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# ELECTROMAGNETIC FIELDS SCATTERED BY A CHARGE MOVING ON THE AXIS OF A SEMI-INFINITE CIRCULAR WAVEGUIDE: RADIATION SPECTRUM AND LONGITUDINAL IMPEDANCE

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Abstract

The electromagnetic field symmetric about the axis and excited by a charged particle bunch entering (or exiting) an open perfectly conducting smooth waveguide is found. The longitudinal impedance is expressed as the sum of the contributions from the wall and from the open end of the pipe (diffracted fields). The radiation spectrum is calculated for a wide range in frequency and particle energy.

## 1. Introduction

The concept of an impedance which describes the response of the surroundings to a moving bunch of charged particles has proven to be a very useful concept in solving many accelerator (and other) problems. Particularly it has been extensively and successfully used to describe different coherent instabilitites important to modern accelerators. An excellent review of this concept and its applications can be found in Ref. 1. Impedance for many different structures have been calculated using analytic or numerical methods. In these calculations almost all structures considered were closed in the sense that the field radiated by the bunch and diffracted by the surrounding non-uniformities did not extend to infinity. For such closed structures the impedance and its dependance on the Lorentz factor $\gamma$ and the frequency $\omega$ can be found numerically.

For open structures, on the other hand, the behaviour of the impedance is not well understood, and the impedance found using different methods do not agree. It is quite clear that these differences arise because of the initially assumed approximations.

Given this situation it seems appropriate to consider a case where an exact solution can be found. Such an exact solution does exist for the case of a point charge moving on the axis of a semi-infinite perfectly conducting pipe[2]. The concern of the authors of Ref. 2 was mainly the fields in the radiation region, i.e. at large distance from the source. Starting with this solution we modify it in two ways. First, we require that the field be found on the axis of the pipe in the vicinity of the source. Second, the previous requirement in turn necessitates that the radial charge distribution be considered. A new solution is found which is used to calculate the longitudinal impedance in the usual way[1].

Let us consider a bunch with radial charge distribution f(r) and a total charge q moving with constant velocity $V = \beta c$ on the axis into the one open of a circular waveguide. The case of an exiting charge can be considered in a similar way. The radius of the pipe is denoted as "b", and the characteristic radial dimension of the bunch as "a" (see Fig. 1). The resistivity of the pipe wall is neglected.

The problem can be formulated in terms of a set of coupled integral equations for the induced wall current h(z). There is no wall for z < 0, hence:

(1.1) $$h(z) = 0, \qquad z < 0$$

The condition that the wall is a perfect conductor may be written in the following form:

(1.2) $$[E_{zi}(r,z) + E_{zs}(r,z)]_{r=b} = 0, \qquad z > 0$$

where $E_{zs}$ and $E_{zi}$ are the z-components of the electric field produced by the bunch and by the induced current, respectively.

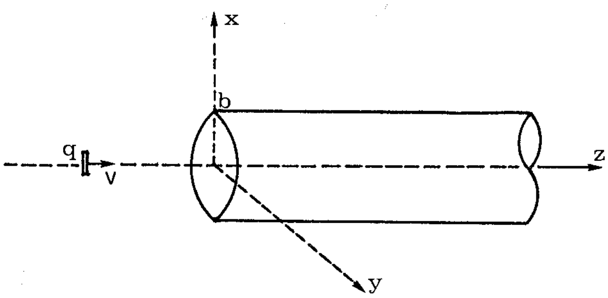


Fig. 1 - The relevant geometry.

Consider now the Fourier transformation $F(\alpha)$ = FT [h(z)]. If for the complex frequency one can find the Green operator $L(\alpha)$, such that for the given geometry, it produces the electric field $E_{zi}$, then Eqs. (1.1) and (1.2) can be written in the form:

(1.3) $$FT^{-1}[F(\alpha)] = 0, \qquad z < 0$$

(1.4) $$FT^{-1}[L(\alpha)F(\alpha)] = -E_{zs}(b,z), \qquad z > 0$$

This set of integral equations can be solved employing a slightly modified Wiener Hopt method[3,4].

In Section 2 the expression for the Green operator $L(\alpha)$ is given as derived in Refs. 3 and 4.

In Section 4 the longitudinal impedance is found and expressed as the sum of contributions from the wall and the open end. In Section 5 we use the obtained formulae to calculate the longitudinal impedance for wide frequency and energy ranges. Additional details of the calculations are included in the Appendices.

