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ZERO MOMENTUM MODES IN GAUGE THEORIES ON A TORUS

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ZERO MOMENTUM MODES IN GAUGE THEORIES ON A TORUS

F. Palumbo
INFN - Laboratori Nazionali di Frascati, Frascati, Italy

ABSTRACT

The canonical quantization of gauge theories on a torus is reported. The main feature introduced by the torus is the presence of zero momentum modes (zmm). It is shown that the unregularized Hamiltonian is not bounded from below due to unlimited growth of zmm amplitude both in the abelian and non abelian case. The only regularization which can prevent such a disease is to give a mass to the gauge field.

Various tricky features of zmm are discussed, related to Schwinger terms, the massive Schwinger model and the Witten index. It is also shown how zmm allow the introduction of an energy scale in abelian gauge theories.

1.- INTRODUCTION

A specific feature of field theories on a torus is the existence of zero momentum modes (zmm). These modes can play a special role in the thermodynamic limit. While the equations involving other modes go smoothly into the continuum when the volume of the torus goes to infinity, the equations involving zmm must be treated separately to investigate wether the measure can develop a singularity at zero momentum. A well known example is Bose-Einstein condensation in statistical mechanics. Another example is provided by abelian gauge theories where due to zmm an energy scale can be introduced(1) through radiative corrections.

In non abelian gauge theories zmm can be expected to be even more important due to the more complicated infrared structure. This point has been fully appreciated by some authors, who have emphasized the possible relevance of classical(*) zmm (torons) in determining the structure of the vacuum[2].

The importance of zmm is also apparent in the reduced model[3] and the quenched model[4]. Finally zmm have been promoted to a central role in the formulation of an effective Hamiltonian in the Fock space generated by them[5].

In this talk I want to report on the canonical quantization of gauge theories on a torus taking zmm into proper account. zmm enter the Gauss constraint and complicate its solution. This difficulty is overcome here by introducing a gauge where the constraint for zmm decouples from the constraint on other modes. The resulting Hamiltonian is unbounded from below when the gauge fields are coupled to fermionic matter both in the abelian and non abelian case. This instability is not present if the coupling is with bosonic matter only, but cannot be prevented if in addition to bosonic matter there is fermionic matter, so that it is relevant to Supersymmetric theories.

This vacuum instability is an infrared problem, and we know that gauge theories must anyhow be regularized in the infrared. As a consequence of the present result, however, the only acceptable regularization on a torus is to give a mass to the gauge field[6] (to replace the gauge multiplet by a massive vector multiplet in Supersymmetric theories).

The instability takes place through the coupling of zmm to the volume average of the fermion current. If such an average vanishes, the Hamiltonian is bounded from below, so that there is no such a problem in the presence of color confinement. The problem remains, however, in perturbation theory.

The same instability related to zmm affects the theory of a spinor and a massless scalar with Yukawa coupling, but in this case it can be removed by adding a quartic scalar self-interaction, which is also necessary to make the theory renormalizable. What makes gauge theories peculiar with respect to zmm is that there is no interaction term which can regularize them.

One might ask: why study gauge theories on a torus if the zmm, which do not exist in an infinite volume, give rise to such a pathology?

One motivation is that continuum gauge theories on a torus should be easier to compare to lattice gauge theories, which are usually studied on a torus in order to have a good statistics.

Another motivation which, though highly speculative appears far more interesting to me, is to make contact with Cosmology.

(*) We call them classical because they are defined by $H_i = E_i = 0$.

If we want to include gravity in our theory, a first step is to quantize in a curved space in which the metric is a background. For a homogeneous Universe, we can assume a Robertson-Walker metric. The normal modes with such a metric are continuous for an open Universe and discrete for a closed Universe. In the flat limit the normal modes of the open Universe are consistent with the normal modes in an infinite volume, while those of a closed Universe have some features in common with the normal modes in a torus, including the existence of zmm. QFT on a torus can be in this respect considered as the flat approximation to QFT in a closed curved space. Therefore either zmm do not introduce any spurious effect but at most make more evident some peculiar feature common to an infinite volume, or we have a way to distinguish between a closed and an open Universe by the infrared properties of gauge theories. Either way it is interesting to investigate gauge theories on a torus.

