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STOCHASTIC QUANTISATION AND KILLING SYMMETRIES OF RANDOM SYSTEMS

Etim Etim
INFN - Laboratori Nazionali di Frascati, P.O.Box 13 - 00044 Frascati, Italy
and
Lothar Schülke*
Fachbereich Physik, Universität - Gesamthochschule Siegen, Siegen, Germany

ABSTRACT

Recent results about the supersymmetries of random systems governed by stochastic differential equations have classical analogues in terms of geometrical symmetries in the configuration space of commuting variables. The related results are: (1) the reversible drift velocity of a stationary stochastic process is a Killing vector of the diffusion metric; (2) the irreversible drift velocity is a geodesic tangent vector; (3) the stationary current conservation equation is a special case of the general result that scalar products of Killing vectors with geodesic tangent vectors are constants of motion. The physical postulate is that microscopic reversibility is to

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the normality of time translation generators what time reversal invariance is to the Hermiticity of Hamiltonians.

A new approach to quantum theory is rapidly taking hold. It is the so-called method of stochastic quantisation[1,2]. The physical principles underlying it, randomised deterministic equations of motion and conservation of probalility, are rooted in classical physics. The underlying picture of the dynamics of random systems is that in a field of force these systems react by redistributing the velocities and position coordinates of their constituents with two different time scales. There is a microscopically short time or random response characterised by a so-called stochastic velocity and a macroscopically long time response characterised by a convection or (drift) current velocity. The formulation of quantum theory based on these ideas is due to Nelson[1]. Parisi and Wu[2] have a variant.

The same picture of stochastic and current velocity response dynamics forms the basis of the well known classical descriptions of random systems in terms of the Langevin and Fokker - Planck equations[3]. They are there identified as irreversible and reversible drift velocities, respectively, with a view to emphasizing the time reversal non-invariance of the macroscopic evolution of these systems. But compared to the Schrödinger equation, the Fokker - Planck equation is relatively very sterile. And for understandable reasons: it expresses only one of the basic physical principles - the conservation of probability. The Fokker - Planck equation is exactly equivalent to a pair of defining equations for the irreversible and reversible velocities. These are the so-called potential conditions[3] which define these velocities in terms of symmetric and antisymmetric tensor potentials, respectively[*]. The dynamics governing their motion must still be specified. Nelson[1] gives a definite prescription which leads to

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(*) see later

the Schrödinger equation.

Parisi and Sourlas[(4)] have made it easier to see that Nelson's approach cannot be the most general theory of random systems. They showed that systems described by stochastic differential equations are associated with supersymmetries. de Alfaro et al[(5)] have supplied the near converse of this result: a system described by a supersymmetric Lagrangian satisfies a set of stochastic identities characteristic of a random Gaussian process. These results do not come totally as a surprise. The surprise, if any, resides in the fact that the symmetry involved is so large (i.e. supersymmetry). Almost by definition, a random system, as a disordered assembly, should possess more symmetries than an ordered one. The importance of the results of Parisi and Sourlas and of de Alfaro et al lies therefore in the fact that these symmetries may actually be discovered within the framework of stochastic quantisation. They are really finding the converses of Noether's theorems for which there exists a standard procedure in classical mechanics[(6)].

The purpose of this letter is therefore to present this standard procedure as it applies to random systems modelled by the Fokker - Planck equation, and in as simple a way as possible. The method is based on the concepts of Killing forms and isometries of Riemannian metrics. In the configuration space of commuting variables, the classical symmetries involved correspond to translations and rotations. More general classical symmetries are also accessible through affine, projective and conformal Killing forms. Mathematical details will however be avoided.

We shall first give a brief review of stochastic quantisation in a flat space with a diffusion matrix. The theory is then covariantised in a curved background with the diffusion matrix as the Riemannian metric. The configuration space is that of a system with N degrees of freedom x^i $(i = 1,2,...N)$.

