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ABSTRACT

The dynamics of the Langevin equation incorporates a set of symmetry constraints related to the existence of two very different time evolutions with different characteristic time scales. The symmetries are generated by the Killing vectors of the diffusion metric and express local fluctuation - dissipation correlations.

In the framework of stochastic quantisation[1,2], the existence of supersymmetries[3] seems to be automatically implied by just the Langevin equation. This is a remarkable result. It is also very surprising. It is not easy to understand why this must be so. For one, a direct connection between the Langevin equation and the symmetry generators has not been established and no indication that perhaps such a connection exists. For another, one ordinarily expects symmetries to emerge as a consequence of some constraints. The question therefore arises: besides its dynamics, does the Langevin equation incorporate a set of symmetry constraints?

It is proposed in this paper to answer this question in the affermative, to indicate the form of the constraints and to qualify them as expressions of local fluctuation - dissipation correlations. The essential idea is that the physical assumptions underlying the separation of the macroscopic drift modes from the microscopic fluctuations in the Langevin equation, implies a very weak or no coupling at all between their dynamics. This suggests that, effectively, the macroscopic drift motion is invariant to perturbations induced by the microscopic fluctuations. This is an old idea. It is familiar, for instance, from the ensemble theory of equilibrium statistical mechanics. In the dynamical framework of the Langevin equation, it assumes peculiar significance, because of the existence in the theory of two very different time evolutions, irreversible and reversible, with different characteristic time scales. It will, in fact, be shown that it is precisely this invariance principle which gives rise to the symmetries which have so far been discovered in the framework of stochastic quantisation(3,4). It can be formulated rigorously.

To this end one observes that the dynamics of the drift and fluctuation modes are different especially in respect of their time reversal properties. The one evolves irreversibly, the other reversibly. There are also differences in the characteristic time scales. The one is on a macroscopic time scale, the other microscopic. The corresponding generators are uniquely defined from the Langevin equation. They are, respectively, Hermitian and anti-Hermitian with respect to the Hilbert space scalar product defined by the normalisation of the stationary probability distribution(4). Here then is the crucial condition: the generators of these two time evolutions commute. For a system of N commuting degrees of freedom $x^i(t)$ $(i=1,2, \ldots N)$, the resulting symmetries are the familiar geometrical ones, such as translations and rotations, generated by Killing vectors(4).

To see, this start with the Langevin equations

$$\Delta x^i(t) = a_+^i(x)\,\Delta t + e_\alpha^i(x)\,\Delta\eta_\alpha(t) \;\; ; \Delta t > 0 \qquad (1a)$$

$$\Delta x^i(t) = a_-^i(x)\,\Delta t + e_\alpha^i(x)\,\Delta\eta_\alpha(t) \;\; ; \Delta t < 0 \qquad (2a)$$

for a stationary stochastic process modeling a system of N degrees of freedom $x^i(t)$ $(i=1,2, \ldots N)$. The $a_\pm^i(x)$ are the forward (+) and backward (–) drift velocities. $e_\alpha^i(x)$ is a non-singular correlation matrix (with inverse $e_{\alpha i}(x)$) coupled to the fluctuating variables $\Delta\eta_\alpha(t)$ $(\alpha=1,2,\ldots)$. Stationarity of the process means that $a_\pm^i(x)$ and $e_\alpha^i(x)$ do not depend explicitly on time. The random variables $\Delta\eta_\alpha(t)$ are assumed to be Gaussian white noise, that is they satisfy

$$< \Delta\eta_\alpha(t) >_o = 0 \qquad (2a)$$

$$< \Delta\eta_\alpha(t)\;\Delta\eta_\beta(t') >_o = \hbar/m\;\delta_{\alpha\beta}\;|t-t'| \qquad (2b)$$

where $<\cdots>_o$ stands for an average over the distribution of the $\Delta\eta_\alpha(t)$. The ratio $\hbar/m$, where $\hbar$ is Planck's constant and m a mass parameter, is the diffusion constant of the process.

Consider the space of differentiable functions f(t,x), with continuous derivatives of, at least, first order in t and of, at least, second order in the x^i. For such functions, one defines the forward (+) and backward (–) derivatives by means of the averages

$$D_\pm f(t,x) = \lim_{\Delta t \to 0_\pm} (1/\Delta t) < f(t + \Delta t, x(t + \Delta t)) - f(t,x) >_o \tag{3}$$

