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G.Parisi: AN INTRODUCTION TO THE STATISTICAL MECHANICS OF AMORPHOUS SYSTEMS

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COURSE 6

AN INTRODUCTION TO THE STATISTICAL MECHANICS OF AMORPHOUS SYSTEMS

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1. Introduction

In recent years we have seen serious progress in the theoretical understanding of the properties of amorphous material (for a review see ref. 1). Many points of principle have been completely (or at least partially) understood. A great part of this progress is due to the combined use of the renormalization group and the "replica" method [2, 3], which allows us to use the field theory language for a class of problems which are not strictly field theory. Here I will try to present some of the recent results for an audience of people working mainly on field theory and on statistical mechanics of non-amorphous systems.

In these lectures we will study the application of the techniques of standard statistical mechanics to random systems, i.e. to systems whose Hamiltonian is not translational invariant and depends locally on random parameters (e.g. electrons in an amorphous material). For reasons of time I will not discuss the phenomenological side (i.e. applications of these ideas to real systems), but I will concentrate on a few theoretically simple examples; magnetic systems in a random magnetic field, spin glasses and the Schrödinger equation with a random potential. In all these cases I will try to stress the limits of our present understanding of the problems.

A few words will be spent on related arguments: the relations between super-symmetric field theories and stochastic differential equations.

2. Random magnetic field (mean field approximation)

It is well known that magnetic systems are well described near the critical point by the following effective hamiltonian [4]

$$\beta H[\phi, h] = \int_V d^D x[\tfrac{1}{2}(\partial_\mu\phi(x))^2 + \tfrac{1}{2}m^2\phi^2(x) + \tfrac{1}{4}g\phi^4(x) - h(x)\phi(x)], \tag{2.1}$$

where the dependence from the temperature is concentrated in m^2 and $h(x)$ is the magnetic field at the point x. The h-dependent free energy is given by:

$$F[h] = -\frac{1}{V}\ln\int d[\phi]\exp(-H[\phi,h]) \tag{2.2}$$

(we have set for simplicity β to 1).

Standard statistical mechanics deals with the problem of computing F for a given h. Here we will suppose that h is not known, but it is a random variable with probability distribution $d\mu[h]$. We are interested in computing the average value of the free energy:

$$\bar{F} = \int d\mu[h]F[h]. \tag{2.3}$$

Standard thermodynamical argument implies that (if the probability distribution $d\mu[h]$ does not contain too long range correlations) in the infinite volume limit the free energy of any magnetic field configuration with non-zero probability coincides with $\bar{F}$. More precisely:

$$\overline{F^2}-(\bar{F})^2 \equiv \int d\mu[h]\,F^2[h]-\left(\int d\mu[h]\,F[h]\right)^2 = O(V^{-1}). \tag{2.4}$$

We can imagine realizing this model by doping a normal ferromagnet with magnetic impurities whose spin is fixed in a random direction by stereo-chemical constraints. If the feedback of the magnetic interaction on the position of the impurities can be neglected, the probability distribution $d\mu[h]$ is not modified by the ϕ field (as in eq. (2.3)): h is quenched. The other possibility (h annealed) corresponds to

$$\tilde{F}_{an} = -V^{-1}\ln\int d\mu[h]\,d[\phi]\exp(-H[\phi,h]). \tag{2.5}$$

The replica trick (which we will discuss later on) allows us to obtain the properties of the quenched system as the analytic continuation of those of a class of annealed systems [2, 3].

Let us introduce some notation: the bar denotes the average over h and the brackets ($\langle\ \rangle$) denote the thermodynamical average over ϕ. There are two kinds of correlation functions: the normal ones:

$$\chi(x_1-x_2) = \overline{\langle\phi(x_1)\phi(x_2)\rangle-\langle\phi(x_1)\rangle\langle\phi(x_2)\rangle} \tag{2.6}$$

and the randomness induced ones:

$$G(x_1-x_2) = \overline{\langle\phi(x_1)\rangle\langle\phi(x_2)\rangle}. \tag{2.7}$$

G will be different from zero also when the spontaneous magnetization $M = \overline{\langle\phi\rangle}$ is equal to zero.

Now we will discuss the mean field approximation:

$$VF[h] = H[\phi_h, h], \tag{2.8}$$

where ϕ_h is the field configuration which minimizes $H[\phi, h]$; ϕ_h satisfies the following differential equation [5]:

$$-\Delta\phi_h(x)+m^2\phi_h(x)+g\phi_h^3(x) = h(x), \tag{2.9}$$

whose solution for m^2 positive, is unique as can be seen from convexity arguments.

In this approximation we find:

$$\begin{aligned} M &= \bar{\phi}_h, \\ G(x_1-x_2) &= \overline{\phi_h(x_1)\phi_h(x_2)}, \\ \chi(x_1-x_2) &= \langle x_1|(-\Delta+m^2+3g\phi^2)^{-1}|x_2\rangle. \end{aligned} \tag{2.10}$$

We will now consider the white noise limit.

$$d\mu[h] = d[h]\exp\left(-\int d^D x \frac{h^2(x)}{2\lambda}\right), \tag{2.11}$$

in other words:

$$\begin{aligned} \overline{h(x)h(y)} &= \lambda\delta^D(x-y), \\ \overline{h(x)h(y)h(z)h(t)} &= \overline{h(x)h(y)}\;\overline{h(z)h(t)}+2 \text{ permutations}. \end{aligned} \tag{2.12}$$

If m^2 is positive, eq. (2.9) can be written as:

$$\phi(x) = \int d^D y\, D(x-y)[h(y)-g\phi^3(y)], \tag{2.13}$$

where

$$D(x-y) = \langle x|(-\Delta+m^2)^{-1}|y\rangle.$$

At $g = 0$ one immediately finds in momentum space

$$\begin{aligned} G(p) &= \lambda/(p^2+m^2)^2, \\ \chi(p) &= 1/(p^2+m^2). \end{aligned} \tag{2.14}$$

The expansion of ϕ in powers of g can be done using eq. (2.13) recursively: in this way, as proved by Symanzik [6], one generates all possible tree diagrams as can be seen in fig. 1.

Similarly the diagrams contributing to the connected correlation function (χ) are shown in fig. 2.

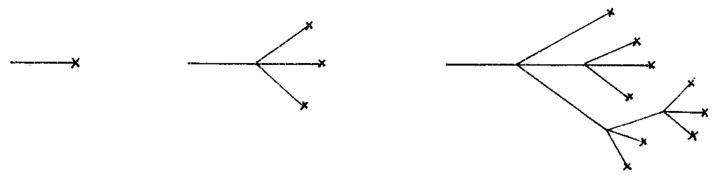

Fig. 1. Diagrams entering in the expansion of ϕ in powers of g. The line and the cross stand for the free propagator and the magnetic field respectively.