As a by-product of the general results on the quantization of gauge theories on a torus I will show how an energy scale can be introduced into abelian gauge theories, and I will comment on the Schwinger terms, the θ-angle of the massive Schwinger model and the Witten index in supersymmetric gauge theories.

2.- THE ABELIAN CASE

The Hamiltonian density is(*)

$$\mathcal{H} = \frac{1}{2} E^2 + \frac{1}{2} H^2 + gA_k j_k + \mathcal{H}_F \tag{1}$$

where j_k and $\mathcal{H}_F$ are the current and Hamiltonian density of the fermion field,

$$H_i = \frac{1}{2} \varepsilon_{ijk} \partial_j A_k \tag{2}$$

and E_k is the momentum conjugate to the gauge field A_k

$$\{ A_h(\vec{x}), E_k(\vec{y}) \} = \delta_{hk} \delta^3(\vec{x} - \vec{y}). \tag{3}$$

These variables are subject to the Gauss constraint

$$\Phi = \partial_k E_k + g j_o = 0. \tag{4}$$

Since we perform the quantization on a torus we can expand A_h and E_k in Fourier series

(*) Repeated indices should always be understood summed over.

$$A_k = \frac{1}{L^{3/2}}\left[Q_k + \sum_{\vec{n}\neq 0} A_{k\vec{n}}\, e^{i\frac{2\pi}{L}\vec{n}\cdot\vec{x}}\right] \overset{def}{=} \frac{1}{L^{3/2}} Q_k + \bar{A}_k \,,$$

$$E_k = \frac{1}{L^{3/2}}\left[P_k + \sum_{\vec{n}\neq 0} E_{k\vec{n}}\, e^{i\frac{2\pi}{L}\vec{n}\cdot\vec{x}}\right] \overset{def}{=} \frac{1}{L^{3/2}} P_k + \bar{E}_k \,. \qquad (5)$$

The Fourier coefficients must satisfy the reality conditions

$$A_{k\vec{n}} = A^*_{k,-\vec{n}} \,, \qquad E_{k\vec{n}} = E^*_{k,-\vec{n}} \,.$$

It follows from eq.(3) that the zmm Q_k and P_k satisfy canonical Poisson brackets.

Introducing also for the current density a decomposition analogous to (5)

$$j_k = \frac{1}{L^{3/2}} I_k - \bar{j}_k \qquad (6)$$

we have for the Hamiltonian

$$H = \frac{1}{2} P^2 + gQ_k I_k + \int d^3x \left[\frac{1}{2}\bar{E}_k \bar{E}_k + \frac{1}{2} H^2 + g\bar{j}_k \bar{A}_k + \mathscr{H}_F\right] \,. \qquad (7)$$

The quantization is now achieved by eliminating the redundant variables by a gauge fixing and by replacing the remaining variables by quantum operators. We do not need to specify the gauge fixing for the present purposes. Let us only emphasize that it cannot act on Q_k which is gauge invariant because Φ commutes with it.

The gauge invariance of Q_k can also be understood in the following way. Suppose that Q_k is changed by a gauge transformation: $Q_k \rightarrow Q_k + c_k$. This means that the parameter $\alpha(x)$ of the gauge transformation can be written as $\alpha(x) = x_k c_k + \bar{\alpha}(x)$. The matter fields ψ would accordingly transform as $\psi' = e^{i[x_k c_k + \bar{\alpha}(x)]}\psi$, and due to the arbitrariness of c_k, they would no longer satisfy periodic boundary conditions.

The above refers to infinitesimal gauge transformations. There are "large" transformations, however,

$$Q_k \rightarrow Q_k + \frac{2\pi}{L}\nu_k \,, \quad \nu_k \text{ integer} \qquad \psi \rightarrow e^{i\frac{2\pi}{L}\nu\cdot x}\psi \,,$$

which do not alter the boundary conditions of the matter fields. They are related to torons[2] in the non abelian case. I will not discuss them because we will see that the only possible regularization breaks the invariance of the Hamilton ian w.r. to such large gauge transformations.

