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TAUBERIAN THEOREMS FOR THE BOREL SUMMABILITY AND ANALYTIC CONTINUATION OF THE ASYMPTOTICS OF QCD SUM RULES

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ABSTRACT

Using QCD sum rules as inputs, the conditions for and proof of the Borel summability and analytic continuation of QCD asymptotic expansions for current propagators are given. The corollary is that duality averages can be "coarse grain", i.e. performed over the mass squared interval $\bar{s} \to \infty$, corresponding to short distances ($t \to 0$), or equivalently "fine grain", i.e. performed over the interval $\bar{s} \to 0$, corresponding to long distances ($t \to \infty$). The former is the usual formulation of duality, the latter, the paradoxical novelty proposed by Shifman, Vainshtein and Zakharov, which pushes duality to the limit of being applicable at a point. The two limits are related so that QCD is, surprisingly, relevant to both. The relationship is a consequence of the covariance of dilatation convolutions, which define duality averages, with respect to the conformal inversion. Underlying both Borel summability and the short distance operator product expansion is SO(2,1) symmetry.

It is claimed by Shifman, Vainshtein and Zakharov (SVZ)[(1)] that QCD asymptotic expansions for flavour current two-point functions ($\alpha,\beta \equiv u,d,s,c,b,\ldots$)

$$\Pi_{\mu\nu}^{(\alpha,\beta)}(q) = -i\int d^4x e^{iqx} \langle 0|T(j_\mu^{(\alpha)}(x) j_\nu^{(\beta)}(0))|0\rangle = \qquad (1)$$

$$= q_\mu q_\nu \Pi_0^{(\alpha,\beta)}(q^2) + (q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi_1^{(\alpha,\beta)}(q^2)$$

admit analytic continuations to low energies. They show, in impressive analyses of the amplitudes $\Pi_{0,1}^{(\alpha,\beta)}(q^2)$, for various currents $j_\mu^{(\alpha)}(x)$ and $j_\nu^{(\beta)}(x)$, that the assumption is consistent with experiments to within about 10-20%. But there are no explicit proofs to validate the claim, apart from more or less successful, phenomenological applications. In particular, there are no indications of the physical principles underlying their principal assumption, namely, that duality can be pushed to the limit of being applicable at a point. And no attempt at a theoretical justification either.

Bell and Bertlmann(2) were the first to react to this novel approach. They argued and concluded that, although it works phenomenologically, the model is, presently, lacking in fundamental basis. The criticism has been answered(3) to the effect that it is not applicable to QCD since it is based on a study of potential models. The answer is incorrect. At issue here is a very well posed problem of summability and analytic continuation, valid for both relativistic and non-relativistic theories. It is: to show, and under which conditions, a given set of asymptotic estimates in the form of sum rules, imply local analyticity properties.

There is a wide class and a long history of such problems in which the asymptotic properties of one function, usually defined by an integral, correspond to the local properties of another and vice versa. Duality(4) is a relationship of this kind, a relationship between asymptotic behaviour and spectral properties. There are many ways of characterising this reciprocal relationship, using either group theory or the theory of analytic functions or both together. Whatever the method, the solutions of these problems reduce, in the end, to finding the correct converses of theorems which predict unique asymptotic behaviour from uniformity and local analyticity properties. The direct theorems are termed Abelian(5). Their converses are, in general, false. The corrected converses are therefore not unconditionally valid. They require the specification of bounds to the local (spectral) functions.

They are known as Tauberian theorems[5,6].

Starting with this paper, and based on the above ideas, we propose to give different proofs of the summability and the analytic continuation of the QCD asymptotic expansions of the amplitudes $\Pi_{0,1}^{(\alpha,\beta)}(q^2)$ (*) to low energies. In particular we derive the result which implements the SVZ Ansatz and other consequences not foreseen by their original prescription. Symmetries and analytic function methods will be used. The symmetries are those relevant to the short distance operator product expansion

