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G. Parisi : SPIN GLASSES

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SPIN GLASSES

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1.- INTRODUCTION

Why spin glasses? Different justification are possible: new concepts and new tools have been introduced in the study of spin glasses[1-4]; while a serious mathematical treatment is not yet available (present day mathematics is not enough advanced), the ideas used in spin glasses may be useful for the study of other amorphous materials like glasses, rubber and (why not?) a random lattice space time or a foam-like space time. If we exclude these last speculations[5], which depend on an unfortunately lacking synthesis of quantum theory and general relativity, the domain of applications is very large, moreover, if we accept the perverse logic of broken replica symmetry, the basic ideas are relatively easy to grasp also from a particle physicist, who will likely be amused (or confused) by the unusual group theory.

In Section 2 we will introduce spin glass, and the concept of frustation.

In Section 3 we will describe the Sherrington Kirkpatrick model for spin glass and present its solution in the high temperature phase.

In Section 4 we will see how to solve the model using the formalism of broken replica symmetry. We will describe the group theoretical implications

of the formalism and we will see how the solution of the model can be reformulated in term of a stochastic differential equation.

Finally, in Section 5 we will present the physical interpretation of the formalism of the previous section.

2.- SPIN GLASSES

Spin glasses belong to a large cathegory of model for which we need to compute the free energy without knowing the Hamiltonian, but only the probability distribution of the Hamiltonian. More precisely we write the Hamiltonian of our system (which for definiteness we suppose to be an Ising model, where the spins σ can take only the ± 1 value) as function of some control parameter J's; the J's are not known but their probability distribution $P[J]$ is known. For each choice of the J's we can compute the partition function Z_J and the free energy desnity F_J defined as usual

$$Z_J = \sum_{\{\sigma\}} \exp\left[-\beta H_J[\sigma]\right] \quad , \quad F_J = -\frac{1}{\beta V} \ln Z_J \tag{1}$$

V being the total number of spins in the system. A typical form of $H_J[\sigma]$ is

$$H_J[\sigma] = \sum_{i,k} J_{ik}\, \sigma_i\, \sigma_k - \Sigma_i h\, \sigma_i \tag{2}$$

h being the external magnetic field.

Our goal is to compute the average of the free energy density, i.e.

$$F = \overline{F_J} = \int dP[J]\; F_J \; . \tag{3}$$

In the infinite volume limit, if J_{ik} decrease faster than $(i-k)^{-3/2}$ when $(i-k)$ goes to infinity, it can be proved that

$$\lim_{V\to\infty} \overline{F_J^2} = \left(\lim_{V\to\infty} \overline{F_J}\right)^2 \; . \tag{4}$$

In other words the free energy density does not fluctuate from sample to sample (i.e. does not depend on the choice of the J's) with probability one for a given form of $P[J]$: as often happens in thermodynamics the most likely values coincide with the average values in the infinite volume limit.

Different model of spin glasses have different form of the J's: in the simplest Sherrington-Kirkpatrick (S-K) model[6] the interaction is infinite

range; all $J_{i,k}$, for any pair i,k, have the same probability:

$$dP[J] = \prod_{(i,k)} dJ_{ik} \exp\left[-\frac{V}{2} \sum_{(i,k)} J_{ik}^2\right] . \tag{5}$$

In other words the J's have a Gaussian distribution with zero average and covariance

$$\overline{J_{ik} J_{i'k'}} = \frac{1}{V} \delta_{ii'} \delta_{kk'} . \tag{6}$$

In the Edward Anderson model[7], which is not very far from being realized in nature, we have the same relations as eqs.(5-6), where now the sum in eq.(5) runs over the nearest neighbour pairs a (i,k) and all $J_{i,k}$ which are not nearest neighbour, are zero; moreover we have (1/Z) at the place of 1/V, Z being the coordination number of the lattice (2D for an hypercubic lattice in D dimensions).

In a very realistic model[8] the position of the spins in real space is random: i.e. we associate to each spin a vector $\vec{x}_i$, which denotes its position, the J's are given by

$$J_{ik} = \frac{\sin(r_{ik} k_F)}{r_{ik}^3} , \qquad r_{ik} = \left| \vec{x}_i - \vec{x}_k \right| . \tag{7}$$

This model represents what is going on in alloyes like Fe_5Au_{95} where the spins of iron interact only by deforming the Fermi surface of gold: k_F in (7) is indeed the Fermi momentum of the host material (in this case gold).