It is obvious that the Hamiltonian (7) is unbounded from below. This can be checked by taking its expectation value in the state

$$\Psi = (\pi Q_0^2)^{-\frac{1}{4}} \exp\left[- \frac{(Q_k - \bar{Q})^2}{2Q_0^2} \right] \chi \qquad (8)$$

where χ is a state functional which does not depend on Q_k, such that

$$< \chi | I_k | \chi > \; = \partial_{k3} I, \qquad I \neq 0.$$

Therefore for $\bar{Q} \to \infty$

$$< \Psi | H | \Psi > \; \to g I \bar{Q} \; .$$

One might object that such a result is not peculiar of gauge theories. The Hamiltonian of a spinor and a massless scalar with a Yukawa coupling, for instance, exhibits the same feature related to zmm. In this latter case, however, the instability can be avoided by adding to the Hamiltonian a selfinteraction φ^4, which is also necessary to make the theory renormalizable. What makes gauge theories peculiar w.r. to zmm is that there is no term analogous to φ^4 which can prevent the infrared instability. The only way to do it is to regularize by giving to the gauge field a mass.

If the gauge field is coupled to a scalar field rather than to a fermion field, a term $g^2 \varphi^* \varphi A^2$ is present in the Hamiltonian, which prevents the infrared instability. Addition of such a coupling to the coupling with a fermion field, however, cannot obviously avoid it. The present result is therefore relevant to Supersymmetric gauge theories on a torus. The only way we can see to regularize these theories is to replace the gauge multiplet by a massive vector multiplet.

Let me now briefly show how an energy scale can appear. Due to the presence of radiation of soft quanta the cross sections can be written

$$\sigma = \sigma_0 \left(\frac{\Delta E - \Lambda W}{E} \right)^{\beta} \theta(\Delta E - \Lambda W) \; .$$

A precise definition of the symbols appearing in the above formula can be found in ref.(1). Here it is sufficient to say that σ_0 is proportional to the cross section evaluated with an infrared cut-off for the momenta of the gauge quanta, β and W are quantities depending on ingoing and outgoing currents, E the experimental energy, ΔE the experimental energy resolution and Λ the new parameter

$$\Lambda = \lim_{\mu \to 0, L \to \infty} \frac{g^2}{\mu^2 L^3} ,$$

where μ is the regulator mass of the gauge field.

Since there is no prescription about how L must go to infinity w.r. to how Λ must go to zero, Λ is an arbitrary parameter. Such a parameter is obviously zero in an infinite quantization volume.

If we make contact with Cosmology, however, we see that Λ is no longer a free parameter and its value is so small that we cannot distinguish an open from a closed Universe. In the presence of matter, in fact, μ must be replaced by the plasma frequency

$$\mu^2 = \frac{g^2 N}{mL^3} ,$$

where N is the total number of charged particles plus antiparticles in the Universe, assumed for simplicity to have all the mass m, and L^3 is the volume of the Universe. It follows that

$$\Lambda = \frac{m}{N} .$$

Let me now come to the subject of the Schwinger terms. Schwinger has observed[7] that the commutator $\left[i\partial_t P_k, P_k\right]$ cannot vanish, as it would result from canonical commutation relations

$$\left[i\partial_t P_k, P_k\right] = \left[\left[P_k, H\right], P_k\right] = - ig \left[I_k, P_k\right] = 0.$$

In fact by taking the vacuum expectation value of the l.h.s. of the above equation we find

$$\left\langle \left[i\partial_t P_k, P_k\right] \right\rangle = 2 \left\langle P_k M P_k \right\rangle > 0 ,$$

if a vacuum exists which is not annihilated by P_k. Schwinger concludes that $\left[I_k, P_k\right] \neq 0$, implying extra terms in the commutators, namely the Schwinger terms.