## 2. The integral equations

In this Section we derive the integral equation for the current induced in the wall of the pipe. The Fourier components of the scalar $\Phi_\omega$ and vector $\bar{A}_\omega$ potentials can be expressed in terms of the Fourier components of the total charge $\varrho_\omega$ and current $\bar{J}_\omega$ densities:

$$\Phi_\omega(\bar{r}) = \int \frac{\rho_\omega(\bar{r}_0)\, e^{ikR}\, d\bar{r}_0}{R} \tag{2.1}$$

$$\bar{A}_\omega(\bar{r}) = \frac{1}{c}\int \frac{\bar{j}_\omega(\bar{r}_0)\, e^{ikR}\, d\bar{r}_0}{R} \tag{2.2}$$

where $k = \omega/c$ (note that this definition is different from the definition for k usually used in circular accelerators), and

$$R = |\bar{r} - \bar{r}_0| \tag{2.3}$$

Due to the symmetry the vector $\bar{r}$ may be chosen to be in the plane (x,z), (see Fig. 2)

$$R = \sqrt{r^2 + r_0^2 - 2rr_0\cos\theta + (z - z_0)^2} \tag{2.4}$$

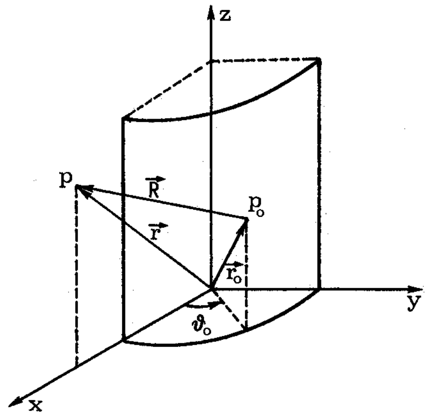


Fig. 2 - Cylindrical coordinate system.

The total charge and current densities are the sums of those of the source and of those induced in the wall. Again, due to the symmetry, all current densities have only a z-component and hence the same is true for the vector potential.

$$\rho_\omega = \rho_{\omega s} + \rho_{\omega i} = \frac{q}{2\pi V} f(r) e^{ikz/\beta} + \frac{q}{4\pi^2 c}\frac{\delta(r-b)}{b} g(z) \tag{2.5}$$

$$j_\omega = j_{\omega s} + j_{\omega i} = \frac{q}{2\pi} f(r) e^{ikz/\beta} + \frac{q}{4\pi^2}\frac{\delta(r-b)}{b} h(z) \tag{2.6}$$

Here two unknown functions g(z) and h(z) describe the induced charge and current densities. There is no wall for $z < 0$, hence these functions should satisfy the conditions:

$$g(z) = 0 \qquad for\ z < 0 \tag{2.7}$$

$$h(z) = 0 \qquad for\ z < 0 \tag{2.8}$$

Using the continuity equation:

$$-i\omega\rho_\omega + \frac{\partial j_\omega}{\partial z} = 0$$

one can relate these two functions by the equation:

$$g(z) = \frac{1}{ik}\frac{\partial h(z)}{\partial z} \tag{2.9}$$

The scalar and vector potentials then become (see Appendix A for relevant integrals and definitions):

$$\Phi_\omega(r,z) = \frac{iq\pi}{V}\int_0^\infty G_{k/\beta}(r,r_0) f(r_0) e^{ikz/\beta} r_0 dr_0 + \frac{iq}{4\pi c}\int_{-\infty}^{+\infty} g(z_0)dz_0 \int_{-\infty}^{+\infty} G_\alpha(r,b) e^{i\alpha(z-z_0)} d\alpha \tag{2.10}$$

$$A_\omega(r,z) = \frac{iq\pi}{c}\int_0^\infty G_{k/\beta}(r,r_0) f(r_0) e^{ikz/\beta} r_0 dr_0 + \frac{iq}{4\pi c}\int_{-\infty}^{+\infty} h(z_0)dz_0 \int_{-\infty}^{+\infty} G_\alpha(r,b) e^{i\alpha(z-z_0)} d\alpha \tag{2.11}$$

where the Green function $G_\alpha(r,b)$ is:

$$G_\alpha(r_1,r_2) = \begin{cases} J_0(vr_1)H_0(vr_2), & r_1 < r_2 \\ J_0(vr_2)H_0(vr_1), & r_1 > r_2 \end{cases} \tag{2.12}$$

Here

$$v = \sqrt{k^2 - \alpha^2} \tag{2.13}$$

where we choose $\mathrm{Im}\, v > 0$.

With the help of (2.9) it is easy to show that the Lorentz condition:

$$-ik\Phi_\omega + \frac{\partial A_\omega}{\partial z} = 0 \tag{2.14}$$

is fulfilled by (2.10) and (2.11). The fields E and B can be obtained from $\Phi_\omega$ and $\bar{A}_\omega$ by

$$\bar{E}_\omega = -\nabla\Phi_\omega + ik\bar{A}_\omega \tag{2.15}$$

$$\bar{B}_\omega = -\nabla \times \bar{A}_\omega \tag{2.16}$$

Using these equations one finds that only $E_r$, $E_z$ and B components are produced (TM type fields). The fields (2.15) and (2.16) satisfy Maxwell's equations by construction.

