_k\right]$$

$$\left.+\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2\right\}. \tag{5.3}$$

In the above equation

$$l'_k = f_{kh}\,L_h(\theta). \tag{5.4}$$

As anticipated the denominators $(\lambda^2{}_m-\lambda^2{}_n)$ should prevent degenerate configurations $\lambda_m=\lambda_n$. We perform the discretization in the standard way

$$\lambda_n(\vec{r})\rightarrow\frac{1}{a}\lambda_n(\vec{r}) \tag{5.5}$$

getting

$$H_{G\infty}=\frac{g^{\frac{2}{3}}}{2}\sum_r\left\{-\sum_r\frac{\partial^2}{\partial\lambda_n^2(\vec{r})}-2\sum_{n>m}\frac{1}{\lambda_n^2(\vec{r})-\lambda_m^2(\vec{r})}\left(\lambda_n(\vec{r})\frac{\partial}{\partial\lambda_n(\vec{r})}-\lambda_m(\vec{r})\frac{\partial}{\partial\lambda_m(\vec{r})}\right)\right.$$

$$+\frac{1}{2}\sum_{m,n}\varepsilon_{mnk}^2\frac{1}{\left(\lambda_n^2(\vec{r})-\lambda_m^2(\vec{r})\right)^2}\cdot\left[\left(\lambda_n^2(\vec{r})+\lambda_m^2(\vec{r})\right)\left(L'^2_k(\vec{r})+l'^2_k(\vec{r})\right)\right.$$

$$\left.+4\,\lambda_m(\vec{r})\,\lambda_n(\vec{r})L'_k(\vec{r})l'_k(\vec{r})\right]+\lambda_1^2(\vec{r})\lambda_2^2(\vec{r})+\lambda_1^2(\vec{r})\lambda_3^2(\vec{r})$$

$$\left.+\lambda_2^2(\vec{r})\lambda_3^2(\vec{r})\right\}. \tag{5.6}$$

For the fermionic part we find with the Wilson method

$$H_{F\infty}=\frac{g^{\frac{2}{3}}}{a}\sum_r\bar{\psi}(\vec{r})u^+(\vec{r})\frac{1}{2}\tau_a\sum_h\gamma_h u(\vec{r})\psi(\vec{r})\bar{A}_{ha} \tag{5.7}$$

and with the Casher-Susskind method

$$H_{F\infty} = \frac{g^{\frac{2}{3}}}{a} \sum_r \psi^+(\vec{r})\, u^+(\vec{r})\, \frac{1}{2}\tau_a \sum_h \sigma_h\, u(\vec{r}+\hat{h})\, \psi(\vec{r}+\hat{h})\, \overline{A}_{ha}(\vec{r}) \tag{5.8}$$

The present result should be compared to the strong coupling limit in the Wilson-Kogut-Susskind formulation. In the latter case both magnetic strength and matter field completely disappear from the Hamiltonian.

For large g we can define also the cartesian components of the gauge field on the sites

$$A_{ka}(\vec{r}) = g^{-\frac{1}{3}}(\vec{r})\, f_{kn}(\vec{r})\, \lambda_n(\vec{r})\, h_{na}(\vec{r}) \tag{5.9}$$

and we can write $H_{G\infty}$ in cartesian variables

$$H_{G\infty} = \frac{g^{\frac{2}{3}}}{2a} \sum_r \; -\frac{\partial^2}{\partial A_{ha}^2(\vec{r})} + \frac{1}{2}\, \varepsilon_{abc}\, \varepsilon_{ade}\, A_h^b(\vec{r})\, A_k^c(\vec{r})\, A_h^d(\vec{r})\, A_k^e(\vec{r}). \tag{5.10}$$

For the Wilson fermionic Hamiltonian we obtain

$$H_{F\infty} = \frac{g^{\frac{2}{3}}}{a} \sum_r \; \overline{\psi}(\vec{r})\, \frac{1}{2}\, \tau_a\, \gamma_h\, \psi(\vec{r})\, A_{ha}(\vec{r}) \tag{5.11}$$

while the Casher-Susskind form cannot be expressed in terms of the cartesian components A_{ka} at a given site.