A similar argument also applies to other commutators, about which I have nothing to say. Concerning the present one, however, we have seen that a vacuum does not exist for the unregularized Hamiltonian. If we regularize by adding a term $\frac{1}{2}\mu^2 A^2$, on the other hand we have

$$\left[i\partial_t P_k, P_k\right] = i\left[- gI_k - \mu^2 Q_k, P_k\right] = - ig\left[I_k, P_k\right] + 3\mu^2$$

whose expectation value in the vacuum is positive even in the absence of Schwinger terms, i.e. $\left[I_k, P_k \right] = 0$.

Finally I will comment on the massive Schwinger model(8) namely an abelian gauge theory in two space time dimensions. Coleman(9) has studied this model and he has concluded that its mass spectrum depends on an angle, which is a new parameter existing only in two space time dimensions. Coleman does not make any explicit assumptions about the quantization volume, but a torus is implicitely assumed in the solution of the Gauss constraint

$$E_1 = - g \, \partial_1^{-1} j_o + P_1 \ ,$$

due to the presence of the constant P_1. The angle which would affect the energy spectrum is

$$\theta = \frac{2\pi P_1}{g} \ .$$

Coleman's result follows from the assumption that P_1 is a constant both w.r. to space and time. P_1, however, is the zmm of the canonical variable E_1, and cannot be taken to be time-independent. From the general treatment just presented we see that there are no new parameters (in addition to mass and coupling constant) in two dimensional gauge theories. The parameter Λ, in fact, is related to radiative corrections which are absent in one space dimension. The present conclusion cannot be possibly affected by Schwinger terms which, if present, will not involve zmm as shown above.

The zmm are not expected to play any role in the Schwinger model. This is because the charge is confined in this model and there being no asymptotic currents, $I_k = 0$, and the zmm decouple.

3.- THE NON ABELIAN CASE

In the non abelian case the Hamiltonian is still given by eq.(1) but eqs. (2) and (4) defining the magnetic strength H_i and the Gauss constraint Φ must be replaced by

$$H_i^a(A) = \frac{1}{2} \varepsilon_{ijk} \left[\partial_j A_k^a - \partial_k A_j^a + g f^{abc} A_j^b A_k^c \right] , \qquad (9)$$

$$\Phi^a = \mathscr{D}_k^{ab}(A) \, E_k^b + g j_o^a = 0 \ , \qquad (10)$$

where f^{abc} are the structure constants of the color group and $\mathscr{D}_k$ the covariant derivative in the adjoint representation

$$\mathcal{D}_k^{ab}(A) = \delta^{ab}\,\partial_k + gf^{abc}\,A_k^c\ . \tag{11}$$

Due to the nonlinearity of H_i and Φ the zmm are coupled to the other modes and we must explicitely define a gauge fixing to perform the quantization. In order to do it we need the following definitions

$$A_{i\vec{n}} = \begin{cases} B_{i\vec{n}} & \text{for } n_3\neq 0\ , \\ C_{i\vec{n}} & \text{for } n_2\neq 0,\ n_3=0, \\ D_{i\vec{n}} & \text{for } n_1\neq 0,\ n_2=n_3=0\ . \end{cases} \tag{12}$$

We will use the obvious notations

$$B_i = \frac{1}{L^{3/2}} \sum_{\vec{n}=0} B_{i\vec{n}} \exp i\,\frac{2\pi}{L}\,\vec{n}\cdot\vec{x}$$

and so on. We fix the gauge for non zero momentum modes by requiring

$$B_3 = C_2 = D_1 = 0\ . \tag{13}$$

We call this gauge the gauge $A_3 \sim 0$, because $A_{3\vec{n}}$ in the continuum limit is zero almost everywhere in momentum space, i.e. everywhere but on the surface $p_3 = 0$.

We define the variables F_i, G_i and H_i conjugate(*) to B_i, C_i and D_i

$$E_{i\vec{n}} = \begin{cases} F_{i\vec{n}} & \text{for } n_3\neq 0\ , \\ G_{i\vec{n}} & \text{for } n_2\neq 0,\ n_3=0\ , \\ H_{i\vec{n}} & \text{for } n_1\neq 0,\ n_2=n_3=0\ . \end{cases} \tag{14}$$

The variables F_3, G_2 and H_1 conjugate to B_3, C_2 and D_1 are obviously not independent.