$$\begin{aligned} j_\mu^{(\alpha)}(x)j_\nu^{(\beta)}(0) &= (\partial_\mu\partial_\nu - g_{\mu\nu}\Box)\sum_{n=0} C_{\alpha_1..\alpha_n}^{(\alpha,\beta)}(x)\,\theta_{\alpha_1..\alpha_n}(0) + \\ &+ (g_{\mu\alpha_1}\partial_\nu\partial_{\alpha_2} + g_{\nu\alpha_1}\partial_\mu\partial_{\alpha_2} - g_{\mu\alpha_1}g_{\nu\alpha_2}\Box - g_{\mu\nu}\partial_{\alpha_1}\partial_{\alpha_2})\cdot \\ &\cdot \sum_{n=0}^{\infty} D_{\alpha_3..\alpha_n}^{(\alpha,\beta)}(x)\,\theta_{\alpha_1..\alpha_n}(0) \end{aligned} \tag{2}$$

The subgroup, SO(2,1), of the conformal group is of interest here. Its most important transformation in this context is the conformal inversion R, the so-called transformation by reciprocal radii. Scale invariance is assumed to be "softly" and spontaneously broken in the form[1]

$$\langle 0\ \mathrm{Trace}(\theta_{\alpha_1..\alpha_n})\ 0\rangle \neq 0\ . \tag{3}$$

The inputs are the QCD sum rules[1,2]

$$\lim_{\bar{s}\to\infty} \frac{s_0^\nu}{\pi}\int_{s_0}^{\bar{s}} \frac{ds}{s^{\nu+1}}\,(\mathrm{Im}\Pi_+(s) - \mathrm{Im}\Pi_-(s)) = O((\frac{s_0}{\bar{s}})^{\nu+1-N_0}) \tag{4}$$

and asymptotic scale invariance. The latter will be cast into a form which allows not only to perform the obvious limit $\bar{s}\to\infty$ in eq.(4) but also the not at all obvious limit $\bar{s}\to 0$ in the same integral.

(*) $\Pi_{0,1}^{(\alpha,\beta)}(q^2) \equiv \Pi(q^2)$ from now on, in order to lighten the notation.

The two limits will then be shown to be related.

$N_o \geq 0$ is an integer, $Re(\nu) > N_o - 1$ and

$$\lim_{\bar{s}\to\infty} (Im\Pi_+(s) - Im\Pi_-(s) = O((\frac{s_o}{\bar{s}})^{1-N_o}) \ . \qquad (5)$$

$Im\,\Pi_-(s)$ is the imaginary part of the amplitude $\Pi_-(q^2) := \Pi(q^2)$, for which there exists a known QCD asymptotic expansion. $Im\,\Pi_+(s)$, on the other hand, is the imaginary part of a comparison amplitude $\Pi_+(q^2)$ with a known Taylor series expansion about $q^2=0$, saturated at low energies by meson resonances. It is parametrised at high energies so that (4) and (5) are satisfied. Also $\Pi_+(q^2) \to (q^2)^{N_o}$ for $q^2 \to 0$.

The basic problem is the following: under what conditions do eqs. (4) and (5) imply the analytic continuation[1,2]

$$RG_+^{(\nu)}(t) - G_-^{(\nu)}(t) := -\frac{s_o^{\nu+N_o}}{\pi} \int_{s_o}^{\infty} \frac{ds}{s^{\nu+N_o+1}} e^{-ist/s_o} \cdot (Im\Pi_+(s) - Im\Pi_-(s)) = 0 \qquad (6)$$

for all complex t and therefore

$$\Pi_+(q^2) = \Pi_-(q^2) \qquad (7)$$

for all q^2? Let $z = q^2/s_o$. Our definitions of the Borel transforms[1] of $\Pi_\pm(q^2)$ are

$$\Pi_+(z) = -iz^{N_o} \int_{-\infty}^{+\infty} dt\, RG_+(t) e^{itz} = -iz^{N_o} \int_{-\infty}^{+\infty} dt\, G_+(t) \cdot Re^{itz} , \qquad (8a)$$

$$\Pi_-(z) = -iz^{N_o} \int_{-\infty}^{+\infty} dt\, G_-(t) e^{itz} = -iz^{N_o} \int_{-\infty}^{+\infty} dt\, RG_-(t) \cdot Re^{itz} . \qquad (8b)$$

$G_\pm(t)$ are the Fourier expansion coefficients of $z^{-N_0}\Pi_\pm(z)$ in the bases of the eigenfunctions Re^{itz} and e^{itz} of the generators of special conformal transformations and translations, respectively. Harmonic analysis on the group SO(2,1) is discussed in detail in ref.(7). Apart from eqs.(8), these details will not be needed in this paper.

