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INFN - Laboratori Nazionali di Frascati, P.O.Box 13, 00044 Frascati, (Italy)

ABSTRACT

The recently discovered symmetries of random systems in the framework of stochastic quantisation may be obtained systematically and much more simply and directly from the Fokker-Planck equations.The associated constants of motion follow, from the same equations, by a standard construction.

Stochastic quantisation[1,2] is a relatively new method of quantisation of field theories. It was first proposed by Nelson[1] in the non-relativistic domain as an alternative method of derivation and interpretation of the Schroedinger equation based entirely on concepts of classical physics. Its underlying physical idea is that a quantum system may be simulated by a classical dynamical system subject to small random perturbations from a large background. This latter is usually referred to as a heat bath (the thermal reservoir) and the random force exerted by it on the system as a noise. The noise induces randomness, or stochastic behaviour, in the dynamical variables of the system. Stochastic quantisation makes use of a correspondence between a dispersive, non-equilibrium system in Minkowski space-time and quantum field theory in Euclidean space-time. The equations of motion for the randomised variables combine two different structures:

(i) A Lagrangian

$$L = \frac{m}{2} g_{ij}(x) \frac{dx^i(t)}{dt} \frac{dx^j(t)}{dt} - V(x) \tag{1}$$

which describes the deterministic motion of the unrandomised variables. The system will be supposed to have N degrees of freedom $x^i(t)$ $(i=1,2,\ldots N)$.

(ii) A probability law which describes the incomplete and uncertain knowledge of the state of the perturbing background and of its effects on the system through a set of time-dependent random variables $\eta_\alpha(t)$ $(\alpha=1,2\ldots N)$. For simplicity it is supposed that there are also N such random variables. These structures are combined in the Langevin equations for the variations $\Delta x^i(t)$ of the fields $x^i(t)$

$$\Delta x^i(t) = a^i_+(x)\,\Delta t + e^i_\alpha(x)\,\Delta\eta_\alpha(t);\ \Delta t > 0 \tag{2a}$$

$$\Delta x^i(t) = a^i_-(x)\,\Delta t + e^i_\alpha(x)\,\Delta\eta_\alpha(t);\ \Delta t < 0 \tag{2b}$$

$a^i_\pm(x)$ are related to the Lagrangian L. They are the so-called forward (+) and backward (–) velocities. $e^i_\alpha(x)$ is a non-singular matrix which couples the system to the background. Eq. (2) describes a stationary stochastic process, a process being simply a random function of time. The stationarity of the process means that $a^i_\pm(x)$ and $e^i_\alpha(x)$ do not depend explicitly on time. The probability law accompanying Eq. (2), specifies the distribution of the random variables $\eta_\alpha(t)$. It is assumed to be such that its variations $\Delta\eta_\alpha(t)$ in Eq. (2) satisfy the equations

$$< \Delta\eta_\alpha(t) >_0 = 0 \tag{3a}$$

$$< \Delta\eta_\alpha(t)\,\Delta\eta_\beta(t') >_0 = \frac{\hbar}{m}\,\delta_{\alpha\beta}\,|t-t'| \tag{3b}$$

where $<\ldots>_0$ stands for an average over the distribution of the $\eta_\alpha(t)$. The parameter $\frac{\hbar}{m}$ is a characteristic constant of the process (its diffusion constant) and is chosen here, for later convenience, to be the ratio of Planck's constant $\hbar$ to the mass m in the Lagrangian L. Eq. (2) is a well known model for the evolution of a non-equilibrium system towards equilibrium. Its distinguishing feature is that the velocities $a^i_\pm(x)$ contain time reversible and irreversible components and consequently dissipative and non-dissipative modes.

Its fluctuating or random terms represent microscopic time reversible dynamics. For a long time, this has been the general understanding of the Langevin equations[3]. Recently, however, Parisi and Sourlas[4] have discovered that the assumptions underlying the Langevin equation invariably imply the existence of symmetries, including supersymmetries, of the system. This is a very remarkable result. This lies not so much in the fact that the symmetry algebra is so large as to include supersymmetries but in the fact that, in formulating eq. (2), the symmetries of the Lagrangian L and/or of the coupling to the random background was never an issue.

In a previous paper[6] we have shown how these symmetries arise and how they may be derived systematically. The arguments in this reference were based mainly on the properties and consequences of time reversal non-invariance of the Langevin equations. Although simple, the method by which one arrives at the symmetries using these arguments are not very familiar. We wish in this paper to give a simpler and more straight-forward derivation based only on the Fokker-Planck equations. These latter are partial differential equations and much more manageable than the Langevin equations. The operators shown in ref. (6) to define the symmetries also define the differential operators of the Fokker-Planck equations.

