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STRONG COUPLING EXPANSION IN THE NONCOMPACT FORMULATION OF YANG–MILLS THEORIES ON A LATTICE

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We study the strong coupling limit of noncompact Yang–Mills theories on a lattice. We find that quarks are superconfined namely single quarks cannot be separated even by one lattice spacing. At large but finite values of the coupling constant superconfinement is relaxed, and the quark–quark wave function decays at large distance not faster than exponentially, a behaviour incompatible with a linear effective potential.

In Wilson's formulation of gauge theories on a lattice [1] the matter fields are defined on the sites while the gauge fields are defined on the links as angular variables. The reason is to make gauge-invariant the product of matter fields at neighbouring sites. This is done according to the point splitting procedure introduced by Schwinger [2], by accompanying such products by an appropriate phase factor

$$\psi^{\dagger}(x)\exp\left(\mathrm{i}\int_{y}^{x}\mathrm{d}x_k A_k\right)\psi(y)\,. \tag{1}$$

In such a way the gauge fields always appear as the arguments of circular functions in the terms describing their interaction with the matter fields. By defining also the pure gauge field lagrangian or hamiltonian by means of circular functions of the gauge fields one obtains a compact formulation of the theory.

As far as gauge-invariance is concerned, however, there is an alternative definition to (1)

$$\psi^{\dagger}(x)\exp\left(\mathrm{i}\int_{y}^{x}\mathrm{d}x_k A_{\mathrm{L}k}\right)\psi(y)\,, \tag{2}$$

where the gauge field A_k has been replaced by its longitudinal part [#1] $A_{\mathrm{L}k}$. If we define the products of matter fields at neighbouring sites in this way, we must add the interaction with the tranverse part of the gauge field, which remains noncompact. Moreover, since A_{L} is a gradient,

$$A_{\mathrm{L}k}=\partial_k\varphi\,,$$

the expression (2) is only apparently nonlocal, since it can be rewritten as

$$\psi^{\dagger}(x)\exp[\mathrm{i}\varphi(x)]\exp[-\mathrm{i}\varphi(y)]\psi(y)\,.$$

The field $\chi=\exp(\mathrm{i}\varphi)\psi$ is gauge invariant, and its use allows us to define all the variables on the sites.

The general formulation of gauge theories on a lattice following the procedure just outlined will be reported somewhere else. In this paper we investigate the strong coupling limit of such a theory, by extending to dynamical quarks the results previously obtained [4] for classical color charges.

We find that in the $g\to\infty$ limit quarks must occur in even number at each site, namely single quarks cannot be separated even by one lattice spacing, which we call superconfinement. At large but finite values of g superconfinement is relaxed, and the quark–quark wave function decays not faster than exponentially at large distance. Note that such a behaviour is incompatible with a linear effective potential.

[#1] I am grateful to Professor Y. Srivastava for calling my attention on the fact that in a previous paper by Schwinger [3] the definition (1.2) is found by solving the equation satisfied by the regularized current in the presence of a gauge field with constant strength. After eq. (3.17) of ref. [3] it is observed that the regularizations (1.1) and (1.2) coincide when the path of integration is a straight line and the field strength is constant, a condition which is not satisfied in the present case.

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The fermion field hamiltonian density and the Gauss constraint read

$$\mathscr{H}_F = i\bar{\psi}\gamma_k(\partial_k - igA_{ka}\tfrac{1}{2}\tau_a)\psi + m\bar{\psi}\psi\,, \tag{3}$$

$$(\partial_k E_{ka} - g\epsilon_{abc}A^c_k E^b_k - gj_{0a})\Psi = 0\,, \tag{4}$$

where

$$j_{0a} = \tfrac{1}{2}\psi^\dagger\tau_a\psi\,, \tag{5}$$

τ being the Pauli matrices. E_{ka} are the momenta canonically conjugate to A_{ka}. We adopt the convention of summation over repeated indices, although in some cases the sum will be explicitly indicated.

We parametrize the fields in such a way as to introduce gauge-invariant and gauge degrees of freedom. For the gauge field we put

$$A_k = u^\dagger\bar{A}_k u + (i/g)u^\dagger\partial_k u\,, \tag{6}$$

or in color components

$$A_{ka} = \bar{A}_{kb}h_{ba} - (1/2g)\epsilon_{abc}h_{db}\partial_k h_{dc}\,. \tag{7}$$

The gauge degrees of freedom are the three angles φ_a which parametrize the orthogonal matrix h_{ba}.