Of interest here is, instead, the representation of the subgroup of dilatations $T_0(\lambda)$ in the function space $L_1(0,\infty)$ of absolutely integrable functions over the half line $0 \leq s < \infty$. It arises in connection with the following alternative formulation of asymptotic scale invariance: physical results are, asymptotically, not affected by and, hence, are independent of duality averaging, provided the weight functions used are reasonable. By "reasonable" we shall mean that the weight function $\varrho(s)$ satisfies the following conditions[5,6]:

(i) $\varrho(s)$ is absolutely integrable

$$\int_0^\infty ds\, |\varrho(s)| < \infty \tag{9}$$

i.e. $\varrho(s) \in L_1(0,\infty)$.

(ii) The set of its dilatations (i.e. its orbit)

$$\varrho_\lambda(s) := T_0(\lambda)\, \varrho(s) = \varrho(\lambda s), \tag{10}$$

$\lambda > 0$, is complete in $L_1(0,\infty)$. In other words, given any function $\bar{\varrho}(s) \in L_1(0,\infty)$ one has the expansion

$$\bar{\varrho}(s) = \underset{\Lambda\to\infty}{\text{L.i.m.}} \int_0^\Lambda \frac{d\lambda}{\lambda} K(\lambda)\, \varrho(\lambda s) \tag{11}$$

valid as limit in the mean (L.i.m.). In particular $\lambda^{-1}K(\lambda) \in L_1(0,\infty)$.

(iii) Its Mellin transform (or power moment)

$$m(\nu) := \int_0^\infty ds\, \varrho(s) \left(\frac{s}{s_0}\right)^{\nu-1} \tag{12}$$

is non-vanishing for $\mathrm{Re}(\nu) = 1$.

This last condition is to ensure that the kernel $K(\lambda)$ in (11) is uniquely determined when $\bar{\varrho}(s)$ and $\varrho(s)$ are given. In fact, taking the Mellin transform of (11) and defining

$$K(\nu) := \int_0^\infty \frac{d\lambda}{\lambda^{\nu+1}} K(\lambda) \tag{13}$$

one obtains

$$K(\nu) = \frac{\bar{m}(\nu)}{m(\nu)} \tag{14}$$

which is possible only on the strength of the last condition. Actually conditions (i) and (iii) are equivalent because of the theorem[6]: The necessary and sufficient condition for the set of dilatations of $\varrho(s)$ to be closed $L_1(0,\infty)$ is that its Mellin transform should not vanish on $\mathrm{Re}(\nu) = 1$.

$L_1(0,\infty)$ is not a Hilbert space. The completeness assumption is on service in situations of the following kind: Let the function $f(s)$ be bounded or of bounded total variation, that is[5],

$$\lim_{\frac{s'}{s}\to 1} \lim_{s\to\infty} \left(f(s') - f(s)\right) = 0 \tag{15}$$

and let

$$\int_0^\infty ds\, \varrho_\lambda(s)\, f(s) = 0 \tag{16}$$

for all $\lambda > 0$, then $f(s)$ is identically zero if (ii) is satisfied. For the proof, one takes the Mellin transform of (16) and then makes use of (iii). $L_1(0,\infty)$ contains, for every weight function $\varrho(s)$, its transform under the inversion R; that is, $R\,\varrho(s) := (s_0/s)^2 \varrho(s_0^2/s) \in L_1(0,\infty)$. It is this circumstance, as we shall see, which allows to implement local duality in the sense of SVZ.