Fig. 2. Diagrams for the expansion of $-\Delta+m^2+3g\phi^2$ in powers of g.

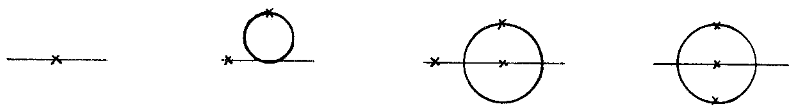

Fig. 3. Diagrams for the expansion of G in powers of g.

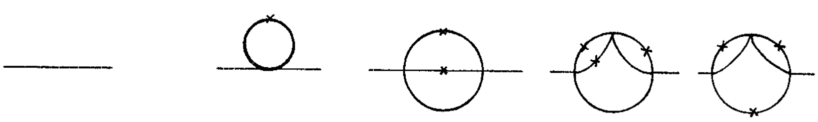

Fig. 4. Diagrams for the expansion of χ in powers of g.

When we average over h field and we take care of the gaussianity of the h distribution we find that all the crosses must coincide at pairs of two. The diagrams contributing to G and χ are shown in figs. 3 and 4 respectively, where the full line is $(p^2+m^2)^{-1}$ and the crossed line is $\lambda(p^2+m^2)^{-2}$. Let us discuss the general rules to obtain all possible diagrams. It is clear that if we neglect the crosses we obtain all possible diagrams of the ϕ^4 theory. We have only to understand where to put crosses. In the case of G the crosses must be such that if we cut the diagram along the corsses, we remain with two disconnected diagrams without loops.

Similarly the crosses for χ must be such that if we cut the diagram along the crosses, we remain with one connected diagram, without loops.

Now let us state the main theorem [5, 7]. Let us consider a conventional field theory with coupling constant g (let us set $\lambda = 1$ for the moment) in dimensions $d = D-2$ with hamiltonian

$$H = \int d^d x \left[\tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2}m^2\phi^2 + \tfrac{1}{2}g\phi^4\right]. \tag{2.15}$$

The correlation function will be defined as usual:

$$C(x-y) = \int d[\phi]\, \phi(x)\phi(y) \exp(-H) \Big/ \int d[\phi] \exp(-H). \tag{2.16}$$

At all orders in g we get:

$$\begin{cases} \tilde{\chi}(p) = \tilde{C}(p), \\ G(x) = C(x), \end{cases} \tag{2.17}$$

where $\tilde{\chi}(p)$ and $\tilde{C}(p)$ are the Fourier transforms of $\chi(x)$ and $C(x)$ done in D and d dimensions respectively.

In other words we have

$$\tilde{C}(p) = \int \frac{d\rho(\mu^2)}{\mu^2 + p^2} = \tilde{\chi}(p),$$

$$\tilde{G}(p) = \int \frac{d\rho(\mu^2)}{(\mu^2 + p^2)^2}. \tag{2.18}$$

The proof is very simple after some non-trivial preliminaries which date back to more than twenty years ago [8].

Let us use the Schwinger representation for the propagator:

$$(p^2 + m^2)^{-1} = \int_0^\infty d\alpha \exp[-\alpha(p^2 + m^2)],$$

$$(p^2 + m^2)^{-2} = \int_0^\infty d\alpha\, \alpha \exp[-\alpha(p^2 + m^2)]. \tag{2.19}$$

If we associate an α_i to each line of the diagram, the contribution of a conventional Feynman diagram can be written as

$$\int dp_i \prod_V \delta(p) \int \prod_L d\alpha_i \exp[-\alpha_i(p^2 + m^2)], \tag{2.20}$$

where $\prod_V \delta(p)$ stands for all the δ functions which are needed to enforce momentum conservation (one for each vertex).

The diagrams with crosses can be obtained by multiplying the integrand in eq. (2.20) with

$$\prod_{i\in\{S\}} \alpha_i, \tag{2.21}$$

where S is the set of lines which have crosses. The integration over the p's can be done independently of the set $\{S\}$. We finally find for the contribution of all topological equivalent diagrams with different crosses:

$$\int \prod_L \mathrm{d}\alpha_i \Big(\sum_{\{S\}} \prod_{i\in\{S\}} \alpha_i \Big) \mathscr{D}(\alpha)^{-D/2} \exp[-N(\alpha)/\mathscr{D}(\alpha) p^2]. \tag{2.22}$$

The contribution of the same diagram in a d-dimensional field theory would be

$$\int \prod_L \mathrm{d}\alpha_i \, \mathscr{D}(\alpha)^{-d/2} \exp[-N(\alpha)/\mathscr{D}(\alpha) p^2], \tag{2.23}$$

$\mathscr{D}(\alpha)$ and $N(\alpha)$ are polynomials in α. Let us now consider the self energy diagrams for $\tilde{\chi}(p)$. If we use the magic formula [8]

$$\mathscr{D}(\alpha) = \sum_{\{S\}} \prod_{i\in\{S\}} \alpha_i, \tag{2.24}$$

which has been proved a long time ago, we see that eqs. (2.22) and (2.23) coincide when $D = d+2$. In the same way one gets the result for G.

We have proved eq. (2.17) at all orders in g for m^2 positive. If the equivalence holds beyond perturbation theory and the mean field approximation is justified, we would have proved the equality of the critical exponents of the random magnetic field model in dimension D with those of a pure system in dimension $d = D-2$. In particular as far as the Ising model in dimension 1 has no transition, we would argue that in the random field model no transition is present in $D = 3$ [9]. However this conclusion is not justified.

