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ORDERING AMBIGUITIES IN SUPERSYMMETRIC GAUGE THEORIES

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It is pointed out that the gauge constraints give rise to two types of ordering ambiguities in supersymmetric gauge theories: the first in the constraints themselves, the second in the spinorial generators after use of the constraints. The effects of these ambiguities are illustrated in a quantum-mechanical model.

1. It is generally assumed that there are no ambiguities in the ordering of non-commuting operators in supersymmetric theories. This is due to the fact that there are no such ambiguities in the spinorial generators and therefore in the hamiltonian.

Gauge theories, however, are characterized by constraints, and this gives rise to two types of ordering ambiguities; one in the constraints themselves and the other in the spinorial generators after use of the constraints.

We will illustrate the effects of these ambiguities in a quantum mechanical model which is the galilean limit [1] of the relativistic model of a gauge multiplet coupled to a massless Wess–Zumino multiplet.

In this example the two types of ambiguities have quite different effects. The ambiguities in the spinorial generators break the supersymmetry (SUSY) algebra, while the ambiguities in the constraints do not break the algebra but affect the existence of a supersymmetric ground state. The motivation for studying the quantum mechanical problem instead of its relativistic parent is just the possibility to investigate this latter effect. An exhaustive analysis requires, however, the complete solution of the model. We have only been able to show that there exists at most one ordering for which a supersymmetric ground state can exist.

2. The lagrangian of the model is

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_M \,, \qquad (1)$$

$$\mathcal{L}_G = \tfrac{1}{2}(\partial_t A_k)^2 + i\lambda^* \partial_t \lambda + \tfrac{1}{2}D^2 + xD \,, \qquad (2)$$

$$\mathcal{L}_M = |\mathcal{D}_t \varphi|^2 + i\xi^* \mathcal{D}_t \xi + |h|^2 - g^2 A_k^2 |\varphi|^2 + gA_k \xi^* \sigma_k \xi + ig\sqrt{2}(\varphi^* \xi \eta^* + \varphi \xi^* \eta) + g|\varphi|^2 D^2 . \qquad (3)$$

The meaning of the symbols appearing in the above equations is the usual one, and is specified in ref. [1]. Let us however explicitly mention that $D_t = \partial_t + igV$.

The lagrangian (1) is invariant under chiral transformations

$$\varphi \to \varphi\, e^{-i\alpha} \,, \quad \xi \to \xi\, e^{-i\alpha/2} \,, \quad \eta \to \eta\, e^{i\alpha/2} \,, \qquad (4)$$

and gauge and SUSY transformations given in ref. [1] where it is shown how the present model has been obtained by performing the limit of infinite light velocity in the relativistic model. There are different ways to perform this limit, and the procedure leading to the lagrangian (1) explicitly breaks gauge invariance with respect to gauge transformations whose parameters depend on the spatial variables. For this reason we do not want to stress the heuristic relevance of the model to its relativistic parent.

It is convenient to use a first-order formulation for the matter field lagrangian introducing the field π.

$$\mathcal{L}_M = -|\pi|^2 + \pi\mathcal{D}_t\varphi + \pi^*\mathcal{D}_t^*\varphi^* + i\xi^*\mathcal{D}_t\xi + |h|^2 - g^2A_k^2|\varphi|^2 + gA_k\xi^*\sigma_k\xi + g|\varphi|^2 D + ig\sqrt{2}(\varphi^*\xi\eta^* + \varphi\xi^*\eta) . \quad (5)$$

In this way the canonical variables are $\xi, \eta, A_k, \varphi, \varphi^*$ with canonical momenta $\xi^*, \eta^*, p_k = \partial_t A_k, \pi, \pi^*$, respectively. The variation of the lagrangian with respect to V originates the gauge constraint

$$-i(\pi\varphi - \pi^*\varphi^*) + \xi^*\xi = 0 . \quad (6)$$

We examine now the two canonical quantization procedures [2] [‡1]. In the first one we impose the constraint (6) on physical states. The dynamical variables are therefore unconstrained, and we can verify that the SUSY algebra holds at the quantum level

$$Q = (-iP_k\sigma_k - x - g|\varphi|^2)\, 2^{-1/2}(\lambda + \sigma_2\lambda^*) + (-\pi + ig\varphi^* A_k\sigma_k)\,\xi + (-\pi^* + ig\varphi A_k\sigma_k)\,\sigma_2\xi^* , \quad (7)$$

$$\tfrac{1}{2}\{Q_\alpha^+, Q_\beta\} = \delta_{\alpha\beta}H = [\tfrac{1}{2}P^2 + \pi^*\pi + \tfrac{1}{2}(x + g|\varphi|^2)^2 + g^2|\varphi|^2A^2 - i\sqrt{2}g(\varphi^*\xi\eta^* + \varphi\xi^*\eta) - gA_k\xi^*\sigma_k\xi]\,\delta_{\alpha\beta} . \quad (8)$$

There is an ambiguity, however, in the constraint, due to the non-commutativity of π with φ, π^* with φ^*, and ξ with ξ^*. This ambiguity can be accounted for by writing the constraint in quantum form as

$$[-i(\pi\varphi - \pi^*\varphi^*) + \xi^*\xi + m]\,\Psi = 0 , \quad (9)$$

where m is an integer. We anticipate that SUSY is unbroken unless $m = -1$.