From eqs. (1) - (3), one obtains the following explicit expressions for the operators $D_\pm$,

$$D_\pm = \pm \frac{\partial}{\partial t} + \frac{1}{\hbar} H_\pm \tag{4.a}$$

$$H_\pm = \hbar\, a^i_\pm \frac{\partial}{\partial x^i} + \frac{\hbar^2}{2m} g^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} \tag{4.b}$$

where

$$g^{ij}(x) = e_\alpha{}^i(x)\, e_\alpha{}^j(x) \tag{5.a}$$

with a summation over repeated indices to be understood. The inverse of $g^{ij}(x)$, that is

$$g_{ij}(x) = e_{\alpha i}(x)\, e_{\alpha j}(x) \tag{5b}$$

is the diffusion metric. Very simply, the relationship between the Langevin and the Fokker-Planck equations is that the operators $H_\pm$ are the Lagrange adjoints of the Fokker-Planck operators

$$F_\pm = -\hbar \frac{\partial}{\partial x^i} (a^i_\pm(x) \cdot) + \frac{\hbar^2}{2m} (\frac{\partial^2}{\partial x^i \partial x^j}) (g_{ij}(x) \cdot) \tag{6}$$

The operators $H_\pm$ are thus more directly related to the Langevin equation. It is convenient to continue to work mostly with them. Furthermore it is also useful to adopt a covariant description. This is achieved by introducing the determinant

$$g(x) = \det (g_{ij}(x)) \tag{7}$$

of the diffusion matrix $g_{ij}(x)$ and to consider the latter as the Riemannian metric of the configuration space of the variables x^i. After this, one operates the transformation

$$H_\pm \to (g(x))^{-1/2} H_\pm (g(x))^{1/2} \tag{8}$$

and brings the $H_\pm$ into the covariant form

$$H_{\pm} = -\hbar \ (u^i \pm v^i)\ \nabla_i + (\hbar^2 / 2m)\ \Delta \tag{9}$$

where

$$u^i(x) = 1/2\ (a^i_+(x)) + (a^i_-(x)) + (\hbar^2/2m)\ \ g^{jk}(x)\Gamma^i_{jk}(x) \tag{10a}$$
$$v^i(x) = 1/2\ (a^i_+(x)) - (a^i_-(x)) \tag{10b}$$

are the so-called irreversible (or stochastic) and the reversible (or current) velocities, respectively.

$$\Delta = g^{-1/2}\ (\frac{\partial}{\partial x^i})\ \ g^{1/2}\ g^{ij}(x)\ (\frac{\partial}{\partial x^j}) \tag{11}$$

is the Laplace - Beltrami operator, $\Gamma^i_{jk}(x)$ the connection of the metric $g_{ij}(x)$ and ∇_i the associated covariant derivative. The irreversible and reversible velocities may also be expressed more compactly in terms of the stationary solution of the covariant Fokker-Planck equation

$$\pm\ \hbar\ \ (\partial\rho(t,x))\ /\ \partial t = F_{\pm}\ \ \rho(t,x) \tag{12a}$$
$$F_{\pm}\ \ = -\hbar\nabla_i\ \ ((u^i \pm v^i))\ \ + (\hbar^2/2m)\ \Delta \tag{12b}$$

$\rho(t,x)$ is the probability distribution of the variables x^i at time t; $\rho(x)$ is the corresponding stationary distribution. From eqs. (12) one gets, in fact, the so-called potential conditions[5]

$$u^i(x) = \ (\hbar/2m)\ \ \rho^{-1}(x)\ \nabla_j\ (g^{ij}(x)\ \rho(x)) \tag{13a}$$
$$v^i(x) = \ (\hbar/2m)\ \ \rho^{-1}(x)\ \nabla_j\ (c^{ij}(x)\ \rho(x)) \tag{13a}$$

which express the velocities $u^i(x)$ and $v^i(x)$ in terms of the symmetric and antisymmetric potentials $g_{ij}(x)$ and $c_{ij}(x)$, respectively. $c_{ij}(x) = -c_{ij}(x)$ is an arbitrary antisymmetric tensor and arises from the general solution of the stationary current conservation equation

$$\nabla_i\ \ (\rho(x)\ v^i(x)) = 0 \tag{14}$$

Substituting from (13) into (9), one gets the even more compact expressions for $H_{\pm}$,

$$H_{\pm} = \ (\hbar^2/2m)\ \ \rho^{-1}(x)\ \nabla_i\ (\rho(x)\ b_{\pm}{}^{ij}(x)\nabla_j\ ) \tag{15}$$

where the tensors $b_{\pm}{}^{ij}(x)$ are defined by

$$b_-{}^{ij}(x) = b_+{}^{ji}(x) = g^{ij}(x) + c^{ij}(x) \tag{16}$$

Now with the linear combinations

$$H = 1/2\,(H_+ + H_-) = \hbar u^i(x)\,\nabla_i + (\hbar^2/2m)\,\Delta \qquad (17a)$$
$$L = 1/2\,(H_+ - H_-) = \hbar\; v^i(x)\,\nabla_i \qquad (17b)$$

one decomposes $H_\pm$ into their Hermitian and anti - Hermitian parts relative to the scalar product

$$< \varphi_1 \mid \varphi_2 = \int d^Nx \sqrt{g(x)}\,\rho(x)\,\varphi^*{}_1(x)\,\varphi_2(x) \qquad (18)$$