We shall therefore start with the construction of the differential operators associated with the Langevin equations. Let f(t,x) be a function with continuous derivatives of, at least, first order in t and of, at least, second order in the x^i. The forward (+) and backward (–) derivatives of f(t,x) are defined by means of the equation

$$D_{\pm}f(t,x) = \lim_{\Delta t \to 0_{\pm}} \frac{1}{\Delta t} < f\,(t+\Delta t,\, x\,(t+\Delta t)) - f\,(t+x) >_0 \tag{4}$$

Substituting from eqs. (2) and (3) into (4), one finds the expressions

$$D_{\pm} = \pm \frac{\partial}{\partial t} + \frac{1}{\hbar} H_{\pm} \tag{5a}$$

$$H_{\pm} = \hbar\, a^i_{\pm}(x) \frac{\partial}{\partial x^i} + \frac{\hbar^2}{2m}\, g^{ij}(x) \cdot \frac{\partial^2}{\partial x^i \partial x^j} \tag{5b}$$

where

$$g^{ij}(x) = e^i_{\alpha}(x)\, e^j_{\alpha}(x) \tag{6}$$

with summation over repeated indices to be understood. Eq. (6) establishes a deep connection between the Lagrangian in eq. (1) and the fluctuation term in eq. (2). The inverse of $g^{ij}(x)$ is in fact the kinetic matrix in eq. (1), that is

$$g_{ij}(x) = e_{\alpha i}(x)\, e_{\alpha j}(x) \tag{7}$$

It is well known that from the positivity of the kinetic energy, the matrix $g_{ij}(x)$ may be interpreted as the Riemannian metric of the configuration space of the variables x^i. Eqs. (6) and (7) identify this metric also with the diffusion matrix defined by the fluctuation correlation matrix $e^i{}_\alpha(x)$. This identification allows to covariantise the theory. To this end, one introduces the determinant

$$g(x) = \det (g_{ij}(x)). \tag{8}$$

The transformation

$$H_\pm \to (g(x))^{-1/2} H_\pm (g(x))^{1/2} \tag{9}$$

then brings $H_\pm$ into the covariant forms

$$H_\pm = \hbar (u^i \pm v^i) \nabla_i + \frac{\hbar^2}{2m} \Delta \tag{10}$$

where

$$u^i(x) = \frac{1}{2}\left[a^i_+(x) + a^i_-(x)\right] + \frac{\hbar^2}{2m} g^{jk}(x) \Gamma^i{}_{jk}(x) \tag{11a}$$

$$v^i(x) = \frac{1}{2}\left[a^i_+(x) - a^i_-(x)\right] \tag{11b}$$

are the so-called irreversible (or stochastic) and the reversible (or current) velocities, respectively.

$$\Delta = g^{-1/2} \frac{\partial}{\partial x^i} \left(g^{1/2} g^{ij}(x) \frac{\partial}{\partial x^j}\right) \tag{12}$$

is the Laplace-Beltrami operator, $\Gamma^i{}_{jk}(x)$ the connection of the metric $g_{jk}(x)$ and ∇_i the associated covariant derivative.

Next consider the Hilbert space scalar product

$$< \varphi_1 | \varphi_2 > = \int d^N(x) \sqrt{g(x)}\, \varphi^*_1(x)\, \varphi_2(x) \tag{13}$$

The adjoints of the operators $H_\pm$ with respect to it are the Fokker-Planck operators

$$F_\pm = -\hbar \nabla_i (u^i \pm v^i) \cdot) + \frac{\hbar^2}{2m} \Delta . \tag{14}$$

for forward (+) and backward (–) time evolution.

Let $\rho(t,x)$ be the probability distribution of the x^i at time t. It satisfies the Fokker-Planck equations

$$\pm \hbar \frac{\partial \rho\,(t\,;\,x)}{\partial t} = F_{\pm}\,\rho\,(t,x) \tag{15}$$

Taking the sum and difference of the equations in eq. (15) one gets the equivalent set of equations

$$-\hbar\ \nabla_i\ (u^i\,\rho\,(t,x)\,) + \frac{\hbar^2}{2m}\ \Delta\,\rho\,(t,x) = 0 \tag{16a}$$

$$\frac{\partial \rho\,(t\,;\,x)}{\partial t} + \nabla_i\,(v^i\,\rho\,(t,x)\,) = 0 \tag{16b}$$