The spinor field ψ is replaced by the gauge-invariant spinor

$$\chi = u\psi\,. \tag{8}$$

We must now express both hamiltonian and Gauss constraint in terms of these variables.

As explained below only the $g\to\infty$ limit of the gauge field hamiltonian is needed for our purposes. This has been derived by Simonov [#2] and will be reported below in discretized form.

The fermionic part of the hamiltonian density in the new variables reads

$$\mathscr{H}_F = i\bar{\chi}\gamma_k(\partial_k - ig\tfrac{1}{2}\tau_a\bar{A}_{ka})\chi + m\bar{\chi}\chi\,. \tag{9}$$

The field χ is gauge-invariant, so that we can define all the variables on the sites, in spite of the presence of products of χ at a site times χ at a neighbouring site.

No difficulty comes from discretization of the Gauss law. Using the angles φ_a the spatial derivatives disappear and it takes the local form [7]

$$[L_a(\varphi) + h_{ba}\chi^\dagger\tfrac{1}{2}\tau_b\chi]\Psi = 0\,, \tag{10}$$

[#2] The polar representation has been introduced for the electric strength in ref. [5] and then for the gauge field in ref. [6].

where L_a are the cartesian components of the orbital angular momentum of a rigid rotator.

We can now perform the discretization in the standard way

$$\varphi_b(\boldsymbol{x})\to\varphi_b(\boldsymbol{\mu}),\quad L_b(\boldsymbol{x})\to a^{-3}L_b(\boldsymbol{\mu}),$$
$$\psi(\boldsymbol{x})\to a^{-3/2}\psi(\boldsymbol{\mu}),\quad \bar{A}_{kb}(\boldsymbol{x})\to a^{-1}\bar{A}_{kb}(\boldsymbol{\mu})\,, \tag{11}$$

where a is the lattice spacing and $\boldsymbol{\mu}$ a vector with integral components.

Let us emphasize that although $\bar{A}_{ka}$ and the gauge angles φ_a have been defined on the sites, the cartesian components A_{ka} do not turn out to be defined on the sites because of the term $\frac{1}{2}g\,\epsilon_{abc}\,h_{db}\partial_k\,h_{dc}$ in eq. (7). This fact has an important consequence which will be discussed later on.

We adopt Wilson's prescription to define the fermion field on the lattice, so that

$$\begin{aligned}\mathscr{H}_F = {} & (1/a)\sum_{\boldsymbol{\mu}} i\bar{\chi}(\boldsymbol{\mu})\sum_h \gamma_h\tfrac{1}{2}[\chi(\boldsymbol{\mu}+\hat{h}) - \chi(\boldsymbol{\mu}-\hat{h})] \\ & + g\bar{\chi}(\boldsymbol{\mu})\tfrac{1}{2}\tau_a\gamma_h\chi(\boldsymbol{\mu})\bar{A}_{ka} + am\bar{\chi}(\boldsymbol{\mu})\chi(\boldsymbol{\mu}) \\ & + \sum_h K\bar{\chi}(\boldsymbol{\mu})[\chi(\boldsymbol{\mu}) - \chi(\boldsymbol{\mu}+\hat{h}) - \chi(\boldsymbol{\mu}-\hat{h})]\,,\end{aligned} \tag{12}$$

where K is the Wilson gap parameter.

Finally the discretized version of the Gauss law reads

$$[L_a(\boldsymbol{\mu}) + h_{ba}\chi^\dagger(\boldsymbol{\mu})\tfrac{1}{2}\tau_b\chi(\boldsymbol{\mu})]\Psi = 0\,. \tag{13}$$

The Gauss operator is the sum of a spin and an orbital angular momentum in color space. The spin takes on integral or half-integral values at each site according to the occurrence of an even or odd number of quarks at that site. As for the orbital angular momentum one would expect naively that it should take on integral values at each site. If this were the case, the Gauss law would require an even number of quarks at each site; and a single quark could not be separated by another one even by a small distance, which we shall call superconfinement.

In order to determine the content of the Gauss law it is necessary to understand the role of the curvilinear coordinates which give the form of an angular momentum to the Gauss operator.