Integrating Φ over the volume of the torus and using the gauge fixing we get

$$f^{abc} Q_k^c P_k^b + R^a = 0\ , \tag{15}$$

(*) There will be no occasion of confusion between this latter component of $\overline{E}_i$ and the magnetic strength H_i.

where

$$R^a = \int d^3x \left\{ j_o^a + f^{abc}\left[D_2^c H_2^b + D_3^c H_3^b + C_1^c G_1^b + C_3^c G_3^b + B_1^c F_1^b + B_2^c F_2^b \right] \right\} \quad . \tag{16}$$

We see that R contains only independent variables, so that the volume average of Φ is a constraint on zmm, decoupled from the constraints on the other modes. Such a decoupling does not occur in the Coulomb gauge, and it is the advantage of the gauge $A_3 \sim 0$.

Lüsher[5] has studied the Coulomb gauge. In this case it is convenient to introduce the Fourier transform of the Gauss constraint

$$\Phi^a_{\vec{n}} = i\,\frac{2\pi}{L}\, n_k E^a_{kn} + gf^{abc} L^{-3/2}\left[\sum_{\vec{m}\neq 0} A^b_{k\vec{m}} E^b_{k,\vec{n}-\vec{m}} + Q^c_k E^b_{k\vec{n}} + A^c_{k\vec{n}} P^b_k \right] + gj^a_{o\vec{n}} = 0, \qquad n\neq 0 \ , \tag{17}$$

$$\Phi^a_o = gf^{abc} L^{-3/2}\Big(Q^c_k P^b_k + \sum_{\vec{m}\neq 0} A^c_{k\vec{m}} E^b_{k,-\vec{m}}\Big) + gj^a_{oo} \ . \tag{18}$$

In the Coulomb gauge the longitudinal components of E^a_k are dependent variables and, unlike the gauge $A_3 \sim 0$, such dependent variables appear also in Φ^a_o. As a consequence the constraint for zmm does not decouple from the equations for the other modes, and the P^a_k appearing in eq.(17) should be reguarded as functions of the longitudinal components of E^a_k as determined by eq.(18). Lüsher has suggested to solve $\Phi_{\vec{n}}$ only for $\vec{n}\neq 0$, leaving a residual invariance of the Hamiltonian w.r. to spatially constant gauge transformations and the corresponding generator Φ^a_o as a constraint on the states. In his actual calculations, however, he does not take into account the term in the sum in eq.(18).

The explicit solution of the constraint for F_3, G_2 and M_1, will be reported somwerelse. We will only need here a property of the solution which will be mentioned later and can be verified almost by inspection. We need instead the solution of the constraint (15) for the zmm, and we will report it for the color group SU(2). In this case it is convenient to introduce new variables through the polar representation[10]

$$Q^a_i = f_{in}(\theta)\ \lambda_n h^a_n(\varphi)$$

where f and h are orthogonal matrices, which implies that the sum over n extends from 1 to 3. The new variables are three angles θ_t which parametrize f, three angles φ_a which parametrize h and the three λ_n's.

We can now express the constraints and the Hamiltonian in terms of the new variables. The constraint for zmm becomes

$$\varepsilon^{abc} Q^c_k P^b_k + R^a = \eta^{-1ab} P_b + R^a = 0 \tag{19}$$

where

$$\eta_{an} = -\frac{1}{2}\,\varepsilon_{nbc} h_{kb} \frac{\partial h^c_k}{\partial \varphi_a}$$

showing that the angles φ_a are pure gauge variables. We choose for these angles the gauge fixing

$$h^a_n = \delta^a_n \; . \tag{20}$$

We next perform the transformation for the different pieces of the Hamiltonian. We first rewrite each piece in terms of zmm and other modes. We have for the electric part

$$\int d^3x \, \frac{1}{2} E^a_k E^a_k = \frac{1}{2} P^a_k P^a_k + \frac{1}{2}\int d^3x \, \bar{E}^a_k \bar{E}^a_k \; . \tag{21}$$