The expert reader has certainly realized that at h = 0 (if the probability distribution of the J's is symmetric) the theory is invariant under the local Z_2 gauge transformations: at fixed i we can change

$$J_{ik} \to - J_{ik} \quad \forall k , \qquad \sigma_i \to - \sigma_i \tag{8}$$

without any effects on the free energy. The average of quantities which are not invariant under the transformation (8) are obvious zero, e.g. $\langle \sigma_i \rangle = 0$. Examples of gauge invariant quantities are (apart from obvious quantities like internal energy, magnetic susceptibility and so on):

$$q_{EA} = \overline{\langle \sigma_i \rangle^2} , \qquad \chi_R = \overline{\sum_k \left| \langle \sigma_i \sigma_k \rangle - \langle \sigma_i \rangle \langle \sigma_k \rangle \right|^2} . \tag{9}$$

It is reasonable to suppose that for each choice of the J's at low temperature (at least in high dimensions as we shall see) the system will have a spontaneous local magnetization and $\langle\sigma_i\rangle = m_i$ will be different from zero; however the sign of m_i will change from point to point and/or from realization to realization of the J's, so that it averages at zero; at low temperature $q_{EA} = m_i^2$ will be different from zero, while it will be zero at higher temperatures. The reader must note that although the definition of q_{EA} seems clear, we shall see later that it is not as unambigous as it looks.

One of typical features of spin glass is that, for a given choice of the J's is very difficult to find the ground state, i.e. configuration of spins σ_1 which minimize the Hamiltonian. Indeed as a consequence of the randomness of the $J_{i,k}$, it is not possible to fix the σ_i's in such a way that

$$J_{i,k}\ \sigma_i\sigma_k > 0\ , \qquad \forall i,k\ . \tag{10}$$

If eq.(10) could be satisfied the product of all J_{ik} along a closed loop (the Toulouse-Wilson loop 9) should be positive, but that is impossible, the J's having zero mean. (If the product of the J's along a loop is negative, the loop is said to be frustrated).

We must decide which bond must be frustated (i.e. is $\sigma_i\sigma_k J_{i,k} < 0$). Different arrangement of the frustated bonds may differ very little in energy but correspond to very different spin configurations. Briefly, in more than two dimensions all known algorithms for finding the ground state of an N-site spin glass take a time proportional to exp aN. This multiplicity of groundstates, which are nearly equivalent from the energetic point of view, is responsible for many of the strange properties of spin glasses.

Different models can be analyzed in terms of the behaviour of the Toulouse-Wilson loop, which are the only gauge invariant characterizations of the J's. If for semplicity we consider only loops in which no J appears twice, we have both in the S-K model and in the E-A model

$$\overline{W(C)} = 0\ , \tag{11}$$

W(C) being the Toulouse-Wilson loop associated to the contour C. We can consider a more general model where the J's are nearest neighbour and they

can be written as

$$J_{ik} = \frac{1}{\sqrt{M}} \sum_{1}^{M} r_i^{(1)} r_k^{(1)} \tag{12}$$

where the r's are random numbers taking only the values ± 1. The M = 1 model is gauge equivalent to the Ising model indeed

$$W(C) = 1 \quad \forall C \ . \tag{13}$$

For generic M we have that

$$W(C) = \frac{1}{(M)^{L/2}} \tag{14}$$

where L is the lenght of the loop: for M going to infinity we recover the original E-A model. It is important to note that in the more realistic models like those corresponding to eq.(7) the expectation values of the Wilson loops are non zero, so that they have an intermediate status between the E-A model and the simplified model (12).

The main approach to spin glasses is based on the introduction of replicas:we consider the system with Hamiltonian

$$H^{(n)} = \sum_{i,k} \sum_{1}^{n}{}_a \ \sigma_i^a \ \sigma_k^a \ J_{ik} \tag{15}$$

where the spin σ's have an additional index a going from 1 to n. We define

$$Z^{(n)} = \int dP[J] \sum_{\{\sigma\}} \exp\left[-\beta H^{(n)}\right] = \int dP[J] \left[Z_J\right]^n \tag{16}$$

$$F^{(n)} = -\frac{1}{n\beta V} \ln Z^{(n)} \ .$$

Eq.(16) allows us to continue analytically over n; it is also evident that $F^{(n)}$ in the limit $n \to 0$ reduces to the F of eq.(3); indeed

$$-\frac{1}{n\beta V} \ln \overline{\left[Z_J\right]^n} \simeq -\frac{1}{n\beta V} \ln \overline{\left[1 + n \ln Z_J\right]} \simeq -\frac{1}{\beta V} \overline{\ln Z_J} \ . \tag{17}$$

The strategy consists in formally use the representations (15-16) also for non integer n by analytically continuing all relevant formulae from integer n to n equal to zero.

Now eq.(16) describes the normal partition function of a system of n replicas of the same spins coupled to the J's. The integration over the J's can often been done; we generate a direct coupling among the replicas; for

example in the case of the E-A model one finds

$$Z_{EA}^{(n)} = \sum_{\{\sigma\}} \exp\left[- \tilde{H}_{EA}^{(n)} \right] \tag{18}$$

$$\tilde{H}_{EA}^{(n)} = - \frac{1}{4D} \beta^2 \sum_{(i,j)} \sum_{1}^{n}{}_a \sum_{1}^{n}{}_b \sigma_i^a \sigma_i^b \sigma_j^a \sigma_j^b$$

and the sum on (i,j) runs over all nearest neighbour pairs. The reader must recall that the integration over the J's was one of the major difficulties in the usual formalism, which is now bypassed by the introduction of the replicas; we have just put the dirty under the carpet: it will come out soon.