In the second quantization procedure we introduce a gauge fixing and use eq. (6) as an equation for the dynamical variables. It is convenient to introduce polar coordinates

$$\varphi = 2^{-1/2} r\, e^{-i\theta} , \quad \varphi^* = 2^{-1/2} r\, e^{i\theta} , \quad (10)$$

$$\pi = 2^{-1/2}\, e^{i\theta}\, [\pi_r + (i/r) M] ,$$

$$\pi^* = 2^{-1/2}\, e^{-i\theta}\, [\pi_r - (i/r) M] , \quad (11)$$

with

$$\pi_r = -i\partial/\partial r , \quad M = -i\partial/\partial\theta . \quad (12)$$

‡1 For a review see ref. [3].

The natural choice for the gauge fixing is therefore $\theta = 0$, and the constraint (6) allows the elimination of M

$$M = -(\xi^*\xi + m') , \quad (13)$$

where the integer m' accounts for the ambiguity in the ordering of ξ and ξ^*. We see that now π and π^* no longer commute with ξ and ξ^*, and this gives rise to an ordering ambiguity in the spinorial generators

$$Q = 2^{-1/2}\{(-iP_k\sigma_k - x - \tfrac{1}{2}gr^2)(\eta + \sigma_2\eta^*) + (-\pi_r + igrA_k\sigma_k)\,\xi + (-\pi_r + igrA_k\sigma_k)\,\sigma_2\xi^* + (i/r)(\xi - \sigma_2\xi^*)(\xi^*\xi + m') - a(i/r)[\xi - \sigma_2\xi^*, \xi^*\xi + m']\} ,$$

$$Q^+ = 2^{-1/2}\{(-iP_k\sigma_k - x - \tfrac{1}{2}gr^2)^*(\eta^* - \sigma_2\eta) - (-\pi_r + igrA_k\sigma_k)^*\xi^* - (-\pi_r + igrA_k\sigma_k)\,\sigma_2\xi - (i/r)(\xi^* + \sigma_2\xi)(\xi^*\xi + m') + b(i/r)[\xi^* + \sigma_2\xi, \xi^*\xi + m']\} . \quad (14)$$

In the above equations $a, b = 0, 1$ according to the ordering chosen, and we have assumed the same ordering for the products $\xi\pi$ and $\xi^*\pi^*$, i.e. either $\xi\pi$, $\xi^*\pi^*$, or $\pi\xi$, $\pi^*\xi^*$, but we have not considered the cases $\xi\pi$, $\pi^*\xi^*$ and $\pi\xi$, $\xi^*\pi^*$ for simplicity.

We now have

$$\tfrac{1}{2}\{Q_\alpha^+, Q_\beta\} = H'\delta_{\alpha\beta} + \tfrac{1}{2}\{ab/r^2 + i[(Q/r)\,\pi_r - \pi_r b/r] + (a/r^2)(1 - \chi^*\chi)\}\,\delta_{\alpha\beta} - \tfrac{1}{2}[(b/r^2)\,\chi^*\sigma_k\chi + g(a+b)\,A_k](\sigma_k)_{\beta\alpha} , \quad (15)$$

where $\chi = (2^{-1/2})(\xi + \sigma_2\xi^*)$ and

$$H' = \tfrac{1}{2}P^2 + \tfrac{1}{2}\pi_r^2 - \tfrac{1}{2}ir^{-1}\pi_r + \tfrac{1}{2}r^{-2}(\xi^*\xi + m')^2 + \tfrac{1}{2}(x + \tfrac{1}{2}gr^2)^2 + \tfrac{1}{2}gr^2A^2 - igr(\xi\eta^* + \xi^*\eta) - gA_k\xi^*\sigma^k\xi . \quad (16)$$

We see that the SUSY algebra is broken unless $a = b = 0$. Note that this implies that Q contains the term $(i/r)(\xi - \sigma_2\xi^*)(\xi^*\xi + m')$ and Q^+ the term $-(i/r) \times (\xi^* + \sigma_2\xi)(\xi^*\xi + m')$ rather than $-(i/r)(\xi^*\xi + m') \times (\xi^* + \sigma_2\xi)$ as expected.

If we introduce polar coordinates in H and use the fact that for physical states $M\psi = -(\xi^*\xi + m)\,\psi$ we see that H and H' are identical if we take $m = m'$.