Indeed let us see what are qualitatively the predictions in the (λ, m^2) plane for $D > 3$: there is a transition which corresponds to the breaking of the $\phi \to -\phi$ symmetry which for $\lambda > 0$ is shifted to negative m^2. However all the previous arguments were done at $m^2 > 0$ and some care must be taken to extrapolate in the region where $m^2 < 0$. In order to see better the difficulties let us start again from eq. (2.10) for $m^2 > 0$, and use the identity

$$\delta(\phi - \phi_h) = \det(-\Delta + m^2 + 3g\phi^2)\delta((-\Delta + m^2 + g\phi^2)\phi - h). \tag{2.25}$$

We get:

$$G(x,y) = \int \mathrm{d}[\phi]\,\delta(\phi-\phi_h)\phi(x)\phi(y)\mathrm{d}\mu[h],$$

$$= \int \mathrm{d}[\phi]\,\phi(x)\phi(y)\det(-\Delta+m^2+3g\phi^2)$$

$$\times \exp\left\{-\int \mathrm{d}^D x \frac{(-\Delta\phi+m^2\phi+g\phi^3)^2}{2\lambda}\right\}. \qquad (2.26)$$

That is correct if $m^2 > 0$ and the stochastic differential equation has only one solution. Let us consider the case in which there are many solutions:

$$\phi_h^i(x), \quad i = 1,\ldots,n[h]$$

ordered with increasing action. The physical quantity is

$$G^{\mathrm{P}}(x-y) = \int \mathrm{d}\mu[h]\,\phi_h^1(x)\phi_h^1(y), \qquad (2.27)$$

while if we call G^{A} the output of eq. (2.26) we get (see Parisi and Sourlas [10], similar conclusions were reached later by Ceccoti and Girardello [10]):

$$G^{\mathrm{A}}(x-y) = \int \mathrm{d}\mu[h] \sum_{i=1}^{n[h]} \phi_h^i(x)\phi_h^i(y)$$

$$\times\ \mathrm{sign}[\det(-\Delta+m^2+3g(\phi_h^i)^2)], \qquad (2.28)$$

as can be seen going backward and using the relation:

$$\delta(ax-1) = |a|^{-1}\delta(x-a^{-1}).$$

Now G^{A} is obviously analytic in m^2 at $m^2 = 0$ and coincides with the result of the $(D-2)$-dimensional field theory. There are no reasons which imply that $G^{\mathrm{A}} = G^{\mathrm{P}}$ for $m^2 < 0$.

An analysis of the zero-dimensional case is useful. We want to find the number ϕ which is the minimum of the function

$$F[\phi,h] = \tfrac{1}{2}m^2\phi^2 + \tfrac{1}{4}g\phi^4 - h\phi, \qquad (2.29)$$

where h has the distribution probability: $\exp -\frac{1}{2}h^2 \mathrm{d}h$. The correct result for $\overline{\langle\phi\rangle^2}$ is

$$G^{\mathrm{P}} = \int \mathrm{d}\phi \exp[-(m^2\phi+g\phi^3)^2/2](m^2+3g\phi^2)\theta(m^2+g\phi^2)\phi^2, \qquad (2.30)$$

while in G^{A} the θ function would be missing. The singular term in G^{P} around $m^2 = 0$ is given by $(-m^2)^{5/2}\theta(-m^2)$.

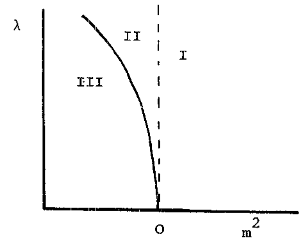

Fig. 5. Schematic phase diagram in the λ, m^2 plane. In regions I and II the spontaneous magnetization is zero, while it is different from zero in region III. The dotted line is at $m^2 = 0$.

The disappointing conclusion is that the dimensional reduction is not valid in the region II of fig. 5, so it does not hold at the transition. The physical solution has a singularity at $m^2 = 0$ which is not present in the dimensionally reduced formulae. The singularity at $m^2 = 0$ follows from general principles: indeed there is a rigorous theorem by Griffith [11] whose generalized heuristic version says: "The free energy of a system with quenched random impurities is singular (C^∞ but not analytic) at the point where the pure system undergoes a phase transition". The physical explanation of this surprising result is that we can have a region of space as large as we want (with very small probability) which is impurity free: the zeros of the partition function can be as near as we want to the real axis and this fact produces the singularity.

We can hope however that the difference between G^{P} and G^{A} will be of order $\exp(-1/g)$ in high dimensions, so that the ε-expansion in $6-\varepsilon$ survives as an asymptotic series. Let us sketch the road to estimate the difference. It is clear that we need to know the minimum value of $\int \mathrm{d}^D x h^2(x)$ for which there are more than one solution to the eq. (2.9). Let us call $\bar{h}(x)$ the field which minimizes $\int \mathrm{h}^2(x)\mathrm{d}^D x$ with the constraint that there is more than one solution, and let us call $\bar{\phi}(x)$ the solution which disappears when we decrease $\int h^2(x)\mathrm{d}^D x$. Let us see what happens for $h' = \bar{h} + \delta h$. We have:

$$\begin{cases} \displaystyle\int \mathrm{d}^D x (h'(x))^2 = \int \mathrm{d}^D x \bar{h}^2(x) + 2\int \mathrm{d}^D x \bar{h}(x)\delta h(x), \\ (-\Delta + m^2 + 3g\bar{\phi}^2)\delta\phi = \delta h, \end{cases} \tag{2.31}$$

where $\phi' = \bar{\phi} + \delta\phi$ is a solution of eq. (2.9) in the field h'.

$\delta\phi$ should not exist for a δh such that

$$\int d^D x \bar{h}(x) \delta h(x) \neq 0;$$

this is impossible only if:

$$(-\Delta + m^2 + 3g\phi^2)\bar{h} = 0. \tag{2.32}$$

We are therefore interested in finding non-trivial solutions of eqs. (2.32) and (2.9) together, i.e. to

$$(-\Delta + m^2 + 3g\phi^2)(-\Delta\phi + m^2\phi + g\phi^3) = 0. \tag{2.33}$$

In other words we must find the stationary points of

$$S[\phi] = \int d^D x (-\Delta\phi + m^2\phi + g\phi^3)^2. \tag{2.34}$$

We have reduced the problem to the estimation of some instanton like solution, however there are still tricky points (e.g. the choice of the boundary condition in the region II) so we shall not push the evaluation of the effect of the Griffith's singularities up to the end. In § 4 we shall see how to obtain eq. (2.33) in the framework of the replica approach.

3. An intermezzo on supersymmetry

We will mention the relation between stochastic differential equations and supersymmetry.

Let us consider eq. (2.26). By introducing fermions ψ and $\bar{\psi}$ it can be written as [5]:

$$G(x-y) = \int d[\phi] d[\psi] d[\bar{\psi}] \exp\left[- \int d^D x S(\phi, \bar{\psi}, \psi) \right] \phi(x)\phi(y),$$

$$S(\phi, \bar{\psi}, \psi) = (-\Delta\phi + m^2\phi + g\phi^3)^2/2\lambda + \bar{\psi}(-\Delta + m^2 + 3g\phi^2)\psi. \tag{3.1}$$

The theory is supersymmetric, i.e. there is a transformation which mixes fermions and bosons and leaves the action invariant. This has been shown explicitly in refs. 5 and 10 where some comments are made on the meaning of supersymmetry breaking. The existence of this supersymmetry allows us to prove the dimensional reduction in a compact way [5, 12]. Unfortunately the theory is not appealing from the point of view of a particle physicist: its fermions have spin 0. However in 2 dimensions it is possible to construct from stochastic differential equations fermions of spin $\frac{1}{2}$.