Now introduce the weight functions

$$\varrho_0(s) := \frac{1}{s}\,\theta(\bar{s} - s) \tag{17a}$$

$$\varrho_-(s,t) := \frac{1}{s}\, e^{-ist/s_0}\,\theta(s - s_0); \quad \mathrm{Im}(t) < 0 \tag{17b}$$

and the abbreviation

$$f_{\nu-1}(s) := \frac{1}{\pi}\left(\frac{s_0}{s}\right)^{\nu}\left(\mathrm{Im}\Pi_+(s) - \mathrm{Im}\Pi_-(s)\right). \tag{18}$$

The function $f_{\nu-1}(s)$ satisfies the condition (15), $\varrho_0(s)$ the conditions (i)-(iii) and $\varrho_-(s,t)$ is a transform of $\varrho_0(s)$ according to the definition in eq.(11), i.e.

$$\varrho_-(s,t) = \int_0^{\infty} \frac{d\lambda}{\lambda}\, K(\lambda,t)\, \varrho_0(\lambda s) \tag{19a}$$

$$K(\lambda,t) = \lambda \frac{d}{d\lambda}\left(\lambda e^{-\bar{s}t/\lambda s_0}\,\theta\left(\frac{\bar{s}}{s_0} - \lambda\right)\right). \tag{19b}$$

Moreover eqs.(4) and (5) may be combined into the single equation

$$f_{\nu-1}(\bar{s}) = \lim_{\bar{s}\to\infty} \frac{1}{\bar{s}} \int_0^{\bar{s}} ds\, f_{\nu-1}(s) = O\left(\left(\frac{s_0}{\bar{s}}\right)^{\nu+1-N_0}\right) \tag{20}$$

Therefore, rewrite eq.(4) (for $\nu \to \nu+N_0$) and the first part of the equality in eq.(6) as

$$\begin{aligned}\lim_{\bar{s}\to\infty} \frac{s_0}{\bar{s}} \int_0^{\infty} ds\, \varrho_0\left(\frac{s_0 s}{\bar{s}}\right) f_{\nu+N_0-1}(s) &= \\ = \frac{1}{\bar{s}} \int_0^{\bar{s}} ds\, f_{\nu+N_0-1}(s) \cdot \int_0^{\infty} ds\, \varrho_0(s)&\end{aligned} \tag{21}$$

$$\frac{s_0}{\bar{s}} \int_0^{\infty} ds\, \varrho_-\left(\frac{s_0 s}{\bar{s}}\right) f_{\nu+N_0-1}(s) = G_-\left(\frac{s_0 t}{\bar{s}}\right) - RG_+\left(\frac{s_0 t}{\bar{s}}\right). \tag{22}$$

Eqs.(20)-(22) allow to activate the following Tauberian theorem[5,6]: Let the function f(s) satisfy the condition of eq.(15), the weight function $\varrho(s)$ the conditions (i)-(iii) and let

$$\lim_{\bar{s}\to\infty} \frac{s_0}{\bar{s}} \int_0^\infty ds\, \varrho(\frac{s_0 s}{\bar{s}}) f(s) = \frac{1}{s} \int_0^{\bar{s}} ds\, f(s) \cdot \int_0^\infty ds\, \varrho(s). \tag{23}$$

Then for all $\bar{\varrho}(s) \in L_1(0,\infty)$, (23) implies

$$\lim_{\bar{s}\to\infty} \frac{s_0}{\bar{s}} \int_0^\infty ds\, \bar{\varrho}(\frac{s_0 s}{\bar{s}}) f(s) = \frac{1}{\bar{s}} \int_0^s ds\, f(s) \cdot \int_0^\infty ds\, \varrho(s) \tag{24}$$

and

$$f(\bar{s}) = \lim_{\bar{s}\to\infty} \frac{1}{\bar{s}} \int_0^{\bar{s}} ds\, f(s). \tag{25}$$

The importance of the QCD inputs is evident from comparing eqs.(20) and (25). The latter equation is a special case of (24) for $\bar{\varrho}(s) = \varrho_0(s)$, since

$$\lim_{\bar{s}\to\infty} \frac{s_0}{\bar{s}^2}\, \theta(\bar{s} - \frac{s_0 s}{\bar{s}}) = \delta(\bar{s} - s)\cdot\theta(s - s_0). \tag{26}$$

But also conversely, and this is the point about the QCD inputs; eq. (24) may be deduced from (25) by setting $\varrho(s) = \varrho_0(s)$ in eq.(23) and using (11) to expand $\bar{\varrho}(s)$ in terms of $\varrho_0(s)$. More elaborate proofs are available in refs.(5) and (6). This theorem is the precise formulation of the assumption of asymptotic scale invariance in the form stated previously, i.e. physical results are independent of duality averaging.

