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THE WITTEN INDEX IN SUPERSYMMETRIC GAUGE THEORIES

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Difficulties in the calculation of the Witten index in gauge theories are pointed out. Such difficulties are due to ambiguities in the identification of *all* the zero energy states and to the fact that some of these states belong to a continuous spectrum.

1. SUSY is broken if and only if the vacuum energy is different from zero. Since in general the vacuum energy cannot be evaluated exactly, it is difficult to establish whether SUSY is or is not broken.

Witten has observed that such a difficulty can be at least in part circumvented, giving a sufficient condition for SUSY not to be broken which does not require the exact evaluation of the vacuum energy [1]. The condition is that the difference between the number of bosonic and fermionic modes of zero energy, the Witten index, be different from zero. The Witten index is not expected to depend on the approximations used in the evaluation of the energy so that it can be calculated easily and reliably in many cases and constitutes a powerful test for SUSY breaking.

The Witten index is well defined only if the energy spectrum is discrete. In order to realize this condition, one takes advantage of the expected independence of the index on the parameters, quantizing in a finite box and keeping the volume finite. The basic assumption is that the energy spectrum in a finite box is discrete, and that the lowest energy states can be constructed in terms of the zero momentum modes of the fields.

It turns out that this assumption is, in general, unjustified for gauge theories. Moreover, the notion of zero momentum mode of the gauge fields is not gauge invariant, with the exception of the abelian case, and this fact gives rise to ambiguities in the identification of *all* the zero energy states, which is necessary to calculate the Witten index.

We will discuss these points for the SU(2) colour group with periodic and twisted boundary conditions (b.c.) [2].

2. To establish the notation, let us introduce the fields of the gauge multiplet in the Wess–Zumino gauge, A^a_μ, λ^a, D^a, with Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \tfrac{1}{2}\mathrm{i}\bar{\lambda}^a\gamma^\mu\mathcal{D}^{ab}_\mu\lambda^b + \tfrac{1}{2}D^aD^a .$$

In the above equation

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu ,$$

$$\mathcal{D}^{ab}_\mu = \delta^{ab}\partial_\mu - g\epsilon^{abc}A^c_\mu , \quad \lambda^a = \begin{pmatrix} \chi^a \\ \sigma^2\chi^{a*} \end{pmatrix} . \tag{1}$$

Keeping only the zero momentum modes of the gauge fields, the lagrangian becomes [3]

$$\begin{aligned}\mathcal{L} = {} & \tfrac{1}{2}(\mathcal{D}_0A^k)^a(\mathcal{D}_0A_k)^a + \mathrm{i}\chi^{a*}\mathcal{D}^{ab}_0\chi^b \\ & + \tfrac{1}{2}D^aD^a - \tfrac{1}{4}g^2(\epsilon^{abc}A^b_iA^c_j)^2 \\ & - \tfrac{1}{2}\mathrm{i}g\epsilon^{abc}A^b_k(\chi^{a*}\sigma^k\sigma^2\chi^{c*} + \chi^a\sigma^2\sigma^k\chi^c) .\end{aligned} \tag{2}$$

This lagrangian can be quantized without difficulty and the resulting hamiltonian is expected to have a discrete spectrum.

The lagrangian (2) can be formally obtained by imposing the constraints

$$\partial_kA^a_i = \partial_k\lambda^a = \partial_kD^a = 0 . \tag{3}$$

These constraints are not gauge invariant, and dealing with gauge theories it is preferable to have gauge invariant constraints, for instance

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$$F^a_{ij} = 0\ , \quad \bar{\lambda}\gamma^k \mathcal{D}_k \lambda = 0\ . \tag{4}$$

With these constraints, the gauge field lagrangian becomes

$$\mathcal{L} = \tfrac{1}{2} F^a_{0i} F^{a0i} - \tfrac{1}{2} \mathrm{i} \chi^{a*} \mathcal{D}^{ab}_0 \chi^b + \tfrac{1}{2} D^a D^a\ . \tag{5}$$

Witten [1] has considered a different class of constraints, namely

$$F^a_{ij} = \gamma^k \mathcal{D}_k \lambda = 0\ , \tag{6}$$

which, of course, also lead to the lagrangian (5).