As usual one starts with the Langevin equations

$$\Delta x^i(t) = a^i_+(t,x)\Delta t + e^i_\alpha(x)\Delta\eta_\alpha(t) \quad ; \quad \Delta t>0 \tag{1.a}$$

$$\Delta x^i(t) = a^i_-(t,x)\Delta t + e^i_\alpha(x)\Delta\eta_\alpha(t) \quad ; \quad \Delta t>0 \tag{1.b}$$

where $a^i_\pm(t,x)$ are the forward (+) and backward (-) velocities, $e^i_\alpha(x)$ a local, non-singular time independent fluctuation correlation matrix coupled to a Gaussian white noise $\Delta\eta_\alpha(t)$. The $\Delta\eta_\alpha(t)$ therefore satisfy

$$<\Delta\eta_\alpha(t)>_0 = 0 \tag{2.a}$$

$$<\Delta\eta_\alpha(t)\Delta\eta_\beta(t')>_0 = \frac{\hbar}{m}\,\delta_{\alpha\beta}\,|t-t'| \tag{2.b}$$

where $<\cdots>_0$ stands for an average over their distribution and $\hbar/m$ is the diffusion constant, $\hbar$ is Planck's constant and m a mass parameter. For a function f(t,x), continuous and differentiable to first order in t and up to second order in x^i, one defines its forward and backward derivatives by[(1)(*)]

$$D_\pm f(t,x) := \lim_{\Delta t \to 0_\pm} \frac{1}{\Delta t} < f(t+\Delta t, x(t+\Delta t)) - f(t,x)>_0 \tag{3}$$

Making use of eqs (1)-(3) one finds for $D_\pm$ the expressions

$$D_\pm = \pm\frac{\partial}{\partial t} + \frac{1}{\hbar}H_\pm \tag{4}$$

where

$$H_\pm = \hbar\, a^i_\pm(t,x)\frac{\partial}{\partial x^i} + \frac{\hbar^2}{2m}\, g^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} \tag{5}$$

and

$$g^{ij}(x) = e^i_\alpha(x)\, e^j_\alpha(x) \tag{6}$$

(*) The symbol: = stands for a definition or the introduction of a new variable or symbol is the inverse of the diffusion matrix $g_{ij}(x)$. Note that the operators $H_\pm$ are the Lagrange adjoints of the Fokker-Planck operators

$$F_\pm = -\hbar \frac{\partial}{\partial x^i}(a^i_\pm(t,x)\cdot) + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^i \partial x^j}(g^{ij}(x)\cdot) \tag{7}$$

Letting $f(t,x) = x^i(t)$ in eq (3), one finds from eqs (4) and (5)

$$D_\pm x^i(t) = a^i_\pm(t,x) \tag{8}$$

whence the meaning of $a^i_\pm(t,x)$ as the average forward and backward velocities. Taking a cue from here Nelson[1] proposes to specify the dynamics of a random system under the influence of a force $f^i(t,x)$ by means of Newton's equation of motion[(*)]

$$\frac{1}{2}(D_+D_- + D_+D_-)x^i(t) = \frac{1}{2}\{D_+,D_-\}\, x^i(t) = \frac{-1}{m} f^i(t,x) \tag{9}$$

in which the anticommutator $\{D_+,D_-\}$ of D_+ and D_- defines the average acceleration. This is the model which, with further simplifying assumptions, yields the Schrödinger equation[1]. Eq (9) and the Fokker-Panck equations

$$\pm \hbar \frac{\partial \rho(t,x)}{\partial t} = F_\pm\, \rho(t,x) \tag{10}$$

for the probability distribution $\rho(t,x)$ of the x^i are the two basic inputs of this theory. Parisi and Wu[2] on the other hand, first define $a^i_\pm(t,x)$ in terms of a Lagrangian and then solve the resulting stochastic differential equations (i.e. eqs (1)) by the method of Green's functions.