Let us consider the following system,

$$\partial_1\phi_1+\partial_2\phi_2+g(\phi_1^2-\phi_2^2) = \eta_1,$$
$$-\partial_1\phi_2+\partial_2\phi_1+2g\phi_1\phi_2 = \eta_2, \tag{3.2}$$

$\eta_1(x)$ and $\eta_2(x)$ are uncorrelated white noises. After some manipulations (similar to the previous ones) we find that

$$\overline{\phi_i(x)\phi_j(y)} = \int \mathrm{d}\phi_1\,\mathrm{d}\phi_2\,\mathrm{d}\psi\,\mathrm{d}\bar{\psi}\exp\left[-\int \mathrm{d}^2x S(\phi_1,\phi_2,\psi,\bar{\psi})\right]\phi_i(x)\phi_j(y), \tag{3.3}$$

where

$$S(\phi_1,\phi_2,\psi,\bar{\psi}) = \bar{\psi}[\partial\!\!\!/+g(\phi_1+\gamma_5\phi_2)]\psi$$
$$+\tfrac{1}{2}g^2(\phi_1^2+\phi_2^2)^2+\tfrac{1}{2}(\partial_\mu\phi_1)^2+\tfrac{1}{2}(\partial_\mu\phi_2)^2. \tag{3.4}$$

The model coincides with the two-dimensional supersymmetric Wess Zumino model (for a review see ref. 13). It is natural to pose the following question: "which are the supersymmetric theories that can be written in terms of local stochastic differential equations?". Moreover it was suggested in ref. 10 that the breaking of supersymmetry is related to the existence of an even number of solutions of the associated stochastic differential equation. This could be an alternative way to use topological theorems [14] to study supersymmetry breaking. The field seems to be rather promising but the solution to these problems does not seem to be easy.

4. Random magnetic field (the replica method)

In order to study the random magnetic field model beyond the mean field approximation it is convenient to use the so-called replica trick [3]. The main idea is the following: we define

$$F_n = -\frac{1}{\beta n V}\ln Z_n,$$

$$Z_n = \int \mathrm{d}\mu[h]\{Z[h]\}^n = \overline{Z^n},$$

$$Z[h] = \int \mathrm{d}[\phi]\exp(-\beta H[\phi,h]). \tag{4.1}$$

It is evident that

$$\bar{F} = \lim_{n\to 0} F_n, \tag{4.2}$$

and that for integer n

$$Z[h]^n = \prod_{a=1}^{n} Z[h] = \int \prod_{a=1}^{n} \mathrm{d}[\phi_a] \exp\left\{-\sum_{a=1}^{n} H[\phi_a, h]\right\}. \tag{4.3}$$

The trick consists in using eq. (4.3) for integer n to define an analytic function of n (F_n) and finally to compute $F_0 = \bar{F}$. The method is very safe in perturbation theory where the dependence of F_n on n is a polynomial, while some difficulties are present (depending on the problem) in the non-perturbative region.

Let us start by studying what happens in perturbation theory and why the mean field approximation of the previous section is justified near the transition.

The main advantage of the replica approach is the possibility of integrating immediately over the h field. We are left with the following effective hamiltonian (see refs. 4, 7 and for a review ref. 15)

$$H_{\mathrm{ef}}[\phi] = \int \mathrm{d}^D x \sum_{a=1}^{n} \left[\tfrac{1}{2}(\partial_\mu \phi_a)^2 + \tfrac{1}{2} m^2 \phi_a^2 - \tfrac{1}{2}\lambda \sum_{b=1}^{n} \phi_a \phi_b + \tfrac{1}{4} g \phi_a^4 \right]. \tag{4.4}$$

Let us first consider the case $g = 0$. The propagator $G_{ab} = \langle \phi_a \phi_b \rangle$ satisfies in momentum space the following equation:

$$(p^2 + m^2) G_{ab}(p) - \lambda \sum_a G_{ab}(p) = \delta_{ab}, \tag{4.5}$$

whose solution is given by

$$G_{ab}(p) = \delta_{ab}/(p^2 + m^2) + \lambda/(p^2 + m^2)(p^2 + m^2 - n\lambda). \tag{4.6}$$

Generally speaking the theory is invariant under the group P_n of permutation of n objects; from symmetry arguments it follows that:

$$G_{ab}(p) = \delta_{ab}\chi + G. \tag{4.7}$$

It is easy to verify that the χ and the G introduced in eq. (4.7) coincide with those introduced in eqs. (2.6) and (2.7), in the limit $n \to 0$. It is also clear that the nature of the transition is very different for $n \neq 0$ and $n = 0$. In the first case only G develops a pole at the transition, while in the second case χ and G become critical together. We can now start to do the expansion in powers of g: for generic n the diagrams have a multiplicity factor which is a polynomial in n. The continuation at $n = 0$ does not present problems.

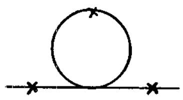

Fig. 6. A diagram having a multiplicity proportional to n.

Fig. 7. Diagrams for χ not included in the mean field approximation.

In accordance with the notation of the previous sections we indicate with a full line the free χ and by a crossed line the free G. Neglecting diagrams which have zero weight when $n \to 0$ (see fig. 6), one finds the same diagrams of the previous section plus other diagrams with less crosses (see fig. 7). However it is evident by dimensional counting that the leading infrared divergent diagrams will have the maximum number of crosses; that justifies the mean field approximation where only these diagrams have been considered. The argument (which produces an upper critical dimension equal to 6) is good near 6 dimensions. Decreasing the dimensions, in particular near 4 dimensions, the other diagrams start to be infrared divergent. A careful analysis of the anomalous dimensions of ϕ^4-like operators is needed (e.g. ref. 16) to decide if these extra diagrams have the effect of changing the critical exponents and destroying dimensional reduction. This problem may be solved in the ε expansion (for $\varepsilon \sim 2$!) or in the loop expansion, using the standard techniques for computing anomalous dimensions of composite operators [17].

Let us now try to see what kind of non-perturbative effects are expected. As is clear from Zinn-Justin's lectures in this Volume, we have to look for instantons, i.e. for localized stationary points of the effective hamiltonian (4.4) [18].