The expression of the cartesian components of the gauge field in terms of gauge angles plus other variables bears some resemblance to the transformation

from cartesian to polar coordinates in quantum mechanics. In order to analyze in detail such an analogy, let us consider the problem of a particle moving in a plane and interacting with a fermion. Let the hamiltonian be

$$H=-\tfrac{1}{2}p^2+\psi^*\psi+V(r,\Sigma_3,L_3)\,, \tag{14}$$

where

$$r=\sqrt{x^2+y^2}\,,\quad L_3=xp_y-yp_x\,,\quad \Sigma_3=\psi^\dagger\tfrac{1}{2}\tau_3\psi\,. \tag{15}$$

The above hamiltonian acts in the configuration space of the particle and in the Fock space of the fermion, ψ, $\psi^\dagger$ being destruction and creation operators respectively.

Let us consider the motion of the particle subject to the constraint

$$(L_3+\Sigma_3)\Psi=0\,, \tag{16}$$

which obviously commutes with the hamiltonian. Introducing polar coordinates H becomes

$$H=-\frac{1}{2}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{2}\frac{L_3^2}{r^2}+V(r,\Sigma_3,L_3)\,. \tag{17}$$

As far as this form of the hamiltonian is concerned, any number of fermions is allowed. We have the additional requirement, however, that the eigenfunctions of the hamiltonian in polar coordinates be also eigenfunctions of the hamiltonian in cartesian coordinates. This requirement rules out the states with an odd number of fermions which, due to the constraint, belong to a half-integral eigenvalue of L_3 and are therefore multivalued.

Here is the limit of the analogy between this quantum mechanical model and the present formulation of gauge theories on a lattice. In the latter case only the hamiltonian in curvilinear coordinates is defined on the sites, and there is no corresponding cartesian form. As a consequence multivalued functions of the gauge angles cannot be ruled out, and are actually required by the structure of the hamiltonian. Suppose in fact that we start with the wave function

$$\Psi=\sum_s(-1)^{(s+1/2)}\psi_s^\dagger(\mu)\psi_{-s}^\dagger(\mu)|0\rangle\,, \tag{18}$$

where s is the third component of color spin.

By applying the hamiltonian to such a state we get states of the type

$$\Psi=\sum_s(-1)^{s+1/2}\times[\psi^\dagger(\mu+\hat{h})u^\dagger(\mu+\hat{h})\gamma_h u(\mu)]_s\psi_{-s}^\dagger(\mu)\,, \tag{19}$$

which are multivalued functions of the gauge angles, because u belongs to the spinorial representation of O(3).

As already said the hamiltonian cannot be defined on the sites in terms of the cartesian components of the gauge fields because these components are not defined on the sites due to the term $(\mathrm{i}/g)u^\dagger\partial_k u$ in eq. (6). In the limit of large g such a term disappears and also the cartesian components A_{ka} turn out to be defined on the sites.

In order to study the large g limit it is convenient to express the nine gauge-invariant components by six independent variables. We chose the polar representation [7] for the gauge field

$$A_{ka}=g^{-1/3}(f_{kn}\lambda_n h_{na}-g^{-2/3}\tfrac{1}{2}\epsilon_{abc}h_{nb}\partial_k h_{nc})\,, \tag{20}$$

so that

$$\bar{A}_{ka}=f_{ka}\lambda_a\,. \tag{21}$$

The six independent variables are three angles θ_i which parametrize the orthogonal matrix f_{kn} and the three λ_n. The factor $g^{-1/3}$ has been introduced for convenience.

The large g limit of the gauge field hamiltonian has been reported in ref. [4].

$$\begin{aligned}H_{G\infty}=\frac{g^{2/3}}{2a}\sum_\mu\Bigg[&-\sum_n\frac{\partial^2}{\partial\lambda_n^2(\mu)}\\&-2\sum_{n>m}\frac{1}{\lambda_n^2(\mu)-\lambda_m^2(\mu)}\left(\lambda_n(\mu)\frac{\partial}{\partial\lambda_n(\mu)}\right.\\&\left.-\lambda_m(\mu)\frac{\partial}{\partial\lambda_m(\mu)}\right)+\tfrac{1}{2}\sum_{m,n}\epsilon_{mnk}^2\frac{1}{[\lambda_n^2(\mu)-\lambda_m^2(\mu)]^2}\\&\times\{[\lambda_n^2(\mu)+\lambda_m^2(\mu)][L_k'^2(\mu)+l_k'^2(\mu)]\\&+4\lambda_m(\mu)\lambda_n(\mu)L_k'(\mu)l_k'(\mu)\}\\&+[\lambda_1^2(\mu)\lambda_2^2(\mu)+\lambda_1^2(\mu)\lambda_3^2(\mu)+\lambda_2^2(\mu)\lambda_3^2(\mu)]\Bigg]\,.\end{aligned} \tag{22}$$