In addition the electric field must satisfy the boundary condition $E_{tang} = 0$ at all metallic surfaces. For our case this condition can be written as:

$$E_z(b,z) = 0 \qquad for\ z > 0 \tag{2.17}$$

To simplify the calculations in what follows we assume a radially uniform charge distribution within the region a < b:

$$f(r) = \begin{cases} 1/\pi a^2 & r < a \\ 0 & r \geq a \end{cases} \tag{2.18}$$

For any other charge distribution the following results can be obtained in a similar way with only slight changes in the derivation. For the region outside the bunch (b > r > a) we obtain the potentials from eqs. (2.10) and (2.11):

$$\Phi_\omega(r,z) = \frac{q}{2\pi V}\left\{4\pi P_I(k) K_0(kr/\beta\gamma) e^{ikz/\beta} + \frac{\beta}{2}\int_{-\infty}^{+\infty}\frac{\partial h(z_0)}{\partial z_0}dz_0 \int_{-\infty}^{+\infty} J_0(vr)H_0(vb) e^{i\alpha(z-z_0)} d\alpha\right\} \tag{2.19}$$

(2.20) $$A_\omega(r,z) = \frac{q}{2\pi c}\left\{4\pi P_I(k)K_0(kr/\beta\gamma)e^{ikz/\beta} + \frac{i}{2}\int_{-\infty}^{+\infty} h(z_0)dz_0 \int_{-\infty}^{+\infty} J_0(vr)H_0(vb)e^{i\alpha(z-z_0)}d\alpha\right\}$$

where the factor $P_I$ has been introduced:

(2.21) $$P_I(k) = \frac{I_1(ka/\beta\gamma)}{(\pi ka/\beta\gamma)}$$

(see Appendix A for the definitions of functions $J_0$, $H_0$, $I_0$ and $K_0$).

It is now convenient to measure all quantities in units of b: $\varrho = r/b$, $\varkappa = kb$, $y = z/b$, $u = \alpha b$. Using these dimensionless quantities, we find:

(2.22) $$E_{z\omega}(\rho,y) = \frac{2\kappa P_I(\kappa)}{ibc\beta^2\gamma^2}K_0(\kappa\rho/\beta\gamma)e^{i\kappa y/\beta} - \frac{q}{4\pi^2\kappa cb}\int_{-\infty}^{+\infty} h(y_0)dy_0 \int_{-\infty}^{+\infty} L_\rho(u)e^{iu(y-y_0)du}$$

where the notation $L_\varrho(u)$ is introduced:

(2.23) $$L_\rho(u) = \pi v^2 J_0(v\rho)H_0(v)$$

and v is:

(2.24) $$v = \sqrt{\kappa^2 - u^2}$$

Equation (2.17) can be written in the following form:

(2.25)
$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} p(y_0)dy_0 \int_{-\infty}^{+\infty} L(u)e^{iu(y-y_0)}du = M\, e^{i\kappa y/\beta},$$

$$for\ y > 0$$

where

$$L(u) = L_\rho(u)|_{\rho=1}$$

and

(2.26) $$M = \frac{4\pi}{i}\left(\frac{\kappa}{\beta\gamma}\right)^2 P_I(\kappa)K_0\left(\frac{\kappa}{\beta\gamma}\right)$$

Equation (2.25) is the integral equation for the induced current density h(y), h(y) = 0 for y < 0. Let us define the Fourier component of h(y):

(2.27) $$F(u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} h(y)e^{-iyu}dy$$

conversely

(2.28) $$h(y) = \int_{-\infty}^{+\infty} F(u)e^{iyu}du$$

The equation (2.17) is equivalent to the system of the following equations:

(2.29) $$\int_{-\infty}^{+\infty} F(u)L(u)e^{iyu}du = Me^{i\kappa y/\beta},\quad y > 0$$

(2.30) $$\int_{-\infty}^{+\infty} F(u)e^{iyu}du = 0,\quad y < 0$$

Provided the function F(u) is found, one can find h(y) from (2.28) and the problem is then solved. Indeed, all physical quantities of interest can be expressed in terms of the function h(y).

## 3. Factorization method

Let us assume, for the time being, that $\varkappa$ has a small imaginary part $Im(\varkappa) > 0$. In the final result this term will be put to zero. The solution of the system of eqs (2.29), (2.30) is obtained[3,4] by representing L(u) as a product of two funtions chosen in such a way that: 1) F(u)L(u) has one pole in the upper half plane of the complex variable u at the point $u_0 = \varkappa/\beta$. The value of the residue of this pole is $M/2\pi i$. At all other points of the upper half-plane F(u)L(u) is analytic. For $|u| \to \infty$ on the upper half-plane F(u)L(u) tends to zero. 2) F(u) is an analytic function in the lower half-plane of u and tends to zero when $|u| \to \infty$ on the lower half circle. The behaviour for large $|u|$ is related to the continuity of h(y) at y = 0, and is determined by the "edge condition" which requires F(u) to vanish as $|u|^{-3/2}$ for $|u| \to \infty$. The properties of F(u) and F(u)L(u) listed in 1) and 2) are necessary and sufficient for F(u) to be the solution of (2.29), (2.30).