The first of these equations has been used up to now rather limitedly as providing a definition for u^i (x), in the form

$$u^i\,(x) = \frac{\hbar^2}{2m}\,(\rho\,(t,x))^{-1}\,\nabla_j\,(g^{ij}\,(x)\,\rho\,(t,x)) = \frac{\hbar^2}{2m}\,(\rho\,(x))^{-1}\,\nabla_j\,(g^{ij}\,(x)\,\rho\,(x)) \tag{17}$$

where ρ (x) is the stationary limit of ρ (t, x). Similarly, one uses the stationary limit of (16b), that is

$$\nabla_i\,(v^i\,\rho\,(x)) = 0 \tag{18}$$

to provide a definition for $v^i(x)$ in the general form

$$v^i(x) = \frac{\hbar^2}{2m}\,(\rho\,(x))^{-1}\,\nabla_j\,(c^{ij}\,\rho\,(x)) = 0 \tag{19}$$

where c_{ij} (x) = - c_{ji} (x) is an arbitrary antisymmetric tensor. Eqs. (17) and (19) constitute the so-called potential conditions[3,7]. They define the velocities $u^i(x)$ and $v^i(x)$, in terms of the symmetric and antisymmetric tensor potentials g_{ij} (x) and c_{ij} (x), respectively. These potentials are not uniquely defined. This is obvious for c_{ij} (x), from the way it was introduced, but less so for $g_{ij}(x)$. Both are, however, subject to gauge transformations:

$$c_{ij}\,(x) \rightarrow \overline{c}_{ij}\,(x) = c_{ij}\,(x) + \frac{\hbar}{2m\,c}\ \ (\rho(x))^{-1}\,\nabla^k\ (D_{ijk}\,(x)\,(\rho\,(x)) \tag{20}$$

$$g_{ij}\,(x) \rightarrow \overline{g}_{ij}\,(x) = g_{ij}\,(x) + h_{ij}\,(x) \tag{21a}$$

where the gauge function D_{ijk} (x) is totally antisymmetric. The constant c is introduced on dimensional grounds only. It has the dimensions of velocity so that D_{ijk} is dimensionless. For $u^i(x)$ to remain invariant, g_{ij} (x) has to be invariant. Consequently the symmetric gauge function h_{ij} (x) must vanish, that is

$$h_{ij}\,(x) = 0 \tag{21b}$$

Eq. (21b) is not at all trivial.
The functions $h_{ij}(x)$ satisfying it are the vanishing Killing forms of the metric $g_{ij}(x)$, that is

$$h_{ij}(x) = \nabla_i w_j + \nabla_j w_i = 0 \tag{22}$$

The vectors $w^i(x)$ are the generators of the infinitesimal transformations

$$x^i \rightarrow \overline{x^i} = x^i + \epsilon\, w^i(x) + O(\epsilon^2) \tag{23}$$

which leave the metric $g_{ij}(x)$ invariant, that is

$$\overline{g}_{ij}(\overline{x}) = g_{ij}(\overline{x}) \tag{24}$$

Eq. (24) is rigorously equivalent to eqs. (21a) and (21b). The invariance of the metric $g_{ij}(x)$ gives rise to the symmetries (i.e. the isometries) associated with the Langevin equation. Why this is an important result is that the stochastic equations of motion imply that this metric cannot be generic.

We wish here to show how this invariance emerges directly from the Fokker-Planck equations, eqs. (16a) and (16b). These equations are thus more than just defining equations, for the velocities $u^i(x)$ and $v^i(x)$.

To this end one introduces in eqs. (16) the new function $\psi(t,x)$ defined by

$$\rho(t,x) = \rho(x)\, \psi(t,x) \tag{25}$$

$\psi(t,x)$ measures the departure from equilibrium or more generally stationarity. Making use of (17) and (18), one finds that it satisfies the equations

$$H\, \psi(t,x) = 0 \tag{26a}$$

$$\hbar\, \frac{\partial \psi(t\,;\,x)}{\partial t} + L\, \psi(t,x) = 0 \tag{26b}$$

where

$$H = \hbar\, u^i(x) \nabla_i + \frac{\hbar^2}{2m} \Delta = (H_+ + H_-) \tag{27a}$$