In the above equation

$$L_k'=h_{ka}L_a(\varphi),\quad l_k'=f_{kh}L_h(\theta)\,. \tag{23}$$

For the fermionic part we find

$$H_{F\infty} = (g^{2/3}/a) \sum_{\mu} \bar{\psi}(\mu) u^{\dagger}(\mu) \tfrac{1}{2}\tau_a \times \sum_h \gamma_h u(\mu) \psi(\mu) \bar{A}_{ha} . \quad (24)$$

For large g we can define also the cartesian components of the gauge field on the sites

$$A_{ka}(\mu) = g^{-1/3} f_{kn}(\mu) \lambda_n(\mu) h_{na}(\mu) \quad (25)$$

and we can write the hamiltonian in cartesian variables

$$H_{G\infty} = (g^{2/3}/2a) \sum_{\mu} [-\partial^2/\partial A_{ka}^2(\mu) + \tfrac{1}{2} \epsilon_{abc} \epsilon_{ade} A_h^b(\mu) A_k^c(\mu) A_h^d(\mu) A_k^e(\mu)] , \quad (26)$$

$$H_{F\infty} = (g^{2/3}/a) \sum_{\mu} \bar{\psi}(\mu) \tfrac{1}{2}\tau_a \gamma_h \psi(\mu) A_{ha}(\mu) . \quad (27)$$

The hamiltonian contains only fermionic operators at the same site. It is therefore consistent to require that the eigenfunctions of the hamiltonian in polar coordinates be also eigenfunctions of the hamiltonian in cartesian coordinates. Such eigenfunctions must be single-valued, so that only integral values of L_a are admissible and quarks must occur in even number at each site: In the $g \to \infty$ limit we get superconfinement.

We can study how superconfinement is relaxed at finite but large values of g by performing a strong coupling expansion.

Let us evaluate the probability for two quarks to be at relative distance $r = Na$ at the lowest nonvanishing order. We need to consider only those terms in the hamiltonian which can induce transitions of one quark from one site to a neighbouring one. There are no such terms in the expansion of H_G. H_G contains indeed the fermion operators only in $j_{oa}(\mu)$, which cannot give rise to quark hopping. It is therefore sufficient to know $H_{G\infty}$.

Terms of the necessary structure are only present in H_F, and are of zero order in g. Since H_∞ is of order $g^{2/3}$, the expansion parameter is $g^{-2/3}$, in agreement with eq. (20). The desired terms are

$$H'_F = (1/a) \sum_{\mu} \sum_h \bar{\psi}(\mu) [(\tfrac{1}{2} i \gamma_h - K) \psi(\mu + \hat{h}) - (\tfrac{1}{2} i \gamma_h + K) \psi(\mu - \hat{h})] . \quad (28)$$

Now H_∞ is the sum of hamiltonians defined at single sites

$$H_\infty = g^{2/3} \sum_{\mu} h_\infty(\mu) , \quad (29)$$

where the definition of $h_\infty(\mu)$ follows from eqs. (26) and (27).

Let us denote by $\Phi_m^{(1)}(\mu)$ and $\Phi_n^{(0)}(\mu)$ the eigenfunctions of $h_\infty(\mu)$ with one and zero quarks, m, n being all the necessary quantum numbers (we should actually say with fermion number 1 and 0, because h_∞ does not commute with the number of quarks). Let us denote by $\epsilon_m^{(1)}$ and $\epsilon_n^{(0)}$ the corresponding energies.

At zero order we have an eigenfunction Φ with eigenvalue ϵ decribing two quarks at the same site, let us say the origin, due to superconfinement. Such a state is highly degenerate, so that we should use degenerate perturbation theory. It will be clear from the following that this is not necessary for the lowest nonvanishing order.