Let us use the factorization of the product $\pi v\, J_0(v)\, H_0(v)$,

(3.1) $$\pi v J_0(v)H_0(v) = \Gamma_+(u)/\Gamma_-(u)$$

where $\Gamma_+(u)$ is an analytic function in the upper half-plane of u, while $\Gamma_-(u)$ is analytic in the lower half-plane. Suppose that the function L(u) can also be factorized in he following manner:

(3.2) $$L(u) = \frac{L_+(u)L_-(u)}{(u+\frac{\kappa}{\beta})(u-\frac{\kappa}{\beta})}$$

Using expression (3.1),the funtions $L_I(u)$ can be expressed in terms of the functions $\Gamma_+(u)$ and $\Gamma_-(u)$:

(3.3) $$L_+(u) = i\Gamma_+(u)\sqrt{u+\kappa}\,(u+\kappa/\beta)$$

(3.4) $$L_-(u) = -i\sqrt{\kappa-u}\,(u-\kappa/\beta)/\Gamma_-(u)$$

Since $1/L_-(u)$ is an analytic function in the lower half-plane we can assume:

(3.5) $$F(u) = \frac{D}{L_-(u)} = \frac{iD\Gamma_-(u)}{\sqrt{\kappa-u}\,(u-\kappa/\beta)}$$

where D is some constant. This function satisfies eq. (2.30). Multiplying together eqs (3.5) and (3.2) one finds:

$$F(u)L(u) = \frac{DL_+(u)}{(u+\kappa/\beta)(u-\kappa/\beta)} = \tag{3.6}$$

$$i\frac{D\Gamma_+(u)\sqrt{\kappa+u}}{u-\kappa/\beta}$$

This function satisfies equation (2.29) if

$$iD = \frac{M}{2\pi i\Gamma_+(\kappa/\beta)\sqrt{\kappa+\kappa/\beta}} \tag{3.7}$$

and thus

$$F(u) = \frac{M}{2\pi i\ \Gamma_+(\kappa/\beta)\sqrt{\kappa+\kappa/\beta}}\ \frac{\Gamma_-(u)}{(u-\kappa/\beta)\sqrt{u-\kappa/\beta}} \tag{3.8}$$

As shown in Appendix B both functions $\Gamma_+(u)$ and $\Gamma_-(u)$ approach unity as $|u| \rightarrow \infty$. Hence, for $|u| \rightarrow \infty$,

$$F(u) \sim |u|^{-3/2} \tag{3.9}$$

$$F(u)L(u) \sim |u|^{-1/2} \tag{3.10}$$

Hence, functions F(u) and F(u)L(u) satisfy conditions 1) and 2) respectively, and F(u) represents the solution of eqs (2.29) and (2.30). From (2.28) one finds the solution of (2.25):

$$h(y) = \frac{M}{2\pi i\ \Gamma_+(\kappa/\beta)\sqrt{\kappa+\kappa/\beta}} \int_{-\infty}^{+\infty} \frac{\Gamma_-(u)e^{iyu}du}{(u-\kappa/\beta)\sqrt{\kappa-u}} \tag{3.11}$$

The expressions (3.8) and (3.11) were obtained in the reference 2.

## 4. The impedance of an open waveguide

We are now prepared to calculate the longitudinal impedance of an open pipe. To do so we need the z-component of the field on the pipe axis r = 0. Substitute (3.11) into (2.11) and use expression (2.15) to find the longitudinal component of the electric field for r < a:

$$E_{z\omega}(\rho,y) = \frac{q}{2\pi i c\kappa b}\left[\left(\frac{\kappa}{\beta\gamma}\right)^2 4\pi T(\rho)e^{i\kappa y/\beta} + \frac{2\pi i P_I(\kappa)K_0(\kappa/\beta\gamma)\sqrt{\kappa-\kappa/\beta}}{\Gamma_+(\kappa/\beta)}\int_{-\infty}^{+\infty}\frac{\Gamma_-(u)J_0(v\rho)H_0(v)e^{iyu}du}{(u-\kappa/\beta)\sqrt{\kappa-u}}\right] \tag{4.1}$$

where we used the expression for the Green function $G_\alpha(r,a)$ for r < a. To find the impedance, only the value of $E_{z\omega}(\varrho)$ and correspondingly $T(\varrho)$ at $\varrho=0$ is needed and are given below in expression (4.3).

The value of the integral in expression (4.1) can be found by closing the path of integration by a semi-circle of a large radius in the upper half-plane of u. Then the contribution of the pole $u = \varkappa/\beta$ will produce the part of the field which is travelling with the particle. The rest of the integral connected with the poles of the function $\Gamma_-(u)$ is the diffracted part of the field due to the open end of the waveguide. We write for $\varrho = 0$:

$$E_{z\omega}(0,y) = E_{1\omega}(0,y) + E_{2\omega}(0,y) \tag{4.2}$$

Here

$$E_{1\omega}(0,y) = \frac{2q}{icb\kappa}\left(\frac{\kappa}{\beta\gamma}\right)^2 e^{i\kappa y/\beta}\left[P_K(\kappa) - P_I(\kappa)\frac{K_0(\kappa/\beta\gamma)}{I_0(\kappa/\beta\gamma)}\epsilon(y)\right] \tag{4.3.}$$

where $\varepsilon(y)$ is a step function: $\varepsilon(y) = \begin{cases}1 & y>0\\ 0 & y<0\end{cases}$ and constant, $P_K = K_1(ka/\beta\gamma)/(ka/\beta\gamma)$.