$$L = \hbar\, v^i(x)\, \nabla_i = \frac{1}{2}(H_+ - H_-) \tag{27b}$$

We seek solutions of (26) by separation of variables: that is, we put

$$\psi(t,x) = q(t)\, \varphi(x) \tag{28}$$

Eqs. (26) then reduce to

$$H\,\varphi(x) = 0 \tag{29a}$$

$$L\varphi(x) = \hbar\,\omega\varphi(x) \tag{29b}$$

$$q(t) = q(o)\,e^{-\omega t} \tag{29c}$$

$\hbar\,\omega$ is the separation constant in eq. (26b). According to eq. (29b) the function $\varphi(x)$ is in general non-trivial; that is, it is not a constant if the energy eigenvalue $\hbar\,\omega \neq 0$. Eqs. (29a) and 29b) therefore imply that $\varphi(x)$ is a simultaneous eigenfunction of the operators H and L. From this would follow that H and L commute. Much more generally, one gets from combining eqs.(29a) and (29b), that

$$[H, L] = -\lambda(x)\,H \tag{30}$$

where $\lambda(x)$ is a c-number function. The case when H and L commute is obtained for $\lambda(x) = 0$. Eq.(30) is the general condition compatible with eqs. (29a) and (29b). An operator L satisfying eq.(30) with respect to a given operator H is said to be a symmetry of the latter [8]. For $\lambda(x) = 0$, L is said to be an exact symmetry of H; for $\lambda(x) \neq 0$, it is inexact.

Eq. (30) is the condition giving rise to symmetries. We shall next show that $\lambda(x)$ is indeed zero and that this condition is the same as expressed by eq. (22). The direct computation of the commutator on the left hand side of (30), from eqs. (27a) and (27b), is slightly cumbersome. One lightens the task by using eqs. (17) and (19) to re-express the operators $H_\pm$ in the compact forms

$$H_\pm = \frac{\hbar^2}{2m}(\rho(x))^{-1}\,\nabla_i\,(\rho(x)\,b_\pm^{ij}(x)\,\nabla_j) \tag{31}$$

where

$$b_-^{ij}(x) = b_+^{ji} = g^{ij}(x) + c^{ij}(x) \tag{32}$$

From eqs. (27a) and (31) one finds

$$H = \frac{\hbar^2}{2m}(\rho(x))^{-1}\,\nabla_i\,(\rho(x)\,g^{ij}(x)\,\nabla_j) \tag{33}$$

From the identity

$$[H, L] = \frac{1}{2}[H_+,\ H_-] \tag{34}$$

and the definition

$$[H_+, H_-] = 2mc^2 H_0 \tag{35}$$

one finds, by substituting (31) and (32) in the left hand side of (35),

$$[H, L] = mc^2 H_0 \tag{36}$$

where

$$H_0 = \frac{\hbar^2}{2m} (\rho(x))^{-1} \nabla_i (\rho(x)\, b_0^{ij}(x)\, \nabla_j) \tag{37}$$

and

$$b_0^{ij}(x) = \frac{\hbar}{mc^2} (\nabla^i v^j(x) + \nabla^j v^i(x)) \tag{38}$$

Eqs. (30) and (36) together yield,

$$H_o = -\frac{\lambda(x)}{mc^2} H \tag{39}$$

Comparing the coefficients of powers of covariant derivatives on both sides of (39) one gets

$$\frac{\hbar}{2m} (\rho(x))^{-1} \nabla_j (b_o^{ij}(x)\rho(x)) = -\lambda(x)\, u^i(x) \tag{40a}$$

$$b_o^{ij}(x) = -\lambda(x)\, g^{ij}(x) \tag{40b}$$

From these and the definition of $u^i(x)$ in eq. (17), it follows that

$$\lambda(x) = -\frac{2\hbar}{Nmc^2} \nabla_i v^i(x) = \text{const.} \tag{41}$$

If the velocity $v^i(x)$ vanishes at infinity, $\lambda(x) = 0$. In this case, from eqs. (40), $b_o^{ij}(x) = 0$. Making use of this in eq. (38) yields the Killing equations

$$\nabla_i v^i(x) + \nabla_j v_i(x) = 0 \tag{42}$$

in agreement with eq. (22). The solutions $v_\alpha^i(x)$ $(\alpha = 1, 2 ..)$ of eq. (42) generate the symmetries of the system through the operators $L_\alpha = \hbar\, v^i{}_\alpha(x) \nabla_i$ with

$$[L_\alpha, H] = 0 \quad ; \quad \alpha = 1, 2, ... \tag{43}$$

There are a total of 1/2N (N + 1) such solutions $v_\alpha^i(x)$ in general for given gij (x). Eq. (42) expresses a coexistence condition between the long term convectional modes described by the current velocity $v^i(x)$ and the microscopic fluctuations represented by the diffusion metric gij (x). The condition is that the convectional motion leaves the configuration of fluctuations invariant in the sense that the $v^i(x)$ are Killing vectors of the fluctuation metric.