We can now proceede as in the Ising model to derive the mean field approximation. Using the relation $\sigma^2 = 1$ we can write

$$Z_{EA}^{(n)} = \sum_{\{\sigma\}} \int d[Q] \exp\left[\beta^2 \Sigma_i \sum_{1}^{n}{}_a \sum_{1}^{n}{}_b Q_{ab}^i \sigma_i^a \sigma_i^b - \right. \tag{19}$$

$$\left. - \frac{\beta^2}{D} \sum_{i,k} \sum_{1}^{n}{}_a \sum_{1}^{n}{}_b Q_{ab}^i Q_{ab}^k G(i-k) \right] ,$$

where G(i) satisfies the relation

$$\Sigma_j G(i-j) = \delta_{i,0} \tag{20}$$

and the sum over j in eq.(20) runs over the nearest points of i; in other words

$$G(k) = \frac{1}{(2\pi)^D} \int_B d^D p \exp(ipk) \tilde{G}(p) , \qquad \tilde{G}(p) = \sum_{1}^{D}{}_\nu 2 \cos p_\nu . \tag{21}$$

The correctness of eq.(19) can be checked by integrating back over the Q's.

As usual the perturbative expansion can be constructed by starting from the mean field theory: i.e. we set

$$Q_{ab}^i = Q_{ab} + \tilde{Q}_{ab}^i \tag{22}$$

Q_{ab} being a constant (i.e. it does not depend on i).

If $\tilde{Q}_{ab}^i$ is neglected the value of Q_{ab} is found by maximizing the argument of the exponent in eq.(19); we easily find that it can be written as $- VA(Q)$, where A(Q) is given by

$$A[Q] = \frac{\beta^2}{4} \sum_{a,b} Q_{ab}^2 + \ln \left[\sum_{\{\sigma\}} \exp\left[- \sum_{a,b} \beta^2 Q_{ab} \sigma^a \sigma^b \right] \right] . \qquad (23)$$

Where the sum over σ's is done over the 2^n configuration of the n replicas of a single spin. The condition of minimum of $A[Q]$ implies that

$$\frac{\partial A}{\partial Q_{ab}} = 0 = \beta^2 \left[Q_{ab} - \langle \sigma^a \sigma^b \rangle_{\beta^2 Q_{ab}} \right] \qquad (24)$$

which can be written as

$$Q_{ab} = \sum_{\{\sigma\}} \sigma_a \sigma_b \exp\left[- \Sigma \beta^2 Q_{cd} \sigma^c \sigma^d \right] / \sum_{\{\sigma\}} \exp\left[- \sum_{c,d} \beta^2 Q_{cd} \sigma^c \sigma^d \right] . \qquad (25)$$

After that the maximum is found, the correction to this approximation can be computed by taking care of $\tilde{Q}$. As usual these corrections disappear when D goes to infinity. The standard loop expansion can be constructed provided that the Hessian matrix

$$M_{ab,cd} = \frac{\partial^2 A}{\partial Q_{ab} \partial Q_{cd}} \qquad (26)$$

has no negative eigenvalues as should be at the point of minimum.

We shall see that the necessity of the analytic continuation in n mater the evaluation of the point of minimum and of the eigenvalues of the Hessian particular difficult.

3.- THE SHERRINGTON-KIRKPATRICK MODEL

The advantage of the S-K model is that the mean field approximation is exact. Indeed with the same notation as the previous section we find after the integration over the $J_{i,k}$[6]:

$$Z_n = \sum_{\{\sigma_i^a\}} \exp\left[\beta^2 \sum_{i=1}^{N} \sum_{k=1}^{N} \sum_{a=1}^{n} \sum_{b=1}^{n} \sigma_i^a \sigma_k^b + h\beta \sum_{i=1}^{N} \sum_{a=1}^{n} \sigma_i^a \right] =$$

$$= \int_{-\infty}^{-\infty} dQ_{ab} \exp\left[- NA(Q) \right] ,$$

$$A[Q] = - \frac{\beta^2}{4} + \frac{1}{4} \sum_{a=1}^{n} \sum_{b=1}^{n} \beta^2 Q_{ab}^2 + \qquad (27)$$

$$+ \ln \left[\mathrm{tr} \left[\exp\left(\sum_{a=1}^{n} \sum_{b=1}^{n} Q_{ab} S_a S_b + \beta h \sum_{a=1}^{n} S_a \right) \right] \right]$$

where the indices a and b label the different replicas of the spin system; Q_{ab} is an nxn matrix, zero on the diagonal, and tr stands for the sum over the 2^n possible values of the n Ising spin variables S_a.

Up to now, we have done legal operations. When N goes to infinity we would like to use the saddle point method and write:

$$\tilde{F} = -\frac{1}{\beta} \lim_{n\to 0} \left[\min_{Q} A(Q)\right] \tag{28}$$

as discussed in the previous section. This means that if pathological results are found, it cannot be due to the undeguacy of the mean field approximation.

Let us start and evaluate eq.(28). We have been not very careful in defining what we mean when we say that we should solve the problem at integer n and analytically continue the solution in n up to n=0. We will make strong use that A[Q] is invariant under the action of symmetry group P_n, the permutations of n objects: indeed it is evident that if we permute different rows and coulumns of Q, A[Q] does not change;in particular we will pay attention only to the definition of P_n symmetric functionals of Q. Very often the P_n group is called the replica symmetry group because it exchange different replicas.