The anticommutator $\{D_+,D_-\}$ is, unfortunately, an unwieldy fourth order differential operator in the x^i. Except for the similarity of its action on $x^i(t)$ to an average acceleration, it does not readily recommend itself for anything else which is simple. The commutator

$$[D_-,D_+] = \frac{1}{\hbar}[\frac{\partial}{\partial t}, H_- + H_+] + \frac{1}{\hbar^2}[H_-, H_+] := \frac{1}{\hbar}[\frac{\partial}{\partial t}, H_- + H_+] + \frac{2mc^2}{\hbar^2} H_o \tag{11}$$

(*) In a Euclidean space-time

is a different thing. It plays a much more natural and important role in the theory. The parameter c in eq (11) has the dimensions of velocity so that the operator H_0 there defined has the dimensions of energy. The commutator of D_+ and D_- contains the information about correlations between forward and backward motions, in other words, about the properties of symmetry under time reversal. $D_\pm$ are, basically, generators of time translations. If one analyses fluctuations in terms of their corresponding evolution operators one necessarily encounters their commutator through the Baker-Campbell-Hausdorff formula.

The commutator of D_+ and D_- is a simple second order differential operator. In fact, H_0 and $H_\pm$ have the structure of generalised Laplace-Beltrami operators.

Here then is our main idea: In a state of equilibrium, the forward and backward time translation generators H_+ and H_- are normal operators, that is, they commute with their Hilbert space adjoints. With this property, they are then the closest possible to self-adjoint Hamiltonians describing time reversal invariant dynamics. The symmetries of the system, we maintain, follow therefrom. The point is that there is a metric in Hilbert space, dependent on the equilibrium probability distribution $\rho(x)$ (*), with respect to which the $H_\pm$ are adjoints of each other.

From eq (11), their normality forces H_0 to vanish. The condition

$$H_0 = 0 \tag{12}$$

is what determines these symmetries. It is not at all trivial. It is also a criterion for equilibrium.

Its physical basis is microscopic reversibility. The physical postulate then is that these normal operators are to microscopic reversibility what self-adjoint Hamiltonians are to time reversal invariance.

We now pass on to put into formulae what has been described so far.

(*) $\rho(x)$ is the stationary solution of the Fokker-Planck equation, eq (10).

Let

$$g(x) := \det (g_{ij}(x)) \neq 0 \tag{13}$$

be the determinant of the diffusion matrix. The Fokker-Planck equation is covariantised by means of the transformations

$$\rho(t,x) \longrightarrow (g(x))^{-\frac{1}{2}} \quad \rho(t,x) \tag{14.a}$$

$$F_{\pm} \longrightarrow (g(x))^{-\frac{1}{2}} \quad F_{\pm} \ (g(x))^{+\frac{1}{2}} \tag{14.b}$$

We continue to use the same symbols $F_\pm$ for the transformed operators and $\rho(t,x)$ for the transformed density. Substituting from (7) into (14.b) one gets, for the transformed operators

$$F_{\pm} = - \hbar\nabla_{i}((u^{i} \pm v^{i}).) + \frac{\hbar^2}{2m} \Delta \tag{15}$$

where

$$u^{i}(t,x) = \frac{1}{2} (a_{+}^{i}(t,x) + a_{-}^{i}(t,x)) + \frac{\hbar}{2m} g^{jk}(x)\Gamma^{i}_{jk}(x) \tag{16.a}$$

$$v^{i}(t,x) = \frac{1}{2} (a_{+}^{i}(t,x) - a_{-}^{i}(t,x)) \tag{16.b}$$

are, respectively, the stochastic (or irreversible) and current (or reversible) velocities.

$$\Delta := g^{-\frac{1}{2}} \frac{\partial}{\partial x^{i}} (g^{\frac{1}{2}} g^{ij}(x) \frac{\partial}{\partial x^{j}}) \tag{17}$$

is the Laplace-Beltrami operator, $\Gamma^{i}_{jk}(x)$ the connection of the metric $g_{ij}(x)$ and ∇_i the associated covariant derivative. The Fokker-Planck equations now become

$$\hbar \frac{\partial \rho(t,x)}{\partial t} = - \hbar \nabla_i ((u^i + v^i) \rho(t,x)) + \frac{\hbar^2}{2m} \Delta\rho(t,x) \qquad (18.a)$$

$$- \hbar \frac{(t,x)}{\partial t} = - \hbar \nabla_i ((u^i - v^i) \rho(t,x)) + \frac{\hbar^2}{2m} \Delta\rho(t,x) \qquad (18.b)$$

Summing and subtracting (18.a) and (18.b) one gets a definition of the stochastic velocity

$$u^i(t,x) = \frac{\hbar}{2m} (\rho(t,x))^{-1} \nabla_j (g^{ij}(x) \rho(t,x)) \qquad (19.a)$$

in terms of the probability density $\rho(t,x)$ plus the current conservation equation

$$\frac{\partial \rho(t,x)}{\partial t} + \nabla_i (\rho(t,x) v^i(t,x)) = 0 \qquad (19.b)$$

Let $\rho(x)$ be the stationary solution of eqs (18) or equivalently of eqs (19).