The stationary point equations are:

$$-\Delta\phi_a + m\,\phi_a + g\phi_a^3 = \lambda \sum_{b=1}^{n} \phi_b; \qquad (4.8)$$

now we must make an ansatz of the form of the solution: if we consider

$$\phi_a(x) = f(x), \quad a = 1, \ldots, j, \tag{4.9}$$

we find the standard equation:

$$-\Delta f + (m^2 - \lambda j) f + g f^3 = 0, \tag{4.10}$$

which for $D > 1$ does not have localized solutions for $g > 0$.

Let us now consider a more complicated form for the solution:

$$\phi_a(x) = f_1(x) \quad a = 1, \ldots, j; \quad \phi_a(x) = f_2(x) \quad a = j+1, \ldots, n. \tag{4.11}$$

In the limit $n \to 0$ the number of ϕ's equal to f_1 and f_2 will be respectively j and $-j$. (The appearance of a negative number of objects is one of the disturbing features of the replica approach.)

Eq. (4.8) now becomes

$$-\Delta f_1 + m^2 f_1 + g f_1^3 = -\Delta f_2 + m^2 f_2 + g f_2^3 = \lambda j (f_1 - f_2). \tag{4.12}$$

This equation does not have a completely clear meaning: it says that there is a magnetic field such that eq. (2.9) has two solutions, but why must the magnetic field be proportional to the solutions?

However in the limit $j \to \infty$ we recover something familiar; indeed if we call $j(f_1 - f_2) = h$, in the limit $j \to \infty$ (if h remains finite) we find:

$$-\Delta f_1 + m^2 f_1 + g f_1^3 = h, \quad (-\Delta + m^2 + 3 g f_1^2) h = 0, \tag{4.13}$$

which coincides with eqs. (2.9) and (2.32).

Of course we can consider also other solutions: the number of φ equal to f_0, f_1, f_2 being n, j, $-j$, respectively.

It is clear that there is no bound to the number and to the form of the solutions we can consider: when the replica symmetry group P_n is broken, (instantons have a definite orientation in the replica space), Pandora's box is opened, and we gain nothing by closing it again.

We shall not try to classify the solutions and to interpret them from the physical point of view: we limit ourselves to a few comments.

Instantons in the replica space seem to be connected to Griffith singularities [18]; a careful analysis of the dependence of their contribution as a function of the various parameters g, λ, m^2 and D would be welcome.

Formally in the limit $g \to 0$, $\lambda \to \infty$ at $g\lambda$ fixed the mean field approximation of § 2 is correct. Instantons here have the rôle of breaking the dimensional reduction and of allowing the possibility of a transition at $D = 3$. Contrary to what happens in quantum mechanics

[17] instantons seem to restore (not to destroy) the breaking of the symmetry $\phi \leftrightarrow -\phi$.

There is also a peculiar feature of the instantons for $n = 0$ that I would like to stress. The contribution to the hamiltonian of a P_n symmetric configuration goes to zero when $n \to 0$, while the contribution of an instanton (let us consider the case $j = 1$ for definiteness) remains finite. However there is no contradiction with the fact that the free energy F_n defined in eq. (4.1) remains finite when $n \to 0$. Indeed for small n we have:

$$Z_n \simeq \exp(-n\tilde{F}_0) + w\exp(-H_{\rm I}), \tag{4.14}$$

where βV has been set to 1 for simplicity, $\tilde{F}_0$ is the free energy without instantons, $H_{\rm I}$ is the value of the hamiltonian for the instanton configurations and w is a weight factor which contains the number of possible orientations of the instantons in internal space. In the case $j = 1$ we have to pick one direction from n and we obtain a term proportional to:

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{\Gamma(n+1)}{\Gamma(2)\Gamma(n)} \sim n, \quad n \to 0. \tag{4.15}$$

We finally obtain a finite result when $n \to 0$

$$n^{-1} \ln Z_n \to -F_0 + \exp(-H_{\rm I}) \tag{4.16}$$

as it should be.

Although in the dilute gas approximation instantons are well separated and independent of one another, it is not clear if (when the instanton density increases) the instanton–instanton interaction will not be such as to orient all the instantons in the same direction in replica space, producing a spontaneous breaking of the replica symmetry at the global level.

Summarizing, we understand quite well what happens in the region I of fig. 1, but when we start to move left by decreasing the temperature, we pretty soon loose control of the situation.

5. Spin glasses (the naive approach)

Spin glasses have been the object of an intensive study in the last decade, from both the theoretical and experimental points of view. There are good reviews of the subject [19]. Here I will consider only a very simple theoretical model (the Edward–Anderson (EA) model [3]) and I will spend most of the time discussing the infinite range version of this model (the Sherrington–Kirkpatrick (SK) model [20]) where the mean field approximation is supposed to be exact.

The hamiltonian of the EA model is as follows:

$$H[\sigma] = -\sum_{\{i,k\}} \sigma_i\sigma_k J_{ik} - h\sum_i \sigma_i, \tag{5.1}$$

where h is the external magnetic field, σ_i are Ising spins (± 1) defined on a d-dimensional lattice. The sum on i and k runs on all the possible nearest neighbour pairs of the lattice and $J_{i,k}$ are independent random variables which have a gaussian probability distribution:

$$\overline{J_{i,k}} = 0, \quad \overline{J_{i,k}^2} = d^{-1/2}. \tag{5.2}$$

As before, we are interested in computing the quenched average of the correlation functions and of the free energy.

The infinite range SK model has the same hamiltonian, eq. (5.1), where the index i now goes from 1 to N, the sum over i and j runs on all the possible $N(N-1)/2$ pairs of spins, the $J_{i,k}$ are still gaussian random variable with zero mean and covariance:

$$\overline{J_{i,k}^2} = N^{-1}. \tag{5.3}$$

The thermodynamic limit is obtained when $N \to \infty$. The SK model is essentially the EA model restricted on a single cell of a hypertriangular lattice in the limit $d \to \infty$.

The quenched free energy for $h = 0$ is invariant under the local gauge transformation:

$$J_{i,k} \to -J_{i,k}, \quad \forall k; \qquad \sigma_i \to -\sigma_i. \tag{5.4}$$

Quantities which are not invariant under the transformation (5.4) are obviously zero, at $h = 0$.

Apart from quantities like internal energy, susceptibility and so on, it is interesting to consider the following two quantities

$$q_{\mathrm{EA}} = \overline{\langle\sigma_i\rangle^2}, \quad \chi_R = \sum_k \overline{(\langle\sigma_i\sigma_k\rangle - \langle\sigma_i\rangle\langle\sigma_k\rangle)^2}, \tag{5.5}$$

where the bar denotes the average over the $J_{i,k}$ and the brackets the thermodynamic average over the Ising spins at fixed J.