The meaning of eq.(28) is no clearer than that of a Delphic oracle: should we find the minimum at $n\neq 0$ and analytically continue to n=0? No. The analytic continuation of a minimum may be a maximum. The minimum should be found directly at n=0. However the number of variables corresponding to the Q_{ab} is n(n-1) which is negative for $0 < n < 1$: the concept of a minimum of a function depending on negative number of variables is rather subtle! Everybody would say that the minimum of $n^{-1}TrQ^2$ is at Q=0, however if we set

$$Q_{ab} = q \;\forall a,b \;\; (Q_{a,q} = 0) \tag{29}$$

we find

$$n^{-1}TrQ^2 = (n-1)q^2 \; . \tag{30}$$

The point q=0 is (for $n < 1$) a maximum as a function of q. The solution to this apparent paradox is quite simple: the condition that the Hessian matrix (26) has positive eigenvalues does not imply that $\langle x|H|x\rangle$ is positive, if $|x\rangle$ belongs to a negative dimensional space (e.g. the trace of

the identity is equal to the dimension of the space). A moment of reflection is needed to understand that the necessary condition for the use of the saddle point method is that all the eigenvalues of the Hessian must be non-negative. This also guarantees that all the susceptibilities, which are positive definite, are positive indeed.

The final interpretation of eq.(28) is the following: we must consider all possible analytic families of matrices $Q_{ab}^{(n)}$, which may depend on real parameters q_i or integer parameters m_i. An analytic family is an infinite set of matrices (one for each n multiple of n_o), such that the P_n invariant quantities, e.g.:

$$\mathrm{Tr}Q^k \qquad \text{or} \qquad \sum_{a=1}^{n} \sum_{b=1}^{n} (Q_{ab})^k \ , \tag{31}$$

are analytic functions of n. For each analytic family we should compute the analytic continuation in n up to n=0 of the function F(Q) and of the eigenvalues of the Hessian. The final result will be given by that analytic family (hopefully unique) whose elements are stationary points of A[Q], for all n multiplets of n_o, and the eigenvalues of the Hessian analytically continued at n=0 are non negative. As far as one can construct analytic families of matrices which depend analytically on the integer parameters m_i, one is allowed also to consider non-integer values of the m_i's.

While it is not clear if this interpretation gives the correct result, it does make the problem well defined from the mathematical point of view but very hard to control from the practical point of view: the space of all analytical families is very large. Up to now the only approach has been to construct ansatzs. Let us see how this works.

The first case we study is:

$$Q_{ab} = q, \qquad a \neq b \ . \tag{32}$$

After some simple algebraic manipulations[6] we find that:

$$A(q) = -\frac{\beta^2}{4}(1+q^2) - \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dz \left[\exp\left[-z^2/2\right] \cdot \ln\left[2\mathrm{ch}(\beta q^{1/2} z + \beta h)\right]\right] \ . \tag{33}$$

The Hessian will have one eigenvector in the one-dimensional space. One can immediately check that for $n < 1$ the corresponding eigenvalue is posi-

tive if A(q) is a maximum (not a minimum!) as function of q.

One finally finds at h = 0 (after a rescaling of q):

$$
\begin{aligned}
& q = 0, \quad T > T_c = 1, \\
& q_{EA} = q = \int_{-\infty}^{+\infty} \frac{dz}{(2\pi)^{1/2}} \left[\exp\left[-z^2/2\right] \mathrm{th}^2(\beta q^{1/2} z) \right] \neq 0, \quad T < 1, \\
& U(T) \to -\frac{1}{2}\sqrt{\pi} = -0.798, \quad q(T) \sim 1 - \sqrt{\pi} T \qquad (T \sim 0), \\
& C(T) \sim T \qquad (T \sim 0), \\
& \chi(0) \sim \sqrt{\pi}/2, \qquad S(0) = -1/2\pi ,
\end{aligned}
\tag{34}
$$

where U, C, χ and S are the internal energy, the specific heat, the susceptibility and the entropy. Now the Monte Carlo results tell us that $U(0) \simeq -(0.76\text{-}0.77)$ in small but definite disagreement with eq.(34), and the specific heat is quadratic in T; on the other hand the dependence of q(T) is qualitatively correct (apart from the fact that Monte Carlo data suggest $q(T) \sim 1 - aT^2$). Unfortunately the entropy becomes negative at low T and a negative Ising system entropy cannot be tolerated.

The theoretical explanation of this failure has been found by de Almeida and Thouless[10]; they noticed that when $T < \tilde{T}(h)$ where:

$$
\begin{aligned}
& \tilde{T}(h) = T_c - |h|^{2/3} \qquad \text{for} \quad |h| \ll 1 \\
& \tilde{T}(h) \sim 1/\ln|h| \qquad \text{for} \quad |h| \to \infty
\end{aligned}
\tag{35}
$$

one of the eigenvalues of the Hessian becomes negative: this is certainly not acceptable, also because χ_R is proportional to the inverse of this eigenvalues. The previous computation is wrong when $T < \tilde{T}(h)$ and eq.(32) is not the correct choice. Unfortunately eq.(32) is the only possible ansatz which is P_n symmetric, we need therefore to break spontaneously the replica symmetry.