The value of the field $E_{1\omega}$ (4.3) for y > 0 is the same as one produced in an infinite pipe, while the value of the field $E_{1\omega}$(4.3) for y < 0 is the field of the bunch in the vacuum.

$$E_{2\omega}(0,y) = \frac{q}{cb\kappa}P_I(\kappa)\frac{K_0(\kappa/\beta\gamma)(\kappa/\beta-\kappa)\sqrt{\kappa+\kappa/\beta}}{\Gamma_+(\kappa/\beta)}\int_{-\infty}^{+\infty}\left[\frac{S(u)-S(\kappa/\beta)}{u-\kappa/\beta}\right]e^{iyu}du \tag{4.4}$$

where

$$S(u) = \Gamma_-(u)\sqrt{u-\kappa}\ (u+\kappa)K_0(\sqrt{u^2-\kappa^2}) \tag{4.5}$$

The wake function $W(\tau)$ is defined as the integral over z of $-E_z(t)\,|\,r=0,\ t=z/v+\tau$ for a unit charge, i.e. the potential of the field seen by a test particle trailing the source at a distance $V\tau$ (see Ref. 9). Performing the integration one finds for $W(\tau)$:

$$W(\tau) = -\frac{1}{q}\int_{-\infty}^{+\infty} e^{-i\omega\tau}d\omega \int_{-\infty}^{+\infty} E_{z\omega}(0,z)e^{i\kappa y/\beta}dz \tag{4.6}$$

It is clear from this expression, that the impedance (which is the Fourier transform of $W(\tau)^1$) is:

$$Z(\omega) = -\frac{2\pi}{q}\int_{-\infty}^{+\infty} E_{z\omega}(0,z)e^{i\kappa y/\beta}dz \equiv Z_1(\omega) + Z_2(\omega) \tag{4.7}$$

where $Z_1(\omega)$ is the part of the impedance which is produced by the wall of the pipe:

$$Z_1(\omega) = -\frac{4\pi\kappa}{icb\gamma^2\beta^2}\int_{-\infty}^{+\infty}\left[P_K(\kappa) - P_I(\kappa)\frac{K_0(\kappa/\beta\gamma)}{I_0(\kappa/\beta\gamma)}\epsilon(y)\right]dy \tag{4.8}$$

while $Z_2(\omega)$ is the part of the impedance which is produced by the radiation on the open end of the pipe:

$$Z_2(\omega) = -\frac{4\pi^2\kappa}{c\beta^2\gamma^2}P_I(\kappa)\frac{K_0(\kappa/\beta\gamma)}{\Gamma_+(\kappa/\beta)\sqrt{\kappa+\kappa/\beta}}\left(\frac{\partial S}{\partial u}\right)_{u=\kappa/\beta} \tag{4.9}$$

In deriving (4.8) and (4.9) we have used the representation (3.1) in the form:

$$\pi v H_0(v) = \Gamma_+(u)/[\Gamma_-(u)J_0(v)] \tag{4.10}$$

The functions $\Gamma_+(u)$ and $\Gamma_-(\varkappa/\beta)$ can be found in Appendix B. In terms of these functions

$$Z_2(\omega) = \frac{4\pi\kappa K_0(\kappa/\beta\gamma)P_I(\kappa)}{c\beta^2\gamma^2 I_0(\kappa/\beta\gamma)}\left[\frac{\gamma I_1(\kappa/\beta\gamma)}{I_0(\kappa/\beta\gamma)}sgn(\beta) - \frac{\Gamma'_+(\kappa/\beta)}{\Gamma_+(\kappa/\beta)} - \frac{1}{2(\kappa+\kappa/\beta)}\right] \tag{4.11}$$

## 5. Longitudinal impedance and radiation spectrum

We first consider the low frequency behaviour of the term $Z_1$. For large $\gamma$ the argument of the Bessel functions is small. Expanding functions $I_0(x)$ and $K_0(x)$ into the power series and retaining only the first non-vanishing terms we get:

$$P_I = \frac{\pi}{2} \tag{5.1}$$

$$P_K \sim \frac{\pi}{2}\left(\frac{1}{2} + ln\frac{2\gamma\beta}{\kappa a}\right) \tag{5.2}$$

and consequently the well known expression for the pipe impedance per unit length [7] (MKS units):

$$Z_1(\omega) = \frac{iZ_0\kappa}{4\pi\beta^2\gamma^2}\left(1 + 2ln\frac{b}{a}\right) \tag{5.3}$$

where $Z_0$ = 377 Ohms is the vacuum impedance. This part of the impedance is purely reative and goes to zero in the ultrarelativistic limit $\gamma \to \infty$.

To find the contribution from the open end of the pipe one needs to calculate the value of the logarithmic derivative of $\Gamma_+$ at $\varkappa/\beta$. This task is performed in Appendix B with the result:

(5.4)

$$\frac{\Gamma'_+(\kappa/\beta)}{\Gamma_+(\kappa/\beta)} = \frac{1}{2}\left[\frac{\beta}{\kappa(1+\beta)} + \frac{I_1(\kappa/\beta\gamma)\gamma}{I_0(\kappa/\beta\gamma)}sgn(\beta) + \frac{\gamma K_1(\kappa/\beta\gamma)}{K_0(\kappa/\beta\gamma)}\right] + \sum_{n=1}^{\infty}\frac{1}{\alpha_n - \kappa/\beta} + \Delta Re + i\Delta Im$$

