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THE VACUUM POLARIZATION EQUATION OF STATE IN QUANTUM ELECTRODYNAMICS

The Vacuum Polarization Equation of State In Quantum Electrodynamics

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Abstract

The problem of vacuum polarization is discussed with particular emphasis on the nature of logarithmic factors entering into the renormalization group equations for the quantum electrodynamic equations of state. A vacuum magnetic instability is clearly indicated, which is also present in the anomalous magnetic moment of the electron.

1. - Introduction

The purpose of this work is to discuss the problem of QED vacuum polarization with particular emphasis on logarithmic factors entering into the renormalization group equations of state for the polarization. It will be seen that the vacuum has a magnetic instability which enters into the stress tensor. A similar logarithmic dependence is also reflected in the properties of the anomalous magnetic moment of the electron when placed under an external magnetic field. While the magnetic instability per se is confined to vanishingly small magnetic fields and hence practically unobservable, the logarithmic dependence on the applied magnetic field present in the vacuum permeability should be measureable through precision electronic ciruitry.

2. - The Maxwell Equations:

Intrinsic to the notion of vacuum polarization is the fact that the electrodynamic disturbances, characterized by the field-strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{1}$$

which propagate through the vacuum induce a vacuum current J^ν,

$$\partial_\mu F^{\mu\nu} = -(4\pi/c) \, J^\nu \,. \tag{2}$$

Since the current obeys the conservation law

$$\partial_\nu J^\nu = 0, \tag{3}$$

there exists an anti-symmetric polarization tensor $P^{\mu\nu}$ such that

$$J^\nu = c \, \partial_\mu P^{\mu\nu} \,. \tag{4}$$

In terms of the Maxwell displacement fields

$$H^{\mu\nu} = F^{\mu\nu} + 4\pi \, P^{\mu\nu} \,, \tag{5}$$

Eqs.(2) and (4) read

$$\partial_\mu H^{\mu\nu} = 0. \tag{6}$$

Eqs.(1) and (6) completely determine the propagation of electromagnetic disturbances in the vacuum once constitutive equations of state are derived relating $H^{\mu\nu}$ to $F^{\mu\nu}$. From an action principle viewpoint the methods needed to obtain such equations are clear and are discussed next.

3. - Action Principle

Let $\exp\{iW(F)/h\}$ represent the vacuum persistence amplitude under the action of an applied EM field $F^{\mu\nu}$. The constitutive equations of state then follow from the action principle variational equation

$$\delta W = -(1/8\pi c) \int (d^4x) \, H^{\mu\nu} \, \delta F_{\mu\nu} \,. \tag{7}$$

In what follows, we shall take a "thermodynamic equation of state" view in which (for sufficiently slowly varying EM fields) the effective action is described by a local Lagrange density L,

$$cW_{eff} = \int (d^4 x) \, L. \tag{8}$$

From Eqs.(7) and (8) one derives the thermodynamic rule (at zero temperature)

$$8\pi \, dL = - H^{\mu\nu} \, dF_{\mu\nu} . \tag{9}$$

From considerations of Lorentz invariance, the Lagrange density must be a function of

$$\zeta = (1/4) \, F^{\mu\nu} \, F_{\mu\nu} \tag{10}$$

and

$$\eta = (1/4) \, F^{\mu\nu} \, {}^{*}F_{\mu\nu} , \tag{11}$$

where

$${}^{*}F_{\mu\nu} = (1/2) \, \varepsilon_{\mu\nu\lambda\sigma} \, F^{\lambda\sigma} . \tag{12}$$

Eqs. (9-11) imply that

$$H^{\mu\nu} = -4\pi \{ (\partial L/\partial \zeta) \, F^{\mu\nu} + (\partial L/\partial \eta) \, {}^{*}F^{\mu\nu} \} . \tag{13}$$

4. - Polarizabilities

The vacuum polarizability χ and the vacuum magneto-electric coefficient β may be defined by the polarization equation of state

$$P^{\mu\nu} = \chi \, F^{\mu\nu} + \beta \, {}^{*}F^{\mu\nu} . \tag{14}$$

From Eqs. (5),(13) and (14) it follows that

$$-(\partial L/\partial \zeta) = (4\pi)^{-1} + \chi , \tag{15a}$$
$$-(\partial L/\partial \eta) = \beta . \tag{15b}$$