4.- THE SOLUTION

It must be clear that at the present moment the only way to solve eq. (28) is by trying different ansaltz and check if the positivity conditions on the eigenvalues of the Hessian is satisfied.

The only know construction[1,11] (up to equivalences) is the following: as a preliminary step we divide the n replicas in n/m groups of m

replicas (Of course n must be multiple of m). We set $Q_{ab}=q_o$ if a and b belong to the same group $Q_{ab}=q_1$ if a and b belong to different groups (Q_{aa} is always zero). In other words

$$\begin{aligned} Q_{ab} &= q_o \qquad \text{if} \qquad I(a/m) = I(b/m), \\ Q_{ab} &= q_1 \qquad \text{if} \qquad I(a/m) \neq I(b/m), \end{aligned} \tag{36}$$

where I(x) is an integer valued function: its value is the smallest integer greater than or equal to x. Eq.(36) provides us with an example of an analytic family of matrices, depending on the parameters q_o, q_1 and m.

It is evident that:

$$\tilde{\mathrm{Tr}}Q^2 = (1-m)q_o^2 - mq_1^2 \tag{37}$$

is not negative definite if $m > 1$; we must maximize it with respect to q_o and minimize it with respect to q_1: this automatically leads to a free energy worse than the one obtained in the previous section. However if $0 < m < 1$, condition is enforced (obviously m is no more an integer, but we are allowed to do this). After some tedious algebra we get at h=0:

$$A(q_o,q_1,m) = -\frac{\beta^2}{4}\left[1+mq_1^2 + (1-m)q_o^2 - 2q_o^2\right] - \int dp(z)\frac{1}{m}\ln\left[\int dp(y)\mathrm{ch}^m\left[\beta(q_1^{1/2}z-(q_o-q_1)^{1/2}y)\right]\right], \tag{38}$$

$$dp(z) \equiv \exp(-z^2/2)dz/(2\pi)^{1/2}.$$

Maximizing A with respect to q_o and q_1 and m (restricted to the interval 0-1) we obtain the following surprising results: the curves for V, C, and q_{EA} (assuming $q_{EA} = \max Q_{ab} = q_o$) are in very good agreement with the Monte Carlo data (e.g. V(0) = -0.7652); the free energy is obviously higher than that obtained in the previous section. The entropy at zero temperature has collapsed from $S(0) \simeq -0.16$ to $S(0) \simeq -0.01$.

We are clearly on the right track! In order to generalise eq.(36), let us do some unusual group theory. Eq.(36) correspond to breaking the P_n group in the following way

$$P_n \rightarrow (P_m)^{\otimes n/m} \otimes P_{n/m}. \tag{39}$$

Indeed we can permute both the replicas inside each group (and this leads to the product of n/m times P_m) and the groups among themselves (this leads

to $P_{n/m}$). In the limit $n \to 0$ we have the following pattern of symmetry breaking

$$P_o \to (P_m)^{\otimes b} \otimes P_o \ . \tag{40}$$

In other words, P_o contains itself as a subgroup! It is clear now that we can go on and repeat the same operation many times; we introduce a set of integer numbers m_i $(i = 0,..,k+1)$, such that $m_o=0$ and $m_{k+1}=n$ and m_i/m_{i-1} is an integer (for $i=1,..,k+1$). We can divide the n replicas in n/m_k groups of m_k replicas, each group of m_k replicas is divided in m_k/m_{k-1} group of m_{k-1} replicas and so on. The matrix Q will be given by:

$$Q_{ab} = q_i \qquad \text{if} \qquad I(\frac{a}{m_i}) \neq I(\frac{b}{m_i})$$

and (41)

$$I(\frac{a}{m_{i+1}}) = I(\frac{b}{m_{i+1}}), \qquad i = 0,..,k \ ,$$

and the q_i's are a set of k+1 real parameters. For k=1 recover the previous example and for k=0 we recover the unbroken symmetry theory.

An easy computation shows that:

$$-\tilde{\mathrm{Tr}}Q^2 = \sum_{i=1}^{k} (m_i - m_{i+1}) q_i^2 \ . \tag{42}$$

Condition $\mathrm{Tr}Q^2 \geq 0$ is satisfied only if

$$0 \leq m_{i+1} \leq m_i \leq 1. \tag{43}$$

From now on let us assume that condition is valid.

For each value of k, one can compute the free energy by maximising it with respect to the q's and the m's. An explicit computation shows that, near T_c, $F^{(k)}$ contains a term proportional to

$$(T - T_c)^5/(2k+1)^4 \ ; \tag{44}$$

we are naturally led to consider the case $k \to \infty$. In order to keep track of the parameters q_i and m_i it is convenient to consider the function:

$$q(x) = q_i, \qquad m_{i+1} < x < m_i \ . \tag{45}$$

There is a one-to-one correspondence between the piecewise constant functions with k discontinuities and the parameters q_i and m_i. In the limit

$k \to \infty$, $q(x)$ becomes a generic L^2 function on the internal 0-1.