The decomposition is really with respect to the behaviour of these parts under time inversion $t \to - t$. On real c-number functions, time inversion coincides with the time reversal operator T. The latter is, as usual, anti-unitary and defined in terms of the combined action of the time inversion $t \to -t$ and Hermitian conjugation. Thus one gets

$$T\,g_{ij}(x)\,T^{-1} = g_{ji}(x) = g_{ij}(x) \qquad (19a)$$
$$T\,c_{ij}(x)\,T^{-1} = c_{ji}(x) = -\,c_{ij}(x) \qquad (19b)$$

whence

$$T\,u^i\,(x)\,T^{-1} = u^i\,(x) \qquad (20a)$$
$$T\,v^i\,(x)\,T^{-1} = -\,v^i\,(x) \qquad (20b)$$

and

$$T\,H_\pm\,T^{-1} = H_\pm \qquad (21)$$

The time translation generators H and L describe therefore irreversible and reversible dynamics.

Our basic assumption is that because of this very different dynamical properties, these two operators invariably commute. We argue that this is compatible with and is basically a consequence of the physical assumptions underlying the Langevin equation. The vanishing of this commutator is the symmetry constraint. A more general condition, expressing essentially the same idea, reads as follows: the operator L, considered as a perturbation of the Hamiltonian H, is a symmetry of the latter. In formulae,

$$[\,L, H\,] = \lambda\,(x)\,H \qquad (22)$$

with λ (x) a c-number function. The special case, when L and H commute, is obtained with $\lambda(x) = 0$.

Substituting from eqs. (15) - (17) into (22), one gets, with

$$[L, H] = -1/2\, [H_-, H_+] = -mc^2 H_0 \tag{23a}$$
$$H_0 = (\hbar^2/2m)\ \rho^{-1}(x)\, \nabla_i\ (b_o{}^{ij}(x)\, \rho(x) \nabla_j) \tag{23b}$$
$$b_o{}^{ij}(x) = (\hbar/mc^2)\, (\nabla^i v^j(x) + \nabla^j v^i(x)) \tag{23c}$$

the constraint equation

$$H_0 = (\lambda(x) / mc^2)\, H \tag{24}$$

The parameter c has the dimensions of velocity so that the operator H_0 has the dimensions of energy. Substituting in (24) for H and H_0 from (17.a), (23.b) and (23.c) and comparing the coefficients of the powers of the covariant derivatives, one gets

$$(\hbar/2m)\ \rho^{-1}(x) \nabla_j\, (b_o{}^{ij}\, \rho(x)) = -\lambda(x)\, u^i\, (x) \tag{25a}$$
$$b_o{}^{ij} = -\lambda(x)\, g^{ij}\, (x) \tag{25b}$$

From these and from the definition of the vector u^i (x) (cf. eq. (13.a)), it follows that

$$\lambda(x) = -(2\hbar/mc^2 N)\, \nabla_i\, v^i\, (x) = \text{const.} \tag{26}$$

The constant is therefore zero if the current velocity v^i (x) vanishes at infinity*. In this case the operator H_0 is identically zero. The symmetry constraint, eq. (22), reduces then, on account of eq. (23), to the Killing equations

$$\nabla_i\, v_j\, (x) + \nabla_j\, v_i\, (x) = 0 \tag{27}$$

If there are no additional constraints in the theory, (27) admits a maximum number $\bar{N} = 1/2\, N(N+1)$ of solutions $v^i_\alpha(x)$ ($\alpha = 1,2,\ldots \bar{N}$). $\bar{N}$ is the total number of translation and rotation generators in a space of dimension N. These generators are defined by the first order differential operators

* Integrate both sides of (26) over all space to deduce the result.

$$L_\alpha = v^i{}_\alpha(x)\, \nabla_i \quad ; \quad (\alpha = 1,2,\ldots \bar{N}) \tag{28}$$

or linear combinations thereof.

The $v^i{}_\alpha$ (x), solutions of eq. (27), are the so-called Killing vectors of the diffusion metric $g_{ij}(x)$. In particular, the operator L is, obviously, a particular linear combination of the L_α. It therefore follows that the reversible velocity v^i (x) of a stationary stochastic process is a Killing vector of the diffusion metric. This is a new kind of fluctuation - dissipation correlation. It is local in nature. From the above derivation of eq. (27), one concludes that the Langevin equation incorporates these symmetry constraints in its dynamics. We recall that (27) was obtained in ref. (4) from the assumption of microscopic reversibility. By this is meant that the time translation generators $H_\pm$ are to microscopic reversibility what self-adjoint Hamiltonians are to time reversal invariance. Consequently they have to be normal operators, that is commute, such operators being the closest possible to self-adjoint operators in their functional properties. Clearly this is a rather technical assumption which is not readily suggested by the Langevin equation. On the other hand, the assumption adopted in this paper is a more easily accessible implication of the Langevin equation. In either case, one argues that the origin of symmetries in the framework of stochastic quantisation is to be traced to the existence of two different time evolutions with different characteristic time scales. The one is macroscopic and irreversible, the other microscopic and reversible.

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