Although the definition of q_{EA} seems clear, we shall see later on that it is not as unambigous as it looks.

One of typical features of spin glasses is that, for a given choice of $J_{i,k}$, is very difficult to find the ground state, i.e. the configuration of spins σ_i which minimize the hamiltonian. Indeed as a consequence of the randomness of the $J_{i,k}$, it is not possible to fix the σ_i's in such a way that

$$J_{i,k}\sigma_i\sigma_k > 0, \quad \forall i, k. \tag{5.6}$$

If eq.(5.6) could be satisfied the product of all J_{ik} along a closed loop (the Toulouse–Wilson loop [21]) should be positive, but that is impossible, the J's having zero mean. (If the product of the J's along a loop is negative, the loop is said to be frustrated; for the application of the idea of frustration in a different context see ref. 22.)

We must decide which bond must be frustrated, (i.e. is $\sigma_i\sigma_k J_{i,k} < 0$). Different arrangement of the frustrated bonds may differ very little in energy but correspond to very different spin configurations. Briefly, in more than two dimensions all known algorithms for finding the ground state of an N-site spin glass take a time proportional to $\exp aN$ [23]. This multiplicity of groundstates, which are nearly equivalent from the energetic point of view, is responsible for many of the strange properties of spin glasses.

For fixed J's we expect that in the high temperature phase $\langle\sigma_i\rangle = 0$ while, for $T < T_c$, $\langle\sigma_i\rangle$ is different from zero and is oriented in the direction of one of the possible ground states. As far as $\overline{\langle\sigma_i\rangle}$ is obviously zero, the natural order parameter for characterizing the transition is q_{EA}.

At first sight no transition is expected for $h \neq 0$ and χ_R should always be finite (excluding the point $T = T_c$, $h = 0$). However we shall see that this is not the case; the phase diagram is the one shown in fig. 8: χ_R becomes infinite on the transition line which separates regions I and II [24] (and it will remain infinite in the whole of region II). The order

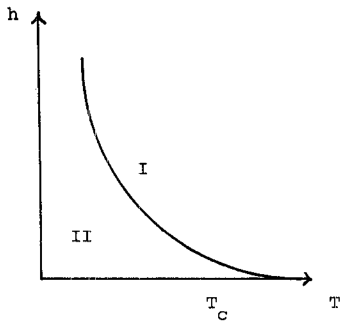

Fig. 8. Region I is the high temperature phase. Region II is the glassy phase where irreversible phenomena are present. The line separating the two regions is the AT line.

parameter characterizing the properties of region II will not be a number but a function $q(x)$ defined on the interval 0–1 [25]. A very slow approach to equilibrium is expected to be present in region II.

In order to see how all this happens, let us start by looking at the solution of the SK model, using the replica trick. With the same notation as the previous section we find after the integration over the $J_{i,k}$ [20]:

$$Z_n = \sum_{\{\sigma_i^a\}} \exp\left\{\frac{\beta^2}{4N}\sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{a=1}^{n}\sum_{b=1}^{n}\sigma_i^a\sigma_i^b\sigma_k^a\sigma_k^b + h\beta\sum_{i=1}^{N}\sum_{a=1}^{n}\sigma_i^a\right\}$$

$$= \int_{-\infty}^{+\infty} \mathrm{d}Q_{ab}\exp[-NA(Q)],$$

$$A(Q) = -\frac{n\beta^2}{4} + \frac{1}{4}\sum_{a=1}^{n}\sum_{b=1}^{n} Q_{ab}^2 - \ln\left\{\mathrm{tr}\left[\exp\left(\frac{\beta}{2}\sum_{a=1}^{n}\sum_{b=1}^{n} Q_{ab}S_aS_b + \beta h\sum_{a=1}^{n} S_a\right)\right]\right\} \tag{5.7}$$

where the indices a and b label the different replicas of the spin system; Q_{ab} is an $n\times n$ matrix, zero on the diagonal, and tr stands for the sum over the 2^n possible values of the n Ising spin variables S_a.

Up to now, we have done legal operations. When N goes to infinity we would like to use the saddle point method and write:

$$\bar{F} = -\frac{1}{\beta}\lim_{n\to 0}\frac{1}{n}\left[\min_{Q} A(Q)\right]. \tag{5.8}$$

The meaning of eq. (5.8) is no clearer than that of a Delphic oracle: should we find the minimum at $n \neq 0$ and analytically continue to $n = 0$? No. The analytic continuation of a minimum may be a maximum. The minimum should be found directly at $n = 0$. However the number of variables corresponding to the Q_{ab} is $n(n-1)$ which is negative for $0 < n < 1$: the concept of a minimum of a function depending on a negative number of variables is rather subtle! Everybody would say that the minimum of $n^{-1}\mathrm{Tr}\,Q^2$ is at $Q = 0$, however if we set

$$Q_{ab} = q \quad \forall a,b \quad (Q_{a,a} = 0),$$

we find

$$n^{-1}\mathrm{Tr}\,Q^2 = (n-1)q^2. \tag{5.9}$$

The point $q = 0$ is (for $n < 1$) a maximum as a function of q. The solution to this apparent paradox is quite simple: the condition that the hessian matrix

$$H_{a,b;c,d} = \frac{\partial^2 A}{\partial Q_{ab}\,\partial Q_{cd}}, \tag{5.10}$$

has positive eigenvalues does not imply that $\langle x|H|x\rangle$ is positive, if $|x\rangle$ belongs to a negative dimensional space (e.g. the trace of the identity is equal to the dimension of the space). A moment of reflection is needed to understand that the necessary condition for the use of the saddle point method is that all the eigenvalues of the hessian must be non-negative. This also guarantees that all the susceptibilities, which are positive definite, are positive indeed.

The final interpretation of eq. (5.8) is the following: we must consider all possible analytic families of matrices $Q_{ab}^{(n)}$, which may depend on real parameters q_i or integer parameters m_i [25]. An analytic family is an infinite set of matrices (one for each n multiple of n_0), such that the P_n invariant quantities, e.g.:

$$\operatorname{Tr} Q^k \quad \text{or} \quad \sum_{a=1}^{n} \sum_{b=1}^{n} (Q_{ab})^k,$$

are analytic functions of n. For each analytic family we should compute the analytic continuation in n up to $n = 0$ of the function $A(Q)$ and of the eigenvalues of the hessian. The final result will be given by that analytic family (hopefully unique) whose elements are stationary points of $A(Q)$, for all n multiples of n_0, and the eigenvalues of the hessian analytically continued at $n = 0$ are non-negative. As far as one can construct analytic families of matrices which depend analytically on the integer parameters m_i, one is allowed also to consider non-integer values of the m_i's.