Let us now compute the free energy: after some calculations one arrives to the surprising result[11]

$$-\beta F = \max_{q(x)} A[q] ,$$

$$A[q] = -\frac{1}{4}\beta^2\left[1 + \int_0^1 q^2(x)dx + 2q(1)\right] - a[q] , \qquad (46)$$

$$a[q] = f(0,h) ,$$

where the function $f(x,y)$ satisfies the following differential equation:

$$\frac{\partial f}{\partial x} = -\frac{1}{2}\frac{dq}{dx}\left[-\frac{\partial^2 f}{\partial y^2} + x\left(\frac{\partial f}{\partial y}\right)^2\right] \qquad (47)$$

with the boundary condition:

$$f(1,y) = \ln[2\mathrm{ch}(\beta y)] . \qquad (48)$$

Eq.(46) is correct only if $q(0) = 0$, otherwise

$$a[q] = \int_{-\infty}^{+\infty} dp(z) f(0, h + \sqrt{q(0)}z). \qquad (49)$$

A long argument shows that $q(0) \sim |h|^{2/3}$ and

$$\chi = 1 - O(h^{4/3}).$$

When we cross the AT line $x_m \to x_M \neq 0$. The following semiempirical rules are exact, or well satisfied:

$$q_m(\beta,h) = q_m(h) , \qquad q_M(\beta,h) = q_M(\beta) , \quad q(x) = q_m , \ x < x_m ,$$
$$q(x,\beta,h) = q(x\beta), \qquad x_m < x < x_M , \quad q(x) = q_M , \ x > x_M . \qquad (50)$$

Numerical investigations support the hypothesis that $S(0) = 0$ in this scheme; the ground state energy estimated is $U(0) \simeq -.07633 \pm 10^{-4}$. Apart from the region of very small temperature, the results are very similar to those obtained for k=1.

Before discussing in the next section the physical implications of the replica symmetry breaking we can associate to eq.(46) a stochastic differential equation.

We first define the function $x(q)$ as the inverse of the function $q(x)$;

we assume for simplicity that $q(0) = q_m = 0$: the function $x(q)$ will be defined in the range $0-q_M$. We consider a function $w(q)$ which satisfies the following stochastic differential equation[4,12-14]:

$$\dot{w}(q) = \eta(q) - \beta x(q) m(w,q), \tag{51}$$

where $\eta(q)$ is a white noise:

$$\overline{\eta(q_1)\eta(q_2)} = \delta(q_1 - q_2) \tag{52}$$

and $w(0) = 0$.

The function $m(h,q)$ is fixed by the self consistency condition

$$m(h,q) = \overline{\left.\frac{q \qquad\qquad q_M}{\text{th}(\beta w(q_M))}\right|_{w(q)=h}} \tag{53}$$

where the strange notation indicates that $m(h,q)$ is the average of the $(\beta w(q))$ over all the trajectories such that $w(q) = h$ the average being done over the noise η in the interval $q-q_M$. This self consistency condition can be solved for any choice of $x(q)$; the function $x(q)$ is fixed by the extra self consistency relation

$$q = \overline{m^2(w(q),q)} = \int^{\overline{\left.\frac{0 \qquad\qquad q}{\left(\text{th}\beta w(q_M)\right|_{w(q)=h}\right)^2}}} dh \ . \tag{54}$$

It is a relative simple exercise in the theory of stochastic differential equations to show the equivalence of eqs.(46-48) and eqs.(51-54).

As it was stressed in ref.(13) the following remarkable relations hold

$$\langle \sigma_i \rangle^k = \overline{m^k(w(q_M),q_M)} = \overline{\frac{0 \qquad\qquad q_M}{\text{th}^k(w(q_M))}} \ . \tag{55}$$

Although these last equations are very suggestive of a simple physical interpretation and they are related to the time evolution of the system, their precise meaning is not yet fully understood.

The most important result is that an explicite evaluation shows that the Hessian has negative eigenvalue and that the scheme here described is free of the inconsistences of the previous replica symmetric approach[15-16].

4.- THE PHYSICAL INTERPRETATION

In this section we show how to interpret in a physical way the strange form of the matrix Q. Although the informations contained in this section

should be enough to derive eqs.(46-48) in a simple straightforward way, I never succeded in finding such a derivation: I sincerely hope that some of the readers will solve this problem.

The first thing to do is to be more careful on the precise definition of $q_{EA} = \langle \sigma_i \rangle^2$. Indeed we must define the magnetization $m_i \equiv \langle \sigma_i \rangle$ for a given choice of the J's.