The contribution of the zmm is[10]

$$\frac{1}{2} P^a_i P^a_i = \frac{1}{2} p^2_{\lambda_n} + \sum_{n<m} \frac{1}{(\lambda^2_n-\lambda^2_m)^2} \Big[(\lambda_n p_{\lambda_n} - \lambda_m p_{\lambda_m})(\lambda^2_n - \lambda^2_m) + \tag{22}$$

$$+ \frac{1}{4}(\lambda^2_m+\lambda^2_n)\,\varepsilon^2_{kmn}(l^2_k+L^2_k) + \lambda_m\lambda_n \varepsilon^2_{kmn} l_k L_k \Big] \; ,$$

where

$$L_k = -h_{km} R^m \; , \quad l_k = f_{mk} \xi^{-1}_{mt} P_{\theta_t} \; , \quad \xi_{mt} = -\frac{1}{2}\,\varepsilon_{cbc} f_{bn} \frac{\partial f_{cn}}{\partial \theta_m} \; . \tag{23}$$

In order to transform the magnetic energy it is convenient to use the following expression of the magnetic strength

$$H^a_i(A) = H^a_i(\bar{A}) + g\varepsilon^{abc}\varepsilon_{ilm} Q^b_l (\bar{A}^c_m + \frac{1}{2} Q^c_m)$$

in which the zmm are separated from the other modes. The magnetic energy becomes

$$\int d^3x \, \frac{1}{2} H^2(A) = W + \int d^3x \Big[H^2(\bar{A}) + g\varepsilon^{abc}\varepsilon_{ilm} Q^b_l \,(\bar{A}^c_m + \frac{1}{2} L^{-3/2} Q^c_m) \cdot \tag{24}$$

$$\cdot H^a_i(\bar{A}) + g^2 \varepsilon^{abc}\varepsilon_{ilm}\varepsilon^{ade}\varepsilon_{ijk} Q^b_l Q^d_j \bar{A}^c_m \bar{A}^e_k \Big]$$

where we have isolated for later convenience the term

$$W = \frac{1}{4} g^2 \varepsilon^{abc} \varepsilon_{ilm} \varepsilon^{ade} \varepsilon_{ijk} Q_l^b Q_m^c Q_j^d Q_k^e = \frac{1}{2} g^2 (\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2) \ . \tag{25}$$

Finally the interaction energy with a current is

$$\int g J_k^a A_k^a = g Q_k^a I_k^a + \int d^3x g \bar{J}_k^a \bar{A}_k^a = g \lambda_n f_{kn} I_k^n + \int d^3x \bar{J}_k^a \bar{A}_k^a \ . \tag{26}$$

The main difference concerning vacuum instability w.r. to the abelian case is the presence of higher powers of the Q_k^a in the pure gauge part of the Hamiltonian. Since the coupling to the current remains linear in the Q_k^a, we must show that the higher powers do not avoid the instability. In order to do it we use the trial state functional

$$\Psi = \frac{\sqrt{\bar{\lambda}}}{\pi^{3/4}\lambda_o^2} \exp\left(-\lambda \frac{\lambda_1^2+\lambda_2^2}{2\lambda_o^3}\right) \exp\left[-\frac{(\lambda_3-\bar{\lambda})^2}{2\lambda_o^2}\right] F(\lambda_1,\lambda_2,\lambda_3)\chi \ . \tag{27}$$

In the above equation F is a correlation function which prevents a divergence in $\frac{1}{2} P_k^a P_k^a$ when $\lambda_i = \lambda_j$, for instance

$$F = \prod_{i<j} \left\{ 1 - \exp\left[-\frac{(\lambda_i^2 - \lambda_j^2)^2}{\lambda_o^2} \right] \right\} ,$$

while χ is such that

$$\langle \chi | I_k^a | \chi \rangle = \delta^{a3} \delta_{k3} I \ . \tag{28}$$

As a consequence for $\bar{\lambda} \to \infty$

$$\left\langle \Psi \middle| \int d^3x \; g J_k^a A_k^a \middle| \Psi \right\rangle \to C_1 \bar{\lambda} \ , \qquad C_1 \text{ a constant} \ ,$$

showing that the coupling to the current behaves as in the abelian case.