The Wilson fermionic Hamiltonian contains only fermionic operators at the same site. It is therefore consistent to require that the eigenfunctions of the Hamiltonian in polar coordinates be also eigenfunctions of the Hamiltonian in cartesian coordinates. Such eigenfunctions must be single-valued, so that only integral values of L_a are admissible and quarks must occur in even number at each site: In the $g\to\infty$ limit we get superconfinement.

If on the other hand we do not make the above requirement, since the energy of two quarks does not depend on their distance, we do not get any kind of confinement[6].

If the fermions are put on the lattice by the Casher-Susskind method, only the polar form of the Hamiltonian is meaningful, and again we do not get confinement[6].

We can study how superconfinement is relaxed at finite but large values of g by performing a strong coupling expansion.

Let us evalutate the probability for two quarks to be at relative distance r=Na at the lowest nonvanishing order. We need to consider only those terms in the Hamiltonian which can induce transitions of one quark from one site to a neighbouring one. There are no such terms in the expansion of H_G. H_G contains indeed the fermion operators only in $j_{oa}(r)$, which cannot give rise to quark hopping.

Terms of the necessary structure are only present in H_F, and are of zero order in g. Since H_∞ is of order $g^{2/3}$, the expansion parameter is $g^{-2/3}$, in agreement with Eq. (5.1). The desired terms are

$$H'_F = \frac{1}{a} \sum_{\vec{r}} \sum_{h} \bar{\psi}(\vec{r}) \left[\left(\frac{i}{2} \gamma_h - K \right) \psi(\vec{r} + \hat{h}) - \left(\frac{i}{2} \gamma_h + K \right) \psi(\vec{r} - \hat{h}) \right] \tag{5.12}$$

Now H_∞ is the sum of hamiltonians defined at single sites

$$H_\infty = g^{\frac{2}{3}} \sum_{\vec{r}} h_\infty(\vec{r}), \tag{5.13}$$

where the definition of $h_\infty(r)$ follows from Eqs. (5.10) and (5.11).

Let us denote by $\Phi^{(1)}{}_m$ (r) and $\Phi^{(0)}{}_n$ (r) the eigenfunctions of h_∞ (r) with one and zero quarks, m, n being all the necessary quantum numbers (we should actually say with fermion number 1 and 0, because h_∞ does not commute with the number of quarks). Let us denote by $\varepsilon^{(1)}{}_m$ and $\varepsilon^{(0)}{}_n$ the corresponding energies.

At zero order we have an eigenfunction Φ with eigenvalue ε describing two quarks at the same site, let us say the origin, due to superconfinement. Such a state is highly degenerate, so that we should use degenerate perturbation theory. It will be clear from the following that this is not necessary for the lowest nonvanishing order.

Let us take the distance r between the quarks along one of the coordinate axes, so that we can indicate the position of one quark by one single coordinate, and write $\Phi^{(1)}{}_{m_i}(i)$ for its wave function. The first term in the perturbative expansion in therefore

$$g^{-\frac{2}{3}} \left[\varepsilon - \varepsilon^{(1)}_{m_0} - \varepsilon^{(1)}_{m_1} \right]^{-1} < \Phi^{(1)}_{m_1}(1) \Phi^{(1)}_{m_0}(0) \mid H' \mid \Phi^{(0)}_0(1) \Phi(0) >, \tag{5.14}$$

describing the transition of one of the two quarks from the origin to the site 1, which was before in the ground state (vacuum) $\Phi^{(0)}{}_0(1)$. The generic term in the expansion is

$$g^{-\frac{2}{3}} \left[\varepsilon - \varepsilon^{(1)}_{m_0} - \varepsilon^{(1)}_{m_h} - \sum_{i=1}^{h-1} \varepsilon^{(0)}_{n_i} \right]^{-1} < \Phi^{(1)}_{m_h}(h) \Phi^{(0)}_{n_{h-1}}(h-1) \mid H' \mid \Phi^{(0)}_0(h) \Phi^{(1)}_{m_{h-1}}(h-1) >, \tag{5.15}$$

because to lowest order the quark must always jump in the direction of the site N. It has to do N jumps to reach the site. It follows that the probability to find two quarks at distance Na is of order $g^{-4/3\,N}$, and it is vanishing for $N \to \infty$ if the perturbative expansion converges. This shows that

confinement persists at finite but large values of g.