Applying the theorem to eqs.(21) and (22) one gets

$$\begin{aligned} &\lim_{t\to 0} (G_-^{(\nu)}(t) - RG_+^{(\nu)}(t)) = \\ &= (\lim_{\bar{s}\to\infty} \frac{1}{\bar{s}} \int_0^{\bar{s}} ds\, f_{\nu+N_0-1}(s))\cdot(\lim_{t\to 0} \lim_{\bar{s}\to\infty} E_1(\frac{i\bar{s}t}{s_0})) \end{aligned} \tag{27}$$

where

$$E_1(t) = \int_1^\infty \frac{dx}{x} e^{-xt} \tag{28}$$

is the exponential integral. It has the properties[8]

$$E_1(t) = -\ln(\gamma t) - \sum_{n=1}^{\infty} \frac{(-t)^n}{n \cdot n!} \tag{29a}$$

$$E_1(t) \xrightarrow[t \to \infty]{} e^{-t}/t \tag{29b}$$

$\gamma \simeq 0.577$ is Euler's number.

But the integrals on the left hand sides of eqs.(23) and (24) of the theorem may also be written as

$$\frac{s_0}{\bar{s}} \int_0^\infty ds\, \varrho\left(\frac{s_0 s}{\bar{s}}\right) f(s) = \frac{s_0}{R\bar{s}} \int_0^\infty ds\, \varrho_R\left(\frac{s_0 s}{R\bar{s}}\right) f_R(s) \tag{30}$$

where $R\bar{s} = s_0^2/\bar{s}$,

$$R\,\varrho(s) := \varrho_R(s) = \left(\frac{s_0}{s}\right)^2 \varrho\left(\frac{s_0^2}{s}\right) \tag{31a}$$

$$R\,f(s) := f_R(s) = f\left(\frac{s_0^2}{s}\right). \tag{31b}$$

The functions $\varrho_R(s)$ and $f_R(s)$ satisfy the same conditions as their respective transforms $\varrho(s)$ and $f(s)$. Consequently, one may take either the limit $\bar{s} \to \infty$ or the limit $\bar{s} \to 0$ in eq.(30) and the conclusions of the Tauberian theorem will remain unchanged. Taking now the latter limit in eq.(22) one gets

$$\lim_{|t| \to \infty} \left(G_-^{(\nu)}(t) - RG_+^{(\nu)}(t)\right) = \left(\lim_{\bar{s} \to 0} \frac{1}{\bar{s}} \int_0^{\bar{s}} ds\, f_{\nu+N_0-1}(s)\right) \cdot \left(\lim_{|t| \to \infty} \lim_{\bar{s} \to 0} E_1\left(\frac{i s_0 t}{\bar{s}}\right)\right) \tag{32}$$

Eqs.(27) and (32), together with (29b) and (30) now lead to our main theorem: For all ν and for $t \to 0$, from the half plane $Im(t) < 0$, and $|t| \to \infty$, we have

$$G_-^{(\nu)}(t) - RG_+^{(\nu)}(t) \xrightarrow[t \to 0]{} 0 \tag{33a}$$

$$G_-^{(\nu)}(t) - RG_+^{(\nu)}(t) \xrightarrow[|t| \to \infty]{} 0 \tag{33b}$$

and in addition

$$f_{\nu+N_0-1}(\bar{s}) = \lim_{\bar{s} \to \infty} \frac{1}{\bar{s}} \int_0^{\bar{s}} ds\, f_{\nu+N_0-1}(s) \tag{34a}$$

$$f_{\nu+N_0-1}(\bar{s}) = \lim_{\bar{s} \to 0} \frac{1}{\bar{s}} \int_0^{\bar{s}} ds\, f_{\nu+N_0-1}(s) = $$

$$= \lim_{R\bar{s} \to \infty} \frac{1}{R\bar{s}} \int_0^{R\bar{s}} ds\, Rf_{\nu+N_0-1}(s) \; . \tag{34b}$$