It is interesting that the Fokker-Planck equations express this condition rather explicitly by their very structure. To better understand this let us enquire as to the constants of motion associated with these symmetries. One of these is given directly by the stationary current conservation equation. Consider eq. (18); it may be rewritten in the form

$$u^i(x)\, v_i\,(x) = 0 \tag{44}$$

on making use of eq. (41) with $\lambda\,(x) = 0$.

Let $u^i\,(x) = u^i\,(x(t))$ define a flow by means of the equation

$$\frac{dx^i(t)}{dt} = u^i\,(x(t)) \tag{45}$$

Taking the derivative of eq. (44) along $u^i\,(x(t))$ and making use of eq. (42), one finds.

$$\left[\frac{du^i(x)}{dt} + \Gamma^i_{jk} u^i\,(x)\, u^k\,(x)\right] v_i\,(x) = 0 \tag{46}$$

Thus in the subspace of configuration space spanned by the Killing vectors v^i_α ($\alpha = 1, 2....$), the irreversible drift velocity $u^i(x)$ is a geodesic tangent vector and satisfies the geodesic equation

$$\frac{du^i(x)}{dt} + \Gamma^i_{jk}\, u^i(x)\; u^k(x) = 0 \tag{47}$$

This is again a coexistence condition between fluctuations and dissipation: the geodesics of the diffusion metric are generated by the irreversible drift vector $u^i\,(x)$. The generalisation of eq. (44) to all geodesic tangent vectors $u^i_\alpha\,(x)$ ($\alpha = 1, 2....$) and Killing vectors v^i_β ($\beta = 1, 2....$), allows to construct the constants of motion of the system in a standard way [9]:

$$E_{\alpha\beta}(x) := \frac{m}{2} u^i_\alpha(x)\;\, v_{\beta i}\,(x),\;\; \alpha, \beta, = 1.., 2.., ... \tag{48}$$

In fact from

$$\frac{dx^i(t)}{dt} = u^i_\alpha\,(x(t)) \tag{49a}$$

$$\frac{du^i_\alpha(x(t))}{dt} + \Gamma^i_{jk}\, u^j_\alpha\, u^k_\alpha = 0 \tag{49b}$$

and

$$\nabla_i v_{\beta j}(x) + \nabla_j v_{\beta i}(x) = 0 \tag{50}$$

one finds that

$$\frac{dE_{\alpha\beta}(x)}{dt} = 0 \tag{51}$$

The $E_{\alpha\beta}(x)$ are thus indeed constants of motion. There may be many more such constants of motion. The $E_{\alpha\beta}$ (x) are the ones which follow directly from the Fokker - Planck equations.

We conclude therefore that:

(i) The symmetries discovered recently in the framework of stochastic quantisation may be obtained in a systematic manner from the basic assumptions of the Langevin equations[6]. The Fokker - Planck equations however provide, the simplest and most straight forward way of doing this as well as of constructing the associated constants of motion.

(ii) These symmetries arise as coexistence conditions in the interplay of fluctuations and dissipation in the dynamics. They express local fluctuation-dissipation correlations.

(iii) The deep connection between these symmetries and quantum theory is that the operator H with respect to which they are defined (eq. (30)) is indeed related to the Schroedinger Hamiltonian. In fact the transformation

$$H_S = -(\rho(x))^{-1/2}\, H\, (\rho(x))^{1/2} \tag{52}$$

allows to pass from H to the Schroedinger Hamiltonian H_S

$$H_S = -\frac{\hbar^2}{2m}\Delta + V_o(x) \tag{53}$$

where

$$V_o(x) = \frac{\hbar}{2}\nabla_i u^i(x) + \frac{m}{2} u^i(x) u_i(x) = \frac{\hbar^2}{2m}\frac{\Delta(\sqrt{\rho(x)})}{\sqrt{\rho(x)}} \tag{54}$$

is the so-called quantum potential. Eq. (54) shows that if $V_o(x)$ where given, one may solve for $\sqrt{\rho(x)}$ as the ground state wave function of the Hamiltonian H_S. Eqs. (53) and (54) achieve the derivation of the Schroldinger equation starting from the Langevin equations in the case in which the reversible velocity v^i (x) is zero.

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