If we consider a real system (or a computer simulation) the magnetization m_i is defined[16] by

$$m_i = \frac{1}{t} \int_0^1 d\tau \ \sigma_i(\tau) \ , \tag{56}$$

where $\sigma_i(\tau)$ is the value that σ_i takes at the time τ and t is a large (macroscopic) but not too large observation time. For example in a d-dimensional ferromagnetic system of size L, t must satisfy the conditions

$$t_m \ll t \ll t_M \approx t_m \exp(L^{d-1}), \tag{57}$$

where t_m is the microscopic relaxation time, e.g. one Monte Carlo step. When we change the initial conditions (e.g. $\sigma_i(\tau)$ at $\tau=0$) we may obtain different results: in the Ising ferromagnetic case below T_c there are essentially two possibilities ($m_i > 0$ or $m_i < 0$); it is important to note that, if the initial state is disordered, the approach to equilibrium for quantities invariant under the global Z_c is slow (there are corrections proportional to powers of t) while one needs a time t at least proportional to the volume ($t > L^d$) in order to establish a translationally invariant state.

What do we expect for a spin glass? There will be many minima of the thermodynamic potential which are separated by very high walls. At relative short times the system will remain near one minimum, later on at a very large (macroscopic?) time the system will start jumping from one minimum to the other one, by thermodynamic tunnel effects[17].

Let us denote by $m_i^{(\alpha)}$ the expectation value of σ_i in the state labelled by α. We have approximately

$$\begin{aligned} q(t) &\equiv \frac{1}{V} \sum_{i \in V} \left[\frac{1}{t} \int_0^1 \sigma_i(t') dt' \right]^2 = \\ &= \frac{1}{V} \sum_{i \in V} \left[\sum_{\alpha=1}^{M(t/t_M)} (m_i^{(\alpha)})^2 / M(t/t_M) \right] , \end{aligned} \tag{58}$$

where t_M is a macroscopic time $M[1] = 1$ and when $t \to \infty$ the sum runs over all possible states of the system. In this language $q(t_M) = q_{EA}$ while $q(\infty)$ is obviously much smaller ($q(\infty)$ is zero at h=0).

The concept of average around a minimum of the thermodynamic potential can be sharpened in more general terms[2]: if we denote by $< >_G$ the expectation values in the Gibbs state obtained by considering the infinite volume limit of the usual canonical distribution, it is possible that the Gibbs state is not clustering, in other words the connected correlation functions do not go to zero at large distances; for example in the usual Ising model at H=0, $T=T_c$ we have

$$< \sigma_i > = 0 \ , \qquad < \sigma_i \sigma_j > \xrightarrow[i-j \to \infty]{} m_s^2 \tag{59}$$

where m_s is the spontaneous magnetization.

Under general hypothesis the Gibbs state can be decomposed as the sum of pure clustering equilibrium state

$$< \ >_G = \Sigma_\alpha w_\alpha < \ >_\alpha \tag{60}$$

where the connected correlation functions vanish at large distances and the DLR equations[18] are satisfied for each state α. In the case of the Ising model we have

$$< \ >_G = \frac{1}{2} < \ >_+ + \frac{1}{2} < \ >_- \ , \qquad < \sigma_i >_+ = - < \sigma_i >_- = m_s \ . \tag{61}$$

Let us assume, as a working hypothesis that the decomposition is non trivial for spin glasses and let us try to see the implications of this assumption. In each state α we can define an E-A order parameter

$$q_\alpha = \lim_{V \to \infty} \frac{1}{V} \sum_1^V {}_i (m_i^\alpha)^2 \ , \qquad m_i^\alpha = < \sigma_i >_\alpha \tag{62}$$

which is likely to be independent from α.

Similarly we would like to define the overlap $q_{\alpha\beta}$ (and the distance $\delta_{\alpha\beta}$) of two states:

$$\begin{aligned} q_{\alpha\beta} &= \lim_{V \to \infty} \frac{1}{V} \sum_1^V {}_i \ m_i^\alpha m_i^\beta \ , \\ \delta_{\alpha\beta} &= \lim_{V \to \infty} \frac{1}{V} \sum_1^V {}_i (m_i^\alpha - m_i^\beta)^2 = q_\alpha + q_\beta - 2q_{\alpha\beta} = 2(q_{EA} - q_{\alpha\beta}) . \end{aligned} \tag{63}$$

Our aim is to compute the probability $P_J(q)$ for two states having overlap q, i.e.

$$P_J(q) = \sum_{\alpha\beta} w_\alpha w_\beta \, \delta(q-q_{\alpha\beta}) \ . \tag{64}$$

We could also define the average over the J's of $P_J(q)$

$$P(q) = \overline{P_J(q)} \ . \tag{65}$$

In principle (and also in practice) the function $P_J(q)$ may change when we change the realization of the J's.

It is convenient to consider the quantities

$$Q_J^{(k)} = \int dq P_J(q) q^k \ . \tag{66}$$

An easy computation shows that

$$\begin{aligned} Q_J^{(k)} &= \frac{1}{V^k} \sum_{i_1} \cdots \sum_{i_k} \left| \langle \sigma_{i_1} \cdots \sigma_{i_k} \rangle_G \right|^2 = \\ &= \frac{1}{V^k} \sum_{i_1} \cdots \sum_{i_k} \sum_\alpha \sum_\beta w_\alpha w_\beta \langle \sigma_{i_1} \cdots \sigma_{i_k} \rangle_\alpha \langle \sigma_{i_1} \cdots \sigma_{i_k} \rangle_\beta \end{aligned} \tag{67}$$

where we have used the clustering properties of the pure state to do the approximation

$$\langle \sigma_{i_1} \cdots \sigma_{i_k} \rangle_\alpha = m^\alpha_{i_1} \cdots m^\alpha_{i_k} \tag{68}$$

which becomes exact when the points $i_1 \ldots i_k$ are widely separated.