In terms of the associated stationary velocity fields $u^i(x)$ and $v^i(x)$, eqs (19) reduce to the pair of defining equations

$$u^i(x) = \frac{\hbar}{2m} (\rho(x))^{-1} \nabla_j (g^{ij}(x) \rho(x)) \qquad (20.a)$$

$$v^i(x) := \frac{\hbar}{2m} (\rho(x))^{-1} \nabla_j (c^{ij}(x) \rho(x)) \qquad (20.b)$$

where $c^{ij}(x) = -c^{ji}(x)$ is an arbitrary antisymmetric tensor.

Eq (20.b) is the general solution of the stationary current conservation equation

$$\nabla_i (\rho(x) v^i(x)) = 0 \qquad (21)$$

Eqs (20) form the so-called potential conditions[(3)]. They define $u^i(x)$ and $v^i(x)$ in terms of symmetric and antisymmetric tensor potentials, respectively. Note that the antisymmetric tensor potential is not uniquely defined. It is subject to gauge transformations

$$c_{ij}(x) \longrightarrow \bar{c}_{ij}(x) = c_{ij}(x) + \frac{\hbar}{mc} (\rho(x))^{-1} \nabla^{k} (D_{ijk}(x)\rho(x)) \qquad (22)$$

where the gauge functions $D_{ijk}(x)$ are totally antisymmetric tensor fields. In the configuration space approach adopted here, with $g_{ij}(x)$ the Riemannian metric of the manifold, the symmetric tensor potential too is subject to the seemingly but really not trivial gauge transformations

$$g_{ij}(x) \longrightarrow \bar{g}_{ij}(x) = g_{ij}(x) + h_{ij}(x) \qquad (23.a)$$

$$h_{ij}(x) = 0 \qquad (23.b)$$

The gauge functions $h_{ij}(x)$ satisfying (23.b) are vanishing killing forms

$$h_{ij}(x) = \nabla_i w_j(x) + \nabla_j w_i(x) = 0 \qquad (24)$$

The vectors $w^i(x)$ are the generators of the infinitesimal transformations

$$x^i \longrightarrow \bar{x}^i = x^i + \varepsilon w^i(x) + 0(\varepsilon^2) \qquad (25)$$

leaving the metric $g_{ij}(x)$ invariant, that is

$$\bar{g}_{ij}(\bar{x}) = g_{ij}(\bar{x}) \qquad (23')$$

Eq (23') is equivalent to eqs (23.a) and (23.b). The $w^i(x)$ generate therefore the isometries of the metric $g_{ij}(x)$ [7,8]. Eq (24) is exactly what the condition in eq (12) achieves. To see this one notes that in covariant form the operators $H_\pm$ become, on making use of (20),

$$H_\pm = \frac{\hbar^2}{2m} (\rho(x))^{-1} \nabla_i (\rho(x) b_\pm^{ij}(x) \nabla_j) \qquad (26)$$

where the tensors $b_{\pm}^{ij}(x)$ are defined by

$$b_{-}^{ij}(x) = b_{+}^{ji}(x) = g^{ij}(x) + c^{ij}(x) \qquad (27)$$

Computing the commutator of H_+ and H_- and making use of (21), one finds for the operator H_0 the expression

$$H_0 = \frac{\hbar^2}{2m} \left(\rho(x) \right)^{-1} \nabla_i \left(\rho(x) b_0^{ij}(x) \nabla_j \right) \qquad (28)$$

where $b_0^{ij}(x)$ is the Killing form

$$b_0^{ij}(x) = \frac{\hbar}{mc^2} \left(\nabla^i v^j(x) + \nabla^j v^i(x) \right) \qquad (29)$$

The operators H_0 and $H_{\pm}$ are thus Laplace-Beltrami type operators. From eq (28) it follows that the condition in eq (12) is indeed equivalent to the Killing equation

$$\nabla_i v_j(x) + \nabla_j v_i(x) = 0 \qquad (30)$$

for the reversible velocity $v^i(x)$. We therefore have the result: the reversible drift velocity of a random system in equilibrium is a Killing vector of the diffusion metric. This is a local form of fluctuation-dissipation correlation. But there is more to it.