While it is not clear if this interpretation gives the correct result, it does make the problem well defined from the mathematical point of view but very hard to control from the practical point of view: the space of all analytical families is very large. Up to now the only approach has been to construct ansatzs. Let us see how this works.

The first case we study is:

$$Q_{ab} = q, \quad a \neq b. \tag{5.11}$$

After some simple algebraic manipulations we find that:

$$A(q) = -\frac{\beta^2}{4}(1+q^2) - \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} dz\{\exp[-z^2/2] \ln[2\mathrm{ch}(\beta q^{1/2} z + \beta h)]\}. \tag{5.12}$$

The hessian will have one eigenvector in the one-dimensional space (5.11). One can immediately check that for $n < 1$ the corresponding eigenvalue is positive if $A(q)$ is a maximum (not a minimum!) as function of q.

One finally finds at $h = 0$:

$$q = 0, \quad T > T_c = 1,$$

$$q_{EA} = q = \int_{-\infty}^{+\infty} \frac{dz}{(2\pi)^{1/2}} \{\exp[-z^2/2]\text{th}^2(\beta q^{1/2}z)\} \neq 0, \quad T < 1,$$

$$U(T) \to -\tfrac{1}{2}\sqrt{\pi} = -0.798, \quad q(T) \sim 1-\sqrt{\pi}\,T \quad (T \sim 0),$$

$$C(T) \sim T \quad (T \sim 0),$$

$$\chi(0) \sim \sqrt{\pi}/2, \quad S(0) = -1/2\pi, \tag{5.13}$$

where U, C, χ and S are the internal energy, the specific heat, the susceptibility and the entropy. Now the Monte Carlo results tell us that $U(0) \simeq -(0.76\text{–}0.77)$ in small but definite disagreement with eq. (5.13) and the specific heat is quadratic in T; on the other hand the dependence of $q(T)$ is qualitatively correct (apart from the fact that Monte Carlo data suggest $q(T) \sim 1-aT^2$). Unfortunately the entropy becomes negative at low T and a negative Ising system entropy cannot be tolerated. The situation is more or less elucidated by fig. 9.

We notice that the curve at $q \neq 0$ for $T < T_c$ is definitely better than the $q = 0$ curve and coincides with the experimental line from Monte Carlo up to $T \simeq 0.4T_c$; it is too small near $T \simeq 0$. The shape of the

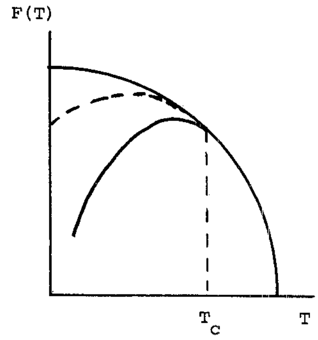

Fig. 9. The lower curve is the free energy for $q = 0$ (i.e. the analytic continuation of the high temperature free energy), the higher curve is the free energy of the computer simulations, the dashed curve in the middle is the free energy given by eq. (5.13).

curves suggests that we have not found the correct saddle point and that if we add extra parameters, they must be such that we have to maximize A as a function of them.

This point has been elucidated by De Almeida and Thouless [24, 26], who computed the other eigenvalues of the hessian: they found an eigenvalue crossing zero on the AT line of fig. 8. They also noticed that χ_R is proportional to the inverse of that eigenvalue, so it becomes infinite on the transition line. The previous computation is therefore wrong in region II of fig. 8, and eq. (5.11) is not the correct choice. Unfortunately eq. (5.11) is the only possible ansatz which is P_n symmetric, we need therefore to break spontaneously the replica symmetry. This approach will be the subject of next section. Let us close this section with a few remarks: The internal energy and the susceptibility at $h = 0$ are given by:

$$U = -\tfrac{1}{4}\beta\,(1-\tilde{\mathrm{T}}\mathrm{r}\, Q^2), \quad \chi = \beta(1-\tilde{\mathrm{T}}\mathrm{r}\, Q),$$

$$\tilde{\mathrm{T}}\mathrm{r} = \lim_{n\to 0} \frac{1}{n}\,\mathrm{Tr}. \tag{5.14}$$

Using eq. (5.11) we get:

$$U = -\tfrac{1}{4}\beta(1-q^2), \quad \chi = \beta(1-q). \tag{5.15}$$

The expression for χ has been derived in full generality by Fisher [27]. Indeed from the linear response theory we know that:

$$\frac{\chi}{\beta} = \sum_k \langle\sigma_i\sigma_k\rangle^c = \sum_k \left[\langle\sigma_i\sigma_k\rangle - \langle\sigma_i\rangle\langle\sigma_k\rangle\right]. \tag{5.16}$$

After the average over the J's, the term with $i \neq k$ gives a zero contribution because of the Z_2 gauge invariance. Only the term with $i = k$ survives. If we recall that $\sigma_i^2 = 1$, we get eq. (5.15) for χ.

It is clear however that the Monte Carlo data [20, 28] do not satisfy eq. (5.15); ($1-q \sim T^2$, $\chi \simeq 1$): the linear response theory does not hold and this is the signal for the breaking of the replica symmetry.

6. Spin glasses (breaking the replica symmetry)

It is clear that we must now find an ansatz different from eq. (5.11). In doing that we will be guided by the following principles [25]:

(a) We should try to remain in the space spanned by the eigenvalues which become negative below the AT line. An explicit computation shows that this condition implies:

$$\sum_b (Q_{ab}-Q_{a'b}) = 0, \quad a \neq a'. \tag{6.1}$$

(b) If we also want to maximize A with respect to the new parameters we shall introduce, we must remain in the space where

$$\tilde{\mathrm{Tr}}\, Q^2 \leqslant 0. \tag{6.2}$$

(c) We should also avoid the possibility

$$\tilde{\mathrm{Tr}}\, Q^2 = \infty. \tag{6.3}$$

A possible way to implement conditions (a) and (c) consists of dividing the n replicas in n/m groups of m replicas. (Of course n must be multiple of m.) We set $Q_{ab} = q_0$ if a and b belong to the same group $Q_{ab} = q_1$ if a and b belong to different groups. (Q_{aa} is always zero.) In other words

$$\begin{aligned} Q_{ab} &= q_0 \quad \text{if } I(a/m) = I(b/m), \\ Q_{ab} &= q_1 \quad \text{if } I(a/m) \neq I(b/m), \end{aligned} \tag{6.4}$$

where $I(x)$ is an integer valued function: its value is the smallest integer greater than or equal to x. Eq. (6.4) provides us with an example of an analytic family of matrices, depending on the parameters q_0, q_1 and m.