Finally from Eqs. (2), (4) and (15) one finds that in a local "thermodynamic" theory, an actual vacuum current exists if and only if the polarizabilities vary in space-time as

$$\{1 + 4\pi \chi\} \, J^{\nu} = c \, \{ F^{\mu\nu} \partial_\mu \chi + {}^{*}F^{\mu\nu} \partial_\mu \beta \} . \tag{16}$$

5. - Stress Tensor

The stress tensor implied by the Lagrange density L is given by[1]

$$T^{\mu\nu} = g^{\mu\nu} \{ L - \eta \, (\partial L/\partial\eta) \} - (\partial L/\partial\zeta) \, F^{\mu}{}_{\lambda} \, F^{\nu\lambda} , \tag{17}$$

whose trace

$$\begin{aligned} t &= T^{\mu}{}_{\mu} \\ &= 4 \{ L - \zeta \, (\partial L/\partial\zeta) - \eta \, (\partial L/\partial\eta) \}. \end{aligned} \tag{18}$$

Eqs. (13) and (18) then yield

$$dt = 4 \{ \zeta \, d\chi + \eta \, d\beta \} . \tag{19}$$

Eqs. (18) and (19) allow one to compute the vacuum polarization directly from the trace of the vacuum stress tensor. The utility of this construction will be evident in the next section where t is discussed at the one-fermion loop level.

6. - Fermion Loops

The Dirac Green's function for an electron moving in the vacuum obeys

$$\{ -i \, \gamma^{\mu} \, d_{\mu} + K \} \, G(x,y;A) = \delta(x-y) , \tag{20}$$

where $hK=mc$ and $d_{\mu} = \partial_{\mu} - (ie/hc) \, A_{\mu}$ .The trace of the stress tensor for electronic vacuum polarization is then (at the one-fermion loop level)

$$t(x;F) = -hcK \, \mathrm{tr}\{ G(x,x;A) - G(x,x;0) \} . \tag{21}$$

In the second order representation

$$G(x,y;A) = \{ i \, \gamma^{\mu} \, d_{\mu} + K \} \, D(x,y;A), \tag{22}$$

so that

$$\{ -d_{\mu} \, d^{\mu} + K^2 - (e/2hc) \, \sigma^{\mu\nu} \, F_{\mu\nu} \} \, D(x,y;A) = \delta(x-y) . \tag{23}$$

In terms of D, Eq.(20) reads

$$t(x;F) = -hcK^2 \, tr\{ D(x,x;A) - D(x,x;0) \} \,. \tag{24}$$

In the Schwinger proper time representation[1]

$$D(x,y;A) = i \int ds \, exp(-iK^2s) \, exp\{is(d_\mu d^\mu + (e/2hc)\sigma_{\mu\nu}F^{\mu\nu})\}, \tag{25}$$

a direct computation (for EM fields uniform in space-time) yields

$$t = (hcK^4/4\pi^2)\int (dz/z^2) \, e^{-z} \{z^2 abcot(az)cot(bz)-1\}, \tag{26}$$

where a and b are defined by

$$(ab) = (\alpha/hcK^4) \, \eta, \tag{27a}$$
$$(b^2-a^2) = 2(\alpha/hcK^4) \, \zeta, \tag{27b}$$

with the coupling strength $\alpha = (e^2/hc)$.

To lowest order in α

$$t = (\alpha/6\pi^2) \, \zeta + \dots . \tag{28}$$

Thus, as first noted by Schwinger {vedi ref.1, Eqs.(3.45) and (5.5)}, there is a stress anomaly <u>independent</u> of the electron mass for "small" EM fields

$$T^\mu_{\ \mu} = (\alpha/24\pi^2) \, F^{\mu\nu} F_{\mu\nu} + \dots \tag{29}$$

The consequences of this anomaly are best discussed in terms of the renormalization group equations of state which will now be considered.