3. Let us now show that a supersymmetric ground state cannot exist for $m \neq -1$. The problem is a little bit simplified by the fact that the chiral transformations commute with the SUSY transformations, and therefore we can look for a state of given chirality. The generator of chiral transformations is

$$q = -M + \tfrac{1}{2}\eta^*\eta - \tfrac{1}{2}\xi^*\xi\,, \qquad (17)$$

ignoring ordering ambiguities which are irrelevant for the present issue. We look for a state of chirality ν:

$$q \cdot \Psi = \gamma\Psi\,. \qquad (18)$$

Using the constraint (13) (with $m' = m$) we can replace the above equation by

$$(\xi^*\xi + \eta^*\eta)\,\Psi = 2(\nu - m)\,\Psi\,, \qquad (19)$$

stating that physical states of definite chirality have a definite number of fermions.

We state without proof that there is no supersymmetric ground state with zero or four fermions, which is the maximum number of fermions we can have with only two fermionic operators η and ξ. A state with two fermions must have the form

$$\Psi = [\tfrac{1}{2}\psi_1\eta^*\sigma_2\eta^* + 2^{-1/2}\psi_2\xi^*\sigma_2\eta^* + 2^{-1/2}\psi_3\xi^*\sigma_k\sigma_2\eta^*A_k + \tfrac{1}{2}\psi_4\xi^*\sigma_2\xi^*]\,|0\rangle\,, \qquad (20)$$

where $\psi_i = \psi_i(|A|, r, \theta)$. The θ-dependence is determined by the constraint (13)

$$\psi_1 = \mathrm{e}^{-\mathrm{i}m\theta}\varphi_1(A,r)\,,\quad \psi_2 = \mathrm{e}^{-\mathrm{i}(m+1)\theta}\varphi_2(A,r)\,,$$
$$\psi_3 = \mathrm{e}^{-\mathrm{i}(m+1)\theta}\varphi_3(A,r)\,,\quad \psi_4 = \mathrm{e}^{-\mathrm{i}(m+2)\theta}\varphi_4(A,r)\,, \qquad (21)$$

and as a result such a state has chirality $m + 1$.

We must now solve the equations $Q_\alpha\psi = Q_\alpha^+\psi = 0$. This is a set of 32 partial differential equations for the φ_i, which can be reduced to the following ones

$$\varphi_4 = -\varphi_1\,,$$
$$[(m+1)/r]\,\varphi_1 = [(m+1)/r]\,\varphi_2 = [(m+1)/r]\,\varphi_3 = 0\,, \qquad (22)$$

$$\mathrm{i}\sqrt{2}gr\varphi_1 + A^{-1}\partial\varphi_2/\partial A - (x + \tfrac{1}{2}gr^2)\,\varphi_3 = 0\,, \qquad (23)$$

$$\mathrm{i}\sqrt{2}(\partial/\partial r + 1/r)\,\varphi_1 + (x + \tfrac{1}{2}gr^2)\,\varphi_2 - (A\,\partial/\partial A + 3)\,\varphi_3 = 0\,, \qquad (24)$$

$$\mathrm{i}\sqrt{2}A^{-1}\partial\varphi_1/\partial A + gr\varphi_2 + \partial\varphi_3/\partial r = 0\,, \qquad (25)$$

$$\mathrm{i}\sqrt{2}(x + \tfrac{1}{2}gr^2)\,\varphi_1 + \partial\varphi_2/\partial r + grA^2\varphi_3 = 0\,. \qquad (26)$$

Eqs. (22) show that there is no solution unless $m = -1$. This condition has been used to derive eqs. (23)–(26). We have not been able to solve these equations, so that we cannot say if a supersymmetric ground state does exist for $m = -1$. We can, however, show that this system of equations is not overdetermined as it seems at first sight, there being four equations for three unknown functions. Note in this connection that in the relativistic model a supersymmetric solution exsists [4] at the tree level for $x/g \leqslant 0$.

We can solve eqs. (23) and (26) for φ_1 and φ_3. Inserting the solutions in eqs. (24) and (25) we find that these equations are identical to each other and to the following equation

$$(\partial^2/\partial r^2 + \partial^2/\partial A^2 + C_1 r^{-1}\partial/\partial r + 2C_2A^{-1}\partial/\partial A - C_3)\varphi_2 = 0\,, \qquad (27)$$

where

$$C_1 = 1 - C_3^{-1}\,2gr^2(2x + gr^2 + gA^2)\,,$$

$$C_2 = 1 + C_3^{-1}gA^2(x - \tfrac{3}{2}gr^2)\,,$$

$$C_3 = (x + \tfrac{1}{2}gr^2)^2 + g^2r^2A^2\,.$$

We finally remark that in the massless Wess–Zumino model in the galilean approximation SUSY is not broken but chirality is spontaneously broken [5], there being two degenerate ground states of different chirality. In the present model, on the contrary, if SUSY is not broken $m = -1$ and therefore all the ground states have chirality zero.

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