Let us now consider the new terms. It is immediate to check that those containing only zmm do not grow faster than $\bar{\lambda}$

$$\left\langle \Psi \middle| \frac{1}{2} P_k^a P_k^a \middle| \Psi \right\rangle \to C_2 \bar{\lambda} \ , \qquad \langle \Psi | W | \Psi \rangle \to C_3 \bar{\lambda} \ . \tag{29}$$

In order to estimate the terms involving also the other modes we perform the transformation

$$\bar{A} \to \frac{1}{\sqrt{\bar{\lambda}}} \bar{A} \ , \qquad \bar{E} \to \sqrt{\bar{\lambda}} \bar{E} \tag{30}$$

which should be understood to hold only for the independent components. Therefore B_3, C_2 and D_1 remain zero, while F_3, G_2 and H_1 are expressed in terms of the independent components rescaled according to (30). It can be checked that also the dependent components of $\bar{E}_k$ do not grow faster than $\sqrt{\bar{\lambda}}$ as the independent ones. Using the rescaled variables we find that there is no term in the Hamiltonian which grows faster than $\bar{\lambda}$

$$\langle \psi | H | \psi \rangle \rightarrow (K + C_1)\bar{\lambda}\,, \qquad K \text{ a constant} \tag{31}$$

Since we can make $K+C_1 < 0$ by choosing I in eq.(28) large enough and $gI < 0$, the Hamiltonian is not bounded from below.

The same considerations made in the abelian case concerning the coupling to bosonic matter and Susy apply obviously also to the non abelian case.

We should note that the unboundedness is due to the existence of an average value of the fermion current over the quantization volume. If there is color confinement such an expectation value must be zero. This situation is best illustrated by the Schwinger model[8] where the same infrared instability would occur, but we know, because the model has been solved, that the current average vanishes. Therefore in the non abelian case the present instability might be an obstacle in the way of perturbation theory just because related to another non-perturbative feature, namely color confinement.

It is perhaps worth while noticing that the Hamiltonian of gauge theories on a lattice, due to the compactness of the gauge variables, is bounded from below. This raises a problem in the way of identifying the thermodynamic limit of gauge theories on a lattice with continuum gauge theories, and frustates one of our motivations for studying gauge theories on a torus, namely the expected ease to compare the continuum to the lattice.

4.- zmm AND THE WITTEN INDEX

As it is known Susy is broken if and only if the vacuum energy is different from zero. Since in general the vacuum energy cannot be evaluated exactely, it is difficult to establish whether Susy is or is not broken.

Witten[11] has observed that such a difficulty can be at least in part circumvented, giving a sufficient condition for Susy not to be broken which does not require the exact evaluation of the vacuum energy. The condition is that the difference between the number of bosonic and fermionic modes of zero energy, the Witten index, be different from zero. The Witten index is not expected to depend on the approximations used in the evaluation of the energy so that it can be cal culated easily and reliably in many cases and constitutes a powerful test for Susy breaking.

The Witten index is well defined only if the energy spectrum is discrete. In order to realize this condition, one takes advantage of the expected independence of the index on the parameters of the theory, quantizing in a finite box and keeping the volume finite. The basic assumption is that the energy spectrum in a finite box is discrete, and that the lowest energy states can be constructed in terms of the zero-momentum modes of the fields. In order to select these modes one can impose the constraints

$$F^a_{ij} = \bar{\lambda}\gamma^k \mathscr{D}_k\lambda = 0 \quad , \tag{32}$$

where λ is the gaugino field. These constraints are gauge invariant but not Lorentz invariant, and one can wonder what is the meaning of using an approximation which breaking Lorentz invariance also breaks Susy in the study of spontaneous Susy breaking.

The answer is that with the above constraints one obtains a Galilean invariant supersymmetric theory. The restriction to Galilean modes implyies a contraction of the Susy algebra to a Galilean Susy, and the Witten index should be the same in the two cases.