In the replica formalism we could naively write

$$\overline{Q_J^{(k)}} = \frac{1}{V^k} \sum_{i_1} \cdots \sum_{i_k} \langle \sigma^1_{i_1} \sigma^2_{i_1} \cdots \sigma^1_{i_k} \sigma^2_{i_k} \rangle = Q_{1,2} \ . \tag{69}$$

However when the replica symmetries broken the off diagonal elements of Q depend on the indices and the choice of the first and second replicas is arbitrary. The correct formula is obtained by summing over all possible choices of the two replicas[19]

$$\overline{Q_J^{(k)}} = \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{ab}^{n} Q_{ab}^k = \int dx q^k(x) \tag{70}$$

where the last equation follows from the assumed form of the matrix Q[20] (cfr. eq.(42).

We finally find that

$$P(q) = \frac{dx}{dq} \tag{71}$$

where $x(q)$ is the inverse of $q(x)$.

When the replica symmetry is exact $q(x)$ is a constant and $P(q)$ is a delta function (only one pure state); otherwise $P(q)$ is not a delta function and the states structure is much more complex.

Many informations can be obtained on the structure of the states in a similar way; for example[3]

$$P^2(q_1,q_2) \equiv \overline{P_J(q_1)\ P_J(q_2)} = \frac{2}{3}P(q_1)P(q_2) + \frac{1}{3}P(q_1)\delta(q_1-q_2). \tag{72}$$

We can also compute the probability for the states $\alpha\ \beta\ \gamma$ of having overlap q_{12}, q_{23}, q_{31}, i.e.

$$\begin{aligned} P(q_{12},q_{23},q_{31}) &= \sum_{\alpha,\beta,\gamma} w_\alpha w_\beta w_\gamma\ \delta(q_{\alpha\beta}-q_{12})\delta(q_{\beta\gamma}-q_{23})\delta(q_{\gamma\alpha}-q_{12}) = \\ &= \lim_{n\to 0} \frac{\sum_{abc}^{n} \delta(q_{12}-Q_{ab})\delta(q_{23}-Q_{bc})\delta(q_{31}-Q_{ca})}{n(n-1)(n-2)} = \\ &= \frac{1}{2}\ P(q_{12})x(q_{12})\delta(q_{12}-q_{23})\delta(q_{12}-q_{13}) + \\ &+ \frac{1}{2}\Big[\ P(q_{12})P(q_{23})\vartheta(q_{12}-q_{23})\delta(q_{23}-q_{13}) + 2\ \text{permutations}\Big]. \end{aligned} \tag{73}$$

If we go from the overlap to distances this last equation implies that

$$P(\delta_{12},\delta_{23},\delta_{13}) = 0 \qquad \text{if} \qquad \delta_{13} > \max(\delta_{12},\delta_{23}). \tag{74}$$

In other words the states form an ultrametric space. Briefly the pure states have very interesting properties when the replica symmetry is broken and this properties can be computed using the same techniques as in eqs. (72-73)[3].

The reader will certainly realize that the whole machinery is two heavy and lot of work is needed to simplyfy the approach.

It is evident that we have in our hands the information needed to compute the corrections to the mean field theory. The first results show that in momentum space[15]

$$\Sigma_{ab} \tilde{Q}_{ab}(k)\tilde{Q}_{ab}(-k) \sim \frac{1}{k^3} \qquad (k \approx 0) \qquad (75)$$

where, for simplicity, we have neglected logaritmic factors. Apparently the $1/k^3$ behaviour is the effect of the "Goldston boson" associated to the spontaneous breaking of the replica symmetry.

Eq.(75) suggests that the lower critical dimension is 3 and that we can associate to the three dimensional spin glass on asyntotically free theory. A careful analysis is needed to prove or to disprove these suggestions.

A very interesting and open field is the generalization of this approach to other systems where the Hamiltonian is fixed. In these cases it is possible that the system has many equilibrium states, the number of these equilibrium states may depend on the volume also on the infinite volume limit. In this case the substitute of eqs.(65,72) should be likely

$$P(q) = \lim_{V\to\infty} \frac{1}{V}\int^{V} dV' P_{V'}(q) \; , \qquad (76)$$

$$P^2(q_1,q_2) = \lim_{V\to\infty} \frac{1}{V} \int^{V} dV' P_{V'}(q_1)P_{V'}(q_2) \; . \qquad (77)$$

It is likely that eqs.(75,76) and eqs.(65,72) are equivalent in the spin glass case.

Apparently we have in our hands a tool which may be used to describe and to predict the behaviour of many amorphous systems. Further investigations are needed to find its precise range of applications.

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