Here $\Delta$Re and $\Delta$Im are defined by expressions (B8) and (B9). Substituting expressions (5.4) into (4.11) one finds:

$$Z_2(\omega) = \frac{Z_0\kappa}{\beta^2\gamma^2}P_I(\kappa)\frac{K_0(\kappa/\beta\gamma)}{I_0(\kappa/\beta\gamma)}F(\kappa,\beta) \tag{5.5}$$

where

$$F(\kappa,\beta) = \gamma\frac{K_1(\kappa/\beta\gamma)}{K_0(\kappa/\beta\gamma)}sgn(\beta) - \frac{\beta}{\kappa(1+\beta)} - \sum_{n=1}^{\infty}\frac{1}{\alpha_n - \kappa/\beta} - \Delta Re - i\Delta Im \tag{5.6}$$

The real part of the impedance $Z_2(\omega)$ as function of the normalized frequency $\varkappa$ presented in Fig. 3 for several different values of the parameter $\gamma$. One can see that as $\gamma$ increases the spectrum reaches a peak and then decreases. The frequency at the peak is given by $\omega = c\,\gamma/b$.

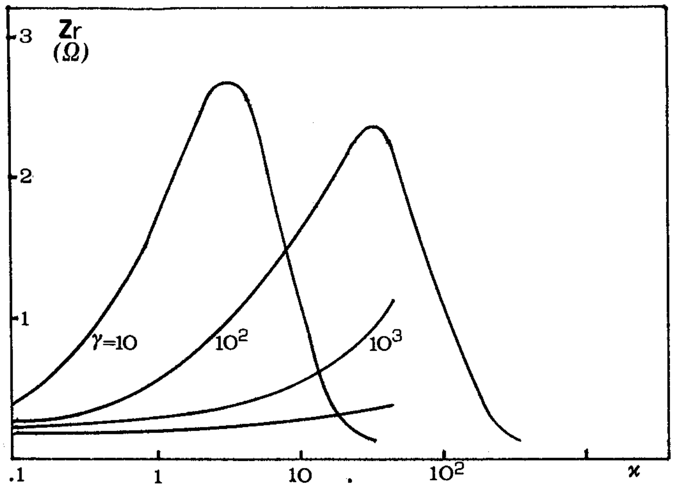


Fig. 3 - Real part of the impedance versus the normalized frequency $\varkappa = \omega b/\beta c$.

This behaviour of the impedance is easily explained. The real part of the impedance is the spectrum of the radiated energy $\Delta U$. For a delta-function charge distribution, the radiated energy can be calculated as[1]:

$$\Delta U = q^2\int_{-\infty}^{+\infty} Z_R(\omega)d\omega \tag{5.7}$$

For a relativistic particle the radiation from the induced charges on the pipe wall will occur mainly when the edge of the pipe is seen by the self-field which is confined within an angle $\vartheta \sim 1/\gamma$. Therefore one can expect the radiation pulse will last over a time interval $b/\gamma_c$, and its spectrum will have a bandwidth $\omega < c\gamma/b$. The low frequency behaviour of the impedance is plotted in Fig.4 versus the Lorentz factor $\gamma$.

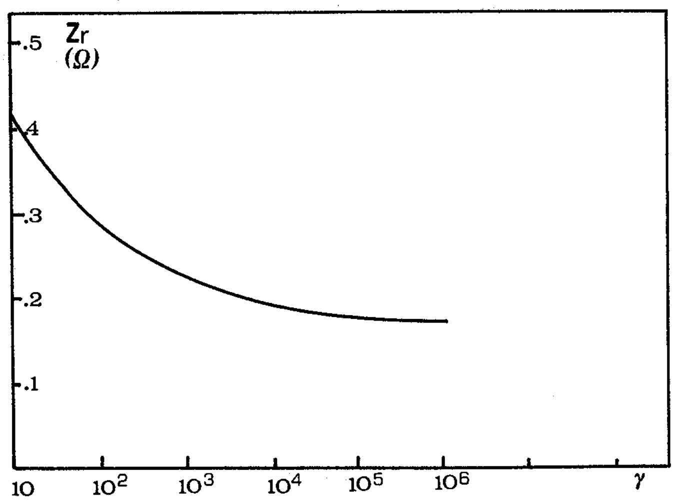


Fig. 4 - Energy dependence of real part of the impedance $Z_R$, calculated at $\varkappa$ =.1.

The curve seems to tend toward a constant value, but there is actually a non zero negative slope at any $\gamma$-value. Therefore the impedance decreases, with the increase of $\gamma$, although much slower than

$1/\gamma$. The reason for this is that in the limit $\gamma \to \infty$, the presence of the perfectly conducting wall does not perturb the field of the axial symmetric charge distribution.

The above calculations refer to the case of a charge entering the semi infinite pipe. The case of the charge exiting the waveguide has also been investigated: the analytical expression of the impedance is basically eq. (5.5), where one should replace $\beta$ with $-\beta$.