Now there is no reason to assume that the zero energy states of the hamiltonians corresponding to constraints (3), (4) and (6) lie in complementary sectors of the full Hilbert space, and therefore we must not include all of them in the calculation of the Witten index. The first problem is therefore how to choose between these possibilities.

The above examples are illustrative of ambiguities which are inherent to the problem of a low approximation to gauge theories. It is natural to require that such an approximation be Galilei invariant. This prescription does not lead to a unique result, however, and in fact the lagrangians (2) and (5) and the constraints (3), (4) and (6) are all Galilei invariant. But if the galilean lagrangian is to be obtained by a suitable limit from the relativistic one, only the constraints (3) and (4) are generated [‡1]. The ambiguity can be further reduced or eliminated by requiring the conservation of all the original symmetries other than Lorentz invariance and SUSY which should contract into Galilei invariance and galilean SUSY. Now full gauge invariance is conserved in the lagrangian (5) and the constraints (4) and (6), while only gauge invariance with respect to x independent transformations is conserved in the lagrangian (2) and the constraints (3). The galilean contraction of SUSY has been discussed only for the lagrangian (2), and it would be interesting to examine it for the lagrangian (5) in relation to both constraints (4) and (6). We will not do it here, but we will confine ourselves to showing that the lagrangian (5) with either constraint, leads to a hamiltonian with continuous spectrum. We will do this in detail for the constraints (4), and we state without proof that there is no difference, as far as the energy spectrum is concerned, if the constraints (6) are assumed.

3. In order to quantize the lagrangian (5), it is convenient to write it in first order formulation

$$\mathcal{L}_{\mathrm{G}} = -\tfrac{1}{2} E^a_i E^{ai} + E_i \partial_t A^i + \mathrm{i}\chi^{a*}\partial_t \chi^a + \tfrac{1}{2} D^a D^a + A^a_0 \Phi^a\ . \tag{7}$$

In the above equation, Φ^a are the gauge constraints

$$\Phi^a = \mathcal{D}^{ab}_i E^{bi} - \tfrac{1}{2} \mathrm{i} g \epsilon^{abc} \chi^{b*} \chi^c\ . \tag{8}$$

The dynamical variables are A^a_i and χ^a_α defined in eq. (1) with canonical momenta $E^a_i = \partial_t A^a_i$ and $\mathrm{i}\chi^{a*}_\alpha$ satisfying the Poisson brackets

$$\{A^a_i(x), E^b_j(y)\}_- = \delta^{ab}\delta_{ij}\delta^3(x-y)\ ,$$
$$\{\chi^a_\alpha(x), \chi^{b*}_\beta(y)\}_+ = -\mathrm{i}\delta^{ab}\delta_{\alpha\beta}\delta^3(x-y)\ . \tag{9}$$

The hamiltonian is

$$H = \int \mathrm{d}^3x \{\tfrac{1}{2} E^a_i E^{ai} - A^a_0 \Phi^a + \varphi^a_{1i}\Phi^{ai}_1 + \varphi^a_2 \Phi^a_2 + \varphi^a_3 \Phi^a_3\}\ , \tag{10}$$

where

$$\Phi^a_{1i} = \epsilon_{ijk} F^a_{jk}\ ,$$
$$\Phi^a_2 = \sigma^k \sigma^2 \mathcal{D}^{ab}_k \chi^{b*}\ , \quad \Phi^a_3 = \sigma^k \mathcal{D}^{ab}_k \chi^b\ , \tag{11}$$

are the constraints (6) and $\varphi^a_{1i}, \varphi^a_2, \varphi^a_3$ are Lagrange multipliers.

The gauge constraints Φ^a commute with the hamiltonian, but the constraints (11) do not, which means that if they are satisfied at some time they will no longer hold at some later time. By applying the Dirac theory of canonical quantization of constrained systems [4] [‡2] we must then impose the vanishing of the commutators of the hamiltonian with these constraints.