Let us take the distance r between the quarks along one of the coordinate axes, so that we can indicate the position of one quark by one single coordinate, and write $\Phi^{(1)}{}_{m_i}(i)$ for its wave function. The first term in the pertubative expansion is therefore

$$g^{-2/3} [\epsilon - \epsilon_{m_0}^{(1)} - \epsilon_{m_1}^{(1)}]^{-1} \times \langle \Phi_{m_1}^{(1)}(1) \Phi_{m_0}^{(1)}(0) | H' | \Phi_0^{(0)}(1) \Phi(0) \rangle , \quad (30)$$

describing the transition of one of the two quarks from the origin to site 1, which was previously in the ground state (vacuum) $\Phi_0^{(0)}(1)$. The generic term in the expansion is

$$g^{-2/3} \left(\epsilon - \epsilon_{m_0}^{(1)} - \epsilon_{m_h}^{(1)} - \sum_{i=1}^{h-1} \epsilon_{n_i}^{(0)} \right)^{-1} \times \langle \Phi_{m_h}^{(1)}(h) \Phi_{n_{h-1}}^{(0)}(h-1) | \times H' | \Phi_0^{(0)}(h) \Phi_{m_{h-1}}^{(1)}(h-1) \rangle , \quad (31)$$

because to lowest order the quark must always jump in the direction of the site N. It has to do N jumps to reach the site. It follows that the probability to find two quarks at distance Na is of order $g^{-4/3N}$, and it is vanishing for $N \to \infty$ if the pertubative expansion converges. This shows that confinement persits at finite but large values of g.

It would be interesting to evaluate the quark–quark wave function in order to find what is the effective potential which can produce it. This would require the evaluation of the Nth order term of the perturbative expansion. We are unable to do it, but we will show that the wave function decays not faster than exponentially at large distance, a behaviour incompatible with a linear effective potential.

To this end let us evaluate the amplitude for one quark to reach the site N by leaving behind itself all the sites in the ground state. Such an amplitude is

$$\Psi = g^{-2/3N} \sum_{m_0 m_1 m_N} (B^{N-1})_{m_N, m_1} C_{m_1 m_0} \times \Phi^{(1)}_{m_0}(0) \Phi^{(1)}_{m_N}(N) \prod_{\mu \neq 0, N\hat{\imath}} \Phi^{(0)}_{0}(\mu) , \quad (32)$$

the matrices B and C being defined by

$$B_{m_h, m_{h-1}} = (\epsilon - \epsilon^{(1)}_{m_0} - \epsilon^{(1)}_{m_h})^{-1} \times \langle \Phi^{(1)}_{m_h}(h) \Phi^{(0)}_{0}(h-1) | \times H' | \Phi^{(0)}_{0}(h) \Phi^{(1)}_{m_{h-1}}(h-1) \rangle ,$$

$$C_{m_1, m_0} = (\epsilon - \epsilon^{(1)}_{m_0} - \epsilon^{(1)}_{m_1})^{-1} \times \langle \Phi^{(1)}_{m_1}(1) \Phi^{(1)}_{m_0}(0) | H' | \Phi^{(0)}_{0}(0) \Phi(0) \rangle . \quad (33)$$

We thus have

$$|\Psi|^2 = g^{-4/3N} \operatorname{Tr} C^{\dagger} (B^{\dagger})^{N-1} B^{N-1} C . \quad (34)$$

Now there exists a real positive number b such that

$$|\Psi|^2 \geqslant (g^{-2/3} b)^{2N} = \exp(-2r/r_0) , \quad (35)$$

where

$$r = Na, \quad r_0 = -a/\ln(g^{-2/3} b) , \quad (36)$$

and r_0 is positive for g large enough. The probability for two quarks to be at distance Na irrespective of the state of sites between them is obviously larger than $|\Psi|^2$, and therefore such a probability decays not faster than exponentially.

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References

[1] K.G. Wilson, Phys. Rev. D 10 (1974) 2455.
J.B. Kogut and L. Susskind, Phys. Rev. D 11 (1975) 395.
[2] J. Schwinger, Phys. Rev. Lett. 3 (1959) 296.
[3] J. Schwinger, Phys. Rev. 82 (1951) 664.
[4] F. Palumbo, Phys. Lett. B 199 (1987) 440; B 201 (1988) 147.
[5] J. Goldstone and R. Jackiw, Phys. Lett. B 74, (1978) 81.
[6] Yu.A. Simonov, Sov. J. Nucl. Phys. 41 (1985) 835.
[7] T.D. Lee, Particle physics and introduction to field theory (Harwood Academic, New York, 1981).