## Aknowledgments

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## APPENDIX A - Relevant mathematics

We collect here, for the convenience of the reader, some relevant integrals and definition which are used in the text. The Hankel function of the first kind has the following integral representation (the superscript (1) is omitted everywhere)[6]:

$$\text{(A.1)} \qquad i\pi H_0(vD) = \int_{-\infty}^{+\infty} \left[\frac{e^{i\kappa\sqrt{D^2+z^2}}}{\sqrt{D^2+z^2}}\right] e^{iuz} dz$$

where D > 0 and

$$\text{(A.2)} \qquad v = \sqrt{\kappa^2 - u^2}$$

Inverting (A.1) one gets:

$$\text{(A.3)} \qquad \frac{e^{i\kappa\sqrt{D^2+z^2}}}{\sqrt{D^2+z^2}} = \frac{i}{2}\int_{-\infty}^{+\infty} H_0(vD) e^{-iuz} du$$

In particular, if $D^2 = \sqrt{r^2 + r_0^2 - 2rr_0\cos\vartheta}$, $H_0(vD)$ may be expanded into a sum of Bessel functions[5].

For $r < r_0$:

$$\text{(A.4)} \qquad H_0(vD) = J_0(vr)H_0(vr_0) + 2\sum_{m=1}^{\infty} J_m(vr)H_m(vr_0)\cos m\theta$$

For $r > r_0$, r and $r_0$ should be exchanged in (A.4). Now average (A.3) over :

$$\text{(A.5)} \qquad \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{ik\sqrt{D^2+z^2}}}{\sqrt{D^2+z^2}} d\theta = \frac{i}{2}\int_{-\infty}^{+\infty} e^{-izu} G_u(r, r_0) du$$

where

$$\text{(A.6)} \qquad G_u(r, r_0) = \begin{cases} J_0(vr)H_0(vr_0), & r < r_0 \\ J_0(vr_0)H_0(vr), & r > r_0 \end{cases}$$

For a purely imaginary argument both the Bessel and the Hankel functions are real, and are called modified Bessel functions of the first second kind:

$$\text{(A.7)} \qquad J_0(ix) = I_0(x)$$

$$\text{(A.8)} \qquad i\pi H_0(ix) = 2K_0(x)$$

## APPENDIX B - Calculation of $\Gamma'_+(u)/\Gamma_+(u)$

Let us consider expression (3.1)

$$\text{(B.1)} \qquad L(u) = \pi v J_0(v) H_0(v) = \Gamma_+(u)/\Gamma_-(u)$$

where $v = \sqrt{\varkappa^2 - u^2}$, $Im(v) > 0$. Assuming that k has a small positive imaginary part, the function L(u) is analytic and without zeroes in the strip $-\varepsilon < Im(u) < \varepsilon$, where $\varepsilon < Im(k)$. Hence, ln L(u) is analytic in this strip and consequently may be represented by a Cauchy integral. We will stretch the integration contour as it is shown in Fig. 5:

(B.2)

$$ln\Gamma_+(u) - ln\Gamma_-(u) = \frac{1}{2\pi i}\int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{ln[\pi\sigma J_0(\sigma)H_0(\sigma)]dt}{t-u} - \frac{1}{2\pi i}\int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{ln[\pi\sigma J_0(\sigma)H_0(\sigma)]dt}{t-u}$$

Here $\sigma = \sqrt{\varkappa^2 - t^2}$, $Im(\sigma) > 0$. Since the first integral is regular for $Im(u) > -\varepsilon$, it can be taken for ln $\Gamma_+(u)$. The mentioned analytic property apply also to the logarithmic derivative of $\Gamma_+$, which is the quantity we need in eq. (4.11).
We get:

(B.3)

$$\frac{\Gamma'_+(\kappa/\beta)}{\Gamma_+(\kappa/\beta)} = \frac{-1}{2\pi i}\int_{\eta^+} \frac{t\,dt}{\sigma(t-\kappa/\beta)}\left[\frac{1}{\sigma} + \frac{J_1(\sigma)}{J_0(\sigma)} + \frac{H_1(\sigma)}{H_0(\sigma)}\right]$$

The integrand function is singular at the points $t = \varkappa/\beta$, $t = \pm\varkappa$ and at the zeros $a_n$ of the Bessel function $J_0(\sigma)$.

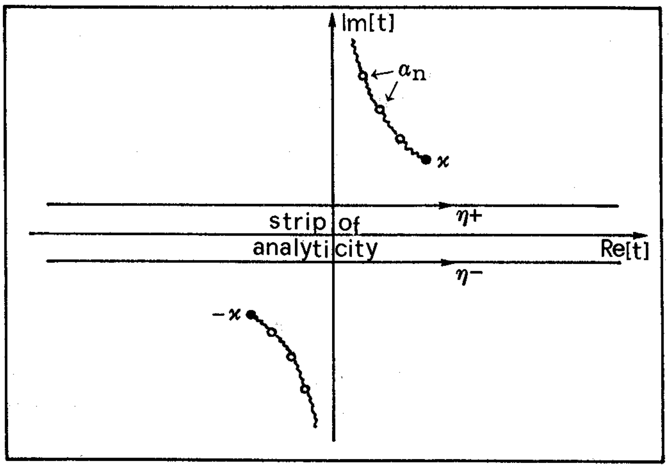


Fig. 5 - Strip of analyticity of L(t) in the complex plane and the integration paths $\eta\pm$.