The vanishing of $[H, \Phi^a_{1i}]$ generates the secondary constraints [6]

$$\Phi^a_{4i} = \epsilon_{ijk} \mathcal{D}^{ab}_j E^b_k = 0\ , \tag{12}$$

while the vanishing of $[H, \Phi^a_2]$ and $[H, \Phi^a_3]$ gives con-

‡1 The constraints arise from an appropriate rescaling [3] of the fields with the velocity of light c. Under such a rescaling $\bar{\lambda}\gamma^\mu D_\mu \lambda \to c\bar{\lambda}\gamma^\mu D_\mu \lambda$. In the limit of infinite c, depending on the rescaling with c assumed for the coupling constant, we get either $\bar{\lambda}\gamma^\kappa \partial_\kappa \lambda = 0$ or $\bar{\lambda}\gamma^\kappa D_\kappa \lambda = 0$. The first constraint is equivalent to $\partial_\kappa \lambda = 0$, but the second is not equivalent to $\bar{\lambda}\gamma^\kappa D_\kappa \lambda = 0$.

‡2 For a review see ref. [5].

ditions on the Lagrange multipliers but does not give rise to new constraints. Also the vanishing of $[H, \Phi^a_{4i}]$ does not give rise to new constraints. The gauge constraints commute weakly with all the constraints and strongly with the hamiltonian, so that they are first class. We can therefore apply the quantization procedure of ref. [4], and using the recursive property of the Dirac brackets we first solve the constraints Φ^a, Φ^a_{1i}, Φ^a_{4i} with the appropriate gauge fixing, getting

$$A^a_i = L^{-3/2} q_i \hat{v}^a\,, \quad \hat{v}^a = v^a \Big(\sum_b v^b v^b \Big)^{-1/2}\,,$$
$$E^a_i = L^{-3/2}(p_i \hat{v}^a + q^{-2} q_i l^a_\perp)\,, \tag{13}$$

where

$$l^a_\perp = (\pi^a - \hat{v}^a \hat{v}^b \pi^b) v\,, \quad l^a_\perp \hat{v}^a = 0\,. \tag{14}$$

The variables q_i, p_i, v^a, π^a satisfy the following commutation relations

$$\{q_i, p_j\}_- = \delta_{ij}\,, \quad \{v^a, \pi^b\}_- = \delta^{ab}\,, \tag{15}$$

all the other commutators vanishing.

Finally, we can write the hamiltonian

$$H = \tfrac{1}{2} p^2 + \tfrac{1}{2} l^2_\perp / q^2\,, \tag{16}$$

which does not contain spinor terms.

Due to the particular procedure adopted in the quantization ‡3, however, physical states are subject to the constraint $\Phi^a_0 \Psi = 0$, Φ^a_0 being the zero momentum component of the gauge field constraint, which contains a spinor term. We must then solve also the spinor constraints. To do this, we make the following expansion

$$\chi^a_\alpha = \xi_\alpha \hat{v}^a + \eta_\alpha \hat{l}^a_\perp + \theta_\alpha \hat{r}^a\,, \tag{17}$$

where $\hat{r}$ is orthogonal to $\hat{v}$ and $\hat{l}_\perp$. Using this expansion in eq. (11), we get

$$\sigma^k \partial_k \xi = 0\,,$$
$$\sigma^k \partial_k \eta + g L^{-3/2} \sigma^k q_k \theta = 0\,,$$
$$\sigma^k \partial_k \theta - g L^{-3/2} \sigma^k q_k \eta = 0\,. \tag{18}$$

The last two equations are equivalent to

$$\Delta \eta + g^2 L^{-3} q^2 \eta = 0\,,$$

which, due to periodic b.c. and the arbitrary value of q, imply $\eta = 0$. It then follows that $\theta = 0$, ξ is independent of $\boldsymbol{x}$ and finally

$$\chi^a_\alpha = L^{-3/2} \hat{v}^a \xi_\alpha\,, \quad \chi^{a*}_\alpha = L^{-3/2} \hat{v}^a \xi^*_\alpha\,. \tag{19}$$

In order to evaluate the Dirac brackets for ξ_α we replace the constraints (16) by

$$\Xi_{i\alpha} = \hat{w}^a_i \chi^a_\alpha = 0\,, \quad \Theta_{i\alpha} = \hat{w}^a_i \chi^{a*}_\alpha = 0\,, \tag{20}$$

where $\hat{w}^a_1 = \hat{r}^a$, $\hat{w}^a_2 = \hat{l}^a_\perp$.