It would be interesting to evaluate the quark-quark wave function in order to find what is the effective potential which can produce it. This would require the evaluation of the N-th order term of the perturbative expansion. We are unable to do it, but we will show that the wave function decays not faster than exponentially at large distance, a behaviour incompatible with a linear effective potential.

To this end let us evaluate the amplitude for one quark to reach the site N by leaving behind itself all the sites in the ground state. Such an amplitude is

$$\Psi = g^{-\frac{2}{3}N} \sum_{m_0 m_1 m_N} \left(B^{N-1}\right)_{m_N,m_1} C_{m_1 m_0} < \Phi^{(1)}_{m_0}(0)\ \Phi^{(1)}_{m_N}(N) \prod_{r\neq 0,Ni} \Phi^{(0)}_0(r), \tag{5.16}$$

the matrices B and C being defined by

$$B_{m_h,m_{h-1}} = \left[\varepsilon - \varepsilon^{(1)}_{m_0} - \varepsilon^{(1)}_{m_h}\right]^{-1} < \Phi^{(1)}_{m_h}(h)\,\Phi^{(0)}_0(h-1)\,|H'|\,\Phi^{(0)}_0(h)\,\Phi^{(1)}_{m_{h-1}}(h-1)>,$$

$$C_{m_1,m_0} = \left[\varepsilon - \varepsilon^{(1)}_{m_0} - \varepsilon^{(1)}_{m_1}\right]^{-1} < \Phi^{(1)}_{m_1}(1)\,\Phi^{(1)}_{m_0}(0)\,|H'|\,\Phi^{(0)}_0(0)\,\Phi(0)>. \tag{5.17}$$

We thus have

$$|\Psi|^2 = g^{-\frac{4}{3}N}\ T_r C^{+}\left(B^{+}\right)^{N-1} B^{N-1} C. \tag{5.18}$$

Now there exists a real positive number b such that

$$|\Psi|^2 \geq \left(g^{-\frac{2}{3}} b\right)^{2N} = e^{-\frac{2r}{r_0}} \tag{5.19}$$

where

$r = Na$

$r_0 = -a/\ln\,(g^{-2/3}b)$

and r_0 is positive for g large enough. The probability for two quarks to be at distance Na irrespective of the state of sites between them is obviously larger than $|\Psi|^2$, and therefore such a probability decays not faster than exponentially.

6. - SUMMARY

We have shown that in abelian compact theories confinement depends, in strong coupling, by the choice of b.c., and we have given an argument whereby the same result should hold in the non-abelian case.

The choice of b.c. which does not confine in the abelian case sugests a non compact formulation of gauge theories. We have performed such a formulation for Yang-Mills theories by expressing all the fields in terms of gauge-invariant and pure gauge degrees of freedom, and then by defining all the variables on the sites.

The cartesian components of the gauge fields do not remain defined on the sites through our procedure. This has the consequence that multivalued functions of the gauge angles are allowed in general.

In the strong coupling limit also the cartesian components of the gauge fields remain defined on the sites. There exists therefore in the Hamiltonian a "kinetic energy" term having the form of a Laplascian in the cartesian components of the gauge fields.

In analogy to quantum mechanics, we can require that the eigenfunctions of the Hamiltonian in polar coordinates be also eigenfunctions of the Hamiltonian in cartesian coordinates, and this excludes multivalued functions of the gauge angles. The gauge color angular momentum takes therefore integral values at each site, and so the color spin of the matter field must do. Accordingly the Gauss law requires occurrence of quarks in even number at each site: Quarks are superconfined.

A strong coupling expansion shows that at finite but large g the quark-quark wave function goes to zero not faster than exponentially at large distance, a behaviour incompatible with a linear effective potential.

If we do not make the above requirement on the wave-function, quarks are not confined at all. In either case there is no linear potential.

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