Due to the square root $\sigma$ the Hankel functions $H_{0,1}(\sigma)$ are multivaluated and branch cuts, starting from $t = \pm\varkappa$, are required to obtain a single-value Riemann plane. The singularities and the cuts are shown in Fig. 5. (For our convenience we choose to cut on the curve where $Im(\sigma) = 0$, and where also $a_n$ lie). The integral (B.3) can be split into two parts: one integral without Bessel functions that we call $I_1$ and a second one with the remaining integrand that we call $I_2$.

The rational function in $I_1$ vanishes as $t^{-2}$ for $|t| \to \infty$ therefore we can add to the integration path a circle of radius $R \to \infty$ without changing the integral value (Jordan Lemma). We can close the path in the upper complex t-plane where the integrand has two singularities at $t = \varkappa$ and $t = \varkappa / \beta$ ; applying the Cauchy theorem we get:

(B.4) $$I_1 = \frac{\beta}{2\kappa(1+\beta)}$$

In order to evaluate $I_2$ we add and subtract the contour "C" shown in Fig. 6 to the original integration path, we get:

(B.5)
$$I_2 = I_{2a} + I_{2b} = \frac{1}{2\pi i}\int_{\eta^+ + C} \frac{t\,dt}{\sigma(t-\kappa/\beta)}\left[\frac{J_1(\sigma)}{J_0(\sigma)} + \frac{H_1(\sigma)}{H_0(\sigma)}\right] - \frac{1}{2\pi i}\int_C \frac{t\,dt}{\sigma(t-\kappa/\beta)}\left[\frac{J_1(\sigma)}{J_0(\sigma)} + \frac{H_1(\sigma)}{H_0(\sigma)}\right]$$

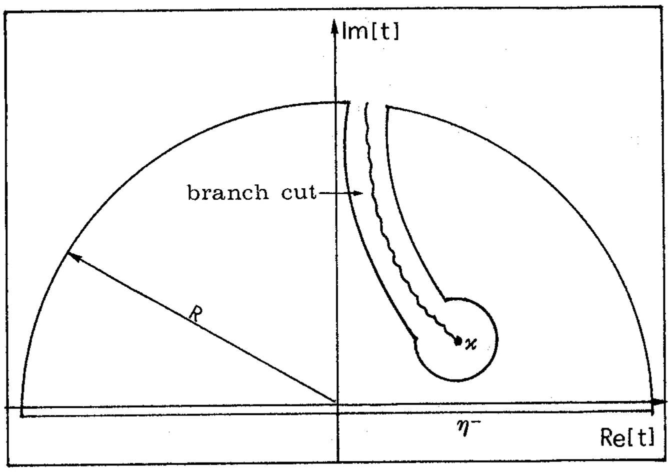


Fig. 6 - The integration contour "C" encompassing the branch cut in the complex t-plane.

The first RHS term can be evaluated again with the residuum theorem yelding:

(B.6) $$I_{2a} = \frac{\kappa}{\beta}\left[\frac{J_1(\sigma)}{\sigma J_0(\sigma)} - \frac{H_1(\sigma)}{\sigma H_0(\sigma)}\right]_{t=\kappa/\beta}$$

For the remaing term $I_{2B}$ one can prove the following results:

1) The integral over the large circle is zero. In fact the "J" and "H" terms give a constant contribution of opposite sign.
2) The integral over the small circle around the point $t = \varkappa$ is zero, (apply again the Jordan Lemma).

Accordingly we will get only the contributions over the two sides of the branch cut. Consider now the "J" and "H" terms separately; the former is a singlevalue function whose singularities lie just on the curve $\mathrm{Im}(\sigma) = 0$ (Fig. 5), therefore the integration over the left and right cut sides gives $2\pi i$ times the residua at the singular points. The "H" term, on the other hand, has no poles but is multivaluated. Making a change of variable $t \to \sigma$ we get the expression:

(B.7) $$I_{2b} = \sum_{n=0}^{\infty} \frac{1}{\alpha_n - \kappa/\beta} + \Delta Re + i\Delta Im$$

where

(B.8)
$$\Delta Re = \frac{\kappa}{\pi\beta}\int_0^\infty \frac{d[argH_0(\sigma)]}{\sigma^2 + (\kappa/\beta\gamma)^2} + \frac{1}{\pi}\int_0^\kappa \frac{\sqrt{\kappa^2-\sigma^2}\; d[argH_0(\sigma)]}{\sigma^2+(\kappa/\beta\gamma)^2}$$

and

(B.9) $$\Delta Im = \frac{1}{\pi}\int_\kappa^\infty \frac{\sqrt{\sigma^2-\kappa^2}\; d[arg\,H_0(\sigma)]}{\sigma^2+(\kappa/\beta\gamma)^2}$$

with

(B.10) $$d[arg\,H_0(\sigma)] = \left[\frac{J_0(\sigma)Y_1(\sigma) - J_1(\sigma)Y_0(\sigma)}{J_0^2(\sigma)+Y_0^2(\sigma)}\right] d\sigma$$

Note that the integrands are regular over the whole integration range while the function $\arg H_0(\sigma)$ is a well behaving function, therefore the quantities (B.8) and (B.9) are easy to compute numerically.

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