These constraints are equivalent to (19) as far as the zero momentum mode of χ^a is concerned. The Poisson matrix of the constraints (20) is

$$\Delta = -\mathrm{i} \left[\begin{array}{cc|cc} & & \delta_{\alpha\beta} & 0 \\ & & 0 & \delta_{\alpha\beta} \\ \hline \delta_{\alpha\beta} & 0 & & \\ 0 & \delta_{\alpha\beta} & & \end{array} \right].$$

The Dirac brackets are

$$\{\chi^a_\alpha, \chi^b_\beta\}^*_+ = \{\chi^{a*}_\alpha, \chi^{b*}_\beta\}^*_- = 0\,,$$
$$\{\chi^a_\alpha, \chi^{b*}_\beta\}^*_+ = \{\xi_\alpha, \xi^*_\beta\}_+ \hat{v}^a \hat{v}^b$$
$$= -\mathrm{i}\delta^{ab}\delta_{\alpha\beta} - \sum_{h,k=1,2} \{\chi^a_\alpha, \Theta_{k\gamma}\}_+ \Delta^{-1}_{k\gamma,h\delta} \{\Xi_{k\delta}, \chi^{b*}_\beta\}_+$$
$$= -\mathrm{i}\delta^{ab}\delta_{\alpha\beta} + \mathrm{i}(\hat{r}^a \hat{r}^b + \hat{l}^a_\perp \hat{l}^b_\perp)\delta_{\alpha\beta}\,,$$

and are consistent with the Poisson brackets

$$\{\xi_\alpha, \xi_\beta\}_+ = \{\xi^*_\alpha, \xi^*_\beta\}_+ = 0\,, \quad \{\xi_\alpha, \xi^*_\beta\}_+ = -\mathrm{i}\delta_{\alpha\beta}\,. \tag{21}$$

The spinor term therefore does not contribute to the constraint on physical states which becomes simply $\epsilon^{abc} v^b \pi^c \Psi = 0$ [with the constraints (4) there would be a spinor contribution].

The variables q_κ are not compact and as a consequence the hamiltonian (16) has a continuous spectrum which makes the Witten index ill defined.

Considering the constraints (3) and (6) at the same time [1] would not change this result, because A^a_i and E^a_i given by eq. (13) do not depend on $\boldsymbol{x}$, so that the constraints (3) are automatically satisfied once the constraints (6) are.

4. Twisted b.c. are incompatible with the constraints (3). It seems therefore that they help to avoid

‡3 In ref. [6] gauge fixing constraints have been imposed only for the components of the gauge constraints ϕ^a of momentum different from zero, and the zero momentum component ϕ^a_0 has been left as a constraint the physical states.

the problem of the zero momentum mode of the gauge fields. To calculate the Witten index one simply counts the number of inequivalent vacua [1], assuming that these vacua belong to a discrete spectrum because of the quantization in a finite box. This unproven assumption is most likely not verified.

Twisted b.c. are in fact not incompatible with the constraints (4) or (6), as shown by the explicit construction of a class of solutions

$$A_i^a = \hat{v}^a \partial_i \left((x_3/L - \tfrac{1}{2}) \sum_{n_1 n_2} q_{n_1 n_2} \right.$$

$$\left. \times \cos((2\pi/L) n_1 x_1) \cos((\pi/L)(2n_2+1) x_2) \right), \quad (22)$$

which satisfy the gauge fixing conditions of ref. [4] and the b.c.

$$A_k(0, x_2, x_3) = A_k(L, x_2, x_3) ,$$
$$A_k(x_1, 0, x_3) = \sigma^2 A_k(x_1, L, x_3) \sigma^2 ,$$
$$A_k(x_1, x_2, 0) = \sigma^3 A_k(x_1, x_2, L) \sigma^3 . \quad (23)$$

Finding the most general solution and performing the quantization is outside the scope of the present paper. It is sufficient for us that the dynamical variables $q_{n_1 n_2}$ are non-compact and therefore the hamiltonian is not expected to have a discrete spectrum. Showing that the spectrum is discrete is, however, necessary before the Witten index can be calculated.

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