7. - Renormalization Group Equations

Under the assumption that a single fermion mass scale exists (i.e., purely electronic vacuum polarization) the trace of the stress tensor may be calculated from the effective Lagrange density using

$$t = 2K^2(\partial L/\partial K^2). \tag{30}$$

From Eqs.(18) and (30), one obtains the RG equation of state

$$L - \zeta(\partial L/\partial\zeta) - \eta(\partial L/\partial\eta) = (1/2)K^2(\partial L/\partial K^2). \tag{31}$$

The general solution of Eq.(31) is given by

$$L = hcK^4\Lambda(\zeta/hcK^4, \eta/hcK^4), \tag{32}$$

where $\Lambda(x,y)$ is an arbitrary function of dimensionless arguments. The functional equation (32) can be extended to include more than one mass scale in the vacuum polarization, but here we shall restrict our discussion to pure electronic vacuum polarization.

8. - Lowest Order Polarizability Corrections

To lowest order in α, Eqs.(19) and (28) yield

$$\zeta \, d\chi = (\alpha/24\pi^2) \, d\zeta + ... \tag{33}$$

or equivalently,

$$\chi = (\alpha/24\pi^2) \, Ln\{C(e^2\zeta/hcK^4)\} + ... , \tag{34}$$

where C is an arbitrary integration constant which should be of order unity. Let us now consider the experimental implications of Eq.(34).

9. - The Ferromagnetic Vacuum

Suppose that a magnetic field B is applied to the vacuum. The vacuum polarizability in Eq.(34) then becomes

$$\chi(B) = (\alpha/24\pi^2) \, Ln\{C(eB/hcK^2)^2\} + ... , \tag{35}$$

so that the induced magnetization

$$-M(B) = B\,\chi(B), \tag{36}$$

implies a field-intensity

$$H = B - 4\pi \, M(B),$$
$$= B\{1+ (\alpha/6\pi)Ln\{C(eB/hcK^2)^2\} +...\}. \tag{37}$$

The above equation of state actually represents a ferromagnetic vacuum with a critical field

$$B^* \approx (hcK^2/e\sqrt{C}) \, exp(-3\pi/\alpha). \tag{38}$$

Note that the critical field is exponentially small so that errors (i.e., fluctuations in B) in laboratory field measurements obey

$$\Delta B \gg B^*, \tag{39}$$

by a very large margin. Nevertheless, the vacuum inverse permeability

$$(dH/dB) = 1 + (\alpha/3\pi)\{1+(1/2)Ln\{C(eB/hcK^2)^2\}+..., \tag{40}$$

exhibits a logarithmic variation in B which should be observable on laboratory electrical engineering inductors, i.e.,

$$1/\mu(B_1) \, -1/\mu(B_2) = (\alpha/3\pi)Ln(B_1/B_2) +..., \tag{41}$$

for vacuum inductors, where

$$\mu = (dB/dH). \tag{42}$$

10. - Anomalous Magnetic Moment

The logarithmic magnetic terms which produce weak vacuum ferromagnetism should also appear in other QED magnetic processes as well. In fact, for the anomalous magnetic moment of an electron, Heitler[2] has discussed the lowest order corrections in a to the magnetic field dependent electronic g-factor as computed by Luttinger[3] and Gupta[4]. According to ref.(4), upto terms of order B^2 and to order a, this correction is

$$(1/2)\Delta g \approx (\alpha/2\pi)\{1 + (47/45)(eB/hcK^2)$$
$$+(8/3)(eB/hcK^2)Ln(2eB/hcK^2)$$
$$+0(B^2) \} \tag{43}$$

As B approaches zero, one gets the Schwinger result[5] $(1/2)\Delta g \approx (\alpha/2\pi)$. The reader should note the similarity between Eqs.(40) and (43) concerning the logarithmic dependences on the magnetic field.

11. - Concluding Remarks

In this work, we have used the electronic Green's function to compute the trace of the energy-momentum tensor for EM fields which are constant (or slowly varying with frequencies much smaller than the electron mass). This requires no renormalization and the so-called trace anomaly term, Eq.(29),is automatically generated. This result is employed to demonstrate the presence of logarithmic corrections to the polarizability. We find a ferromagnetic vacuum i.e., there exists a critical field B* for which the resulting magnetic field intensity H vanishes. Since B* is vanishingly small, for all experimentally accessible fields, H is essentially linearly related to B. However, the magnetic permeability shows a logarithmic variation with B which is in principle measureable through precision resonant ciruits.

Acknowledgement

References

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