Since $G_-^{(\nu)}(t) - RG_+^{(\nu)}(t)$ is analytic in the finite t-plane, it follows from eqs.(33) and Liouville's theorem, that it is identically zero. Hence the formula of analytic continuation

$$G_-^{(\nu)}(t) = RG_+^{(\nu)}(t) \tag{6}$$

or equivalently

$$\Pi_+(z) = \Pi_-(z) := \Pi(z) \tag{35}$$

by putting $\nu = 0$ in (6) and making use of the inverse Fourier transforms of (8).

The connection between the limits $\bar{s} \to (0, \infty)$ and $t \to (\infty, 0)$ is evident from eqs.(27) and (32). In eqs.(32) and (34b) one encounters the local implementation of duality proposed by Shifman, Vainshtein and Zakharov. It is not to be separately assumed. It follows from the Tauberian theorem, and one encounters it, inevitably, in the proof of analytic continuation. Eq.(34b) relates this local version of duality to the usual one so that the one may be defined in terms of the other. This fact is not contained in the original SVZ prescription.

Consequently QCD is relevant to both in a fundamental way and not just approximately. For instance eq.(34b) allows to test the R-invariance of the theory, that is, if

$$Rf(s) = f(s). \tag{35}$$

Eqs. (34a) and (34b) involve, respectively, "coarse" and "fine grain" averaging[2], corresponding to the dominance of short and long distance phenomena. According to eq.(34b) the two kinds of phenomena are related.

Here is a summary of our approach: QCD sum rules are given a wider interpretation as asymptotic averages, similar to ensemble averages. Asymptotic scale invariance is equivalent to the assumption that the fictitious ensemble is "ergodic", that is,

$$\lim_{\bar{s}\to\infty} \frac{s_0}{\bar{s}} \int_0^\infty ds\, \varrho\left(\frac{s_0 s}{\bar{s}}\right) f(s) = \frac{1}{\bar{s}} \int_0^{\bar{s}} ds f(s) \cdot \int_0^\infty ds\, \varrho(s) \tag{23}$$

with respect to the distributions $\varrho(s) \in L_1(0,\infty)$, at all scales $\lambda = \bar{s}/s_0 > 0$. The $\varrho(s)$ are not probabilities. They are all related by scale transformations and are therefore like states which can be reached from an initial one with the help of an evolution operator. The evolution operator is in this case the representation of scale transformations.

Eq.(23) allows to transform inputs about asymptotic averages into outputs about local properties of the functions being averaged. This comes about at twc levels. Firstly through the local form of duality and its relationship to the non-local one (cf. eq.(34b)). Secondly, by means of appropriate choices of the weight functions $\varrho(s)$, one converts the left hand side of (23) into known and simple integral transforms, the properties of which are then completely fixed by the right hand side. Borel transforms result from such simple choices.

It is through the choice of $\varrho(s)$ that symmetries enter the theory. It seems most natural to try those symmetries which have something to do with the asymptotics being investigated. For the QCD asymptotics of the amplitude $\Pi(q^2)$, the relevant symmetry group is SO(2,1). It allows to deduce, rather easily, the formula of analytic continuation and to show that when a proof of this is available, it necessarily leads to the local implementation of duality, first proposed by SVZ. The latter is, therefore, not a separate or independent assumption.

Lastly eq.(23) corresponds to different methods of summability, one for each weight function. For instance, $\varrho(s) = \varrho_-(s,t)$ corresponds to Borel summability and $\varrho(s) = \varrho_0(s)$ to Cesāro summability C(1) (cf. eqs.(34)). Many methods of summability are related to transformation groups and some of the most effective to the group SO(2,1), independently of their relationship to physics. Summability with respect to transformations of SO(2,1) is of peculiar interest to QCD because of the relationship of this group to the short distance operator product expansion. We hope to illustrate this connection better and much more concretely in our next paper[7].

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