In order to show the Galilean invariance of the constrained theory we observe that the relativistic Lagrangian

$$\mathscr{L} = -\frac{1}{4} F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2} i\bar{\lambda}^a\gamma^\mu \mathscr{D}^{ab}\lambda^b + \frac{1}{2} D^a D^a \tag{33}$$

becomes, using the constraints

$$\mathscr{L}_G = -\frac{1}{2} F^a_{oi}F^{aoi} - \frac{1}{2} i\bar{\lambda}^a\gamma^o \mathscr{D}^{ab}_o\lambda^b + \frac{1}{2} D^a D^a \ . \tag{34}$$

Under Galilean transformations(12) of parameters v_k

$$A_i \to A_i \ , \qquad A_o \to A_o + v_k A_k \ , \qquad \lambda \to \lambda \quad , \tag{35}$$

so that

$$F_{ij} \to F_{ij} \ , \qquad F_{oi} \to F_{oi} + v_k F_{ki} \tag{36}$$

and $\mathscr{L}_G$ is Galilei-invariant under the constraints (32).

The Hamiltonian corresponding to $\mathscr{L}_G$ is

$$H_G = \int d^3x \left\{ \frac{1}{2} E^a_i E^a_i - A^a_o \Phi^a + \varphi^a_{1i} \Phi^{ai}_1 + \varphi^a_2 \Phi^a_2 + \varphi^a_3 \Phi^a_3 \right\} \tag{37}$$

where E_i^a are the momenta canonically conjugate to A_i^a, φ_{ki}^a are Lagrange multipliers and

$$\Phi^a = \mathscr{D}_i^{ab} E_i^b - \frac{1}{2} igf^{abc} \lambda^{b*} \lambda^c , \qquad \Phi_{1i}^a = \varepsilon_{ijk} F_{jk}^a ,$$
$$\Phi_2^a = \sigma^k \sigma^2 \mathscr{D}_k^{ab} \chi^{b*} , \qquad \Phi_3^a = \sigma^k \mathscr{D}_k^{ab} \chi^b . \tag{38}$$

are the constraints. Φ^a is the Gauss constraint, Φ_{1i}^a is the first of the primary constraints (32), while Φ_2^a and Φ_3^a are obtained from the second of the primary constraints (32) by putting

$$\lambda = \begin{bmatrix} \chi \\ \sigma^2 \chi^* \end{bmatrix} . \tag{39}$$

The constraints Φ^a, Φ_2^a and Φ_3^a commute with the Hamiltonian, but Φ_1^a does not, which means that if it is satisfied at some time it will no longer hold at a later time. By applying the Dirac theory(13) of constrained systems we must impose the vanishing of $[H, \Phi_{1i}^a]$, which yields the secondary constraint

$$\Phi_{4i}^a = \varepsilon_{ijk} \mathscr{D}_j^{ab} E_k^b = 0 . \tag{40}$$

The general solution(14) of all the constraints with the gauge fixing (13) for SU(2) contains only zmm

$$A_i^a = L^{-3/2} Q_i \hat{V}^a , \qquad E_i^a = L^{-3/2} (P_i \hat{V}^a + Q^{-2} Q_i 1_\perp^a) ,$$
$$\chi_\alpha^a = L^{-3/2} \hat{V}^a \xi_\alpha , \qquad \chi_\alpha^{a*} = L^{-3/2} \hat{V}^a \xi_\alpha^* , \tag{41}$$

where

$$1_\perp^a = (\pi^a - \hat{V}^a \hat{V}^b \pi^b) V , \qquad [Q_i , P_j] = \delta_{ij} , \qquad \hat{V}^a = \frac{V^a}{V} ,$$
$$[V^a , \pi^b] = \delta^{ab} , \qquad \{ \xi_\alpha , \xi_\beta^* \} = \delta_{\alpha\beta} . \tag{42}$$

The gauge fixing(13) does not act on the zmm and correspondingly the zero momentum component of Φ^a, eq.(15) is left as a condition on the physical states Ψ. With the above solutions such condition takes the form

$$\varepsilon^{abc} V^b \pi^c \Psi = 0 . \tag{43}$$

Finally the Hamiltonian is

$$H = \frac{1}{2} P^2 + \frac{1}{2} l_\perp^2 \frac{1}{Q^2} \qquad (44)$$

and has a continuum spectrum which makes the Witten index ill defined.

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