Let

$$\frac{dx^i(t)}{dt} = u^i(\,x(t)\,) \qquad (31)$$

define a flow in the direction of the irreversible velocity. Rewrite the stationary current conservation equation in (21) as

$$u^i(x)\, v_i(x) = 0 \qquad (32)$$

Take the time derivative of (32) along the flow and make use of (30) to find

$$\left[\frac{du^i(x)}{dt} + \Gamma^i_{jk} u^j(x)\, u^k(x) \right] v_i(x) = 0 \tag{33}$$

Eq (33) says that in the sub-space of configuration space spanned by the solutions of the Killing equation (30), the irreversible drift velocity generates a congruence of geodesics defined by the equation

$$\frac{du^i(x)}{dt} + \Gamma^i_{jk} u^j(x)\, u^k(x) = 0 \tag{34}$$

Again this is a local fluctuation-dissipation correlation law.

The overall conclusion is therefore that random systems modelled by the Fokker-Planck equation possess intrinsic geometrical symmetries. Their reversible drift velocities are the isometry-generating Killing vectors of the diffusion metric while the irreversible drift velocities generate geodesics.

The symmetries involved correspond to translations and rotations and combinations of these[7]. The stationary current conservation equation (32) is then a special case of the more general result that the scalar products

$$E_\alpha(x) := \frac{1}{2}\, P^i_\alpha(x)\, v_i(x) \; ; \quad \alpha = 1,2,\dots \tag{35}$$

of Killing vectors $v^i(x)$, that is, solutions of (30), with geodesic tangent vectors

$$\frac{dx^i_\alpha(t)}{dt} = \frac{1}{m}\, P^i_\alpha(x(t)) \tag{36.a}$$

$$\frac{dP^i_\alpha(t)}{dt} + \Gamma^i_{jk} P^j_\alpha(x)\, P^k_\alpha(x) = 0 \tag{36.b}$$

are constants of motion, that is,

$$\frac{dE_\alpha(x(t))}{dt} = 0 \tag{37}$$

The symmetries in question express local fluctuation-dissipation constraints. They derive from a particular formulation of the postulate of microscopic reversibility.

Why are these results specific to the equilibrium configuration? For answer one observes, from eqs (14) and (26), that $H_\pm$ not only are adjoints of each other with respect to the scalar product

$$< \varphi_1 \mid \varphi_2 > = \int d^N x \sqrt{g(x)}\, \rho(x)\, \varphi_1^*(x)\, \varphi_2(x) \tag{38}$$

which depends on the equilibrium distribution, but depend explicitly on (x). Assuming, as usual, that the time reversal operator T is antilinear and is implemented with the help of the adjoint operation, one has, from the properties

$$T\, g_{ij}(x)\, T^{-1} = g_{ji}(x) = g_{ij}(x) \tag{38.a}$$

$$T\, c_{ij}(x)\, T^{-1} = c_{ji}(x) = -\, c_{ij}(x) \tag{38.b}$$

of the tensor potentials, that

$$T\, H_\pm T^{-1} = H_\mp^\dagger = H_\pm \tag{39.a}$$

$$T\, H_0 T^{-1} = -\, H_0^\dagger = -\, H_0 \tag{39.b}$$

We have made use of the fact that eqs (38) imply that $u^i(x)$ and $v^i(x)$ satisfy

$$T\, u^i(x)\, T^{-1} = u^i(x) \tag{40.a}$$

$$T\, v^i(x)\, T^{-1} = -\, v^i(x) \tag{40.b}$$

corresponding to the properties of being irreversible and reversible velocities, respectively. $H_{\pm}$ therefore commute with T but are not self-adjoint. They approach self-adjoint operators the closest by being normal.

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