It is evident that:

$$\tilde{\mathrm{Tr}}\, Q^2 = (1-m)q_0^2 - mq_1^2 \tag{6.5}$$

is not negative definite if $m > 1$; we must maximize it with respect to q_0 and minimize it with respect to q_1: this automatically leads to a free energy worse than the one obtained in the previous section [29, 30]. However if $0 < m < 1$, condition (6.2) is enforced (obviously m is no more an integer, but we are allowed to do this). After some tedious algebra (it has been remarked that the breaking of the replica symmetry has originated a subculture of very complicated arithmetics [31]) we get:

$$\begin{aligned} A(q_0, q_1, m) = &-\frac{\beta^2}{4}[1 + mq_1^2 + (1-m)q_0^2 - 2q_0^2] \\ &- \int \mathrm{d}p(z) \frac{1}{m} \ln\left\{ \int \mathrm{d}p(y)\, \mathrm{ch}^m[\beta(q_1^{1/2} z + (q_0 - q_1)^{1/2} y)] \right\}, \\ \mathrm{d}p(z) &\equiv \exp(-z^2/2)\mathrm{d}z/(2\pi)^{1/2}. \end{aligned} \tag{6.6}$$

Maximizing A with respect to q_0 and q_1 and m (restricted to the interval 0–1) we obtain the following surprising results [32]: the curves for V, C, χ and q_{EA} (assuming $q_{\text{EA}} = \max Q_{ab} = q_0$) are in very good agreement with the Monte Carlo data (e.g. $V(0) = -0.7652$); the free energy is obviously higher than that obtained in the previous section. The entropy at zero temperature has collapsed from $S(0) \simeq -0.16$ to $S(0) \simeq -0.01$.

We are clearly on the right track! In order to generalise eq. (6.4), let us do some unusual group theory. Eq. (6.4) correspond to breaking the P_n group in the following way

$$P_n \to (P_m)^{\otimes n/m} \otimes P_{n/m}. \tag{6.7}$$

Indeed we can permute both the replicas inside each group (and this leads to the product of n/m times P_m) and the groups among themselves (this leads to $P_{n/m}$). In the limit $n \to 0$ we have the following pattern of symmetry breaking [32, 33]:

$$P_0 \to (P_m)^{\otimes 0} \otimes P_0. \tag{6.8}$$

In other words, P_0 contains itself as a subgroup! It is clear now that we can go on and repeat the same operation many times: we introduce a set of integer numbers m_i ($i = 0, \ldots, k+1$), such that $m_0 = 0$ and $m_{k+1} = n$ and m_i/m_{i-1} is an integer (for $i = 1, \ldots, k+1$). We can divide the n replicas in n/m_k groups of m_k replicas, each group of m_k replicas is divided in m_k/m_{k-1} groups of m_{k-1} replicas and so on. The matrix Q will be given by:

$$Q_{ab} = q_i \quad \text{if } I\left(\frac{a}{m_i}\right) \neq I\left(\frac{b}{m_i}\right)$$

and

$$I\left(\frac{a}{m_{i+1}}\right) = I\left(\frac{b}{m_{i+1}}\right), \quad i = 0, \ldots, k, \tag{6.9}$$

and the q_i's are a set of $k+1$ real parameters. For $k = 1$ we recover the previous example and for $k = 0$ we recover the unbroken symmetry theory.

An easy computation shows that:

$$-\tilde{\text{T}}\text{r}\, Q^2 = \sum_{i=1}^{k} (m_i - m_{i+1}) q_i^2. \tag{6.10}$$

Condition (6.2) is satisfied only if

$$0 \leqslant m_{i+1} \lesssim m_i \leqslant 1. \tag{6.11}$$

From now on let us assume that condition (6.11) is valid.

For each value of k, one can compute the free energy by maximizing it with respect to the q's and the m's. An explicit computation shows that, near T_c, $F^{(k)}$ contains a term proportional to

$$(T-T_c)^5/(2k+1)^4;$$

we are naturally led to consider the case $k \to \infty$. In order to keep track of the parameters q_i and m_i it is convenient to consider the function:

$$q(x) = q_i, \quad m_{i+1} < x < m_i. \tag{6.12}$$

There is a one-to-one correspondence between the piecewise constant functions with k discontinuities and the parameters q_i and m_i. In the limit $k \to \infty, q(x)$ becomes a generic L^2 function on the interval 0–1.

The hierarchical construction (6.9) has produced in a strange way an order parameter which is a function. The utility of the parameterization (6.12) can be easily seen from the formulae valid at $h = 0$:

$$-\tilde{\mathrm{Tr}}\, Q^2 = \int_0^1 \mathrm{d}x\, q^2(x),$$

$$U = -\frac{\beta}{2}\int_0^1 (1-q^2(x))\mathrm{d}x, \quad \chi = \beta \cdot \int_0^1 \mathrm{d}x(1-q(x)). \tag{6.13}$$

We will argue in the next section that

$$q_{\mathrm{EA}} = \max_x q(x) \equiv q_{\mathrm{M}}. \tag{6.14}$$

If $q(x)$ is not a constant the Fischer relation is not satisfied.

Let us now compute the free energy: after some calculations one arrives to the surprising result

$$-\beta F = \max_{q(x)} A[q],$$

$$A[q] = -\tfrac{1}{4}\beta^2\left\{1+\int_0^1 q^2(x)\mathrm{d}x + 2q(1)\right\} - a[q],$$

$$a[q] = f(0,h), \tag{6.15}$$

where the function $f(x, y)$ satisfies the following differential equation:

$$\frac{\partial f}{\partial x} = -\frac{1}{2}\frac{\mathrm{d}q}{\mathrm{d}x}\left[-\frac{\partial^2 f}{\partial y^2} + x\left(\frac{\partial f}{\partial y}\right)^2\right] \tag{6.16}$$

with the boundary condition:

$$f(1,y) = \ln[2\,\mathrm{ch}(\beta y)]. \tag{6.17}$$

Eq. (6.15) is correct only if $q(0) = 0$, otherwise

$$a[q] = \int_{-\infty}^{+\infty} \mathrm{d}p(z) f(0, h+\sqrt{q(0)}\, z). \tag{6.18}$$

The generic shape of the function $q(x)$ which maximises eq. (6.15) is shown in fig. 10. There are arguments which suggest that $q(0) \equiv q_m = 0$ for $h = 0$; more precisely:

$$q(0) \sim |h|^{2/3}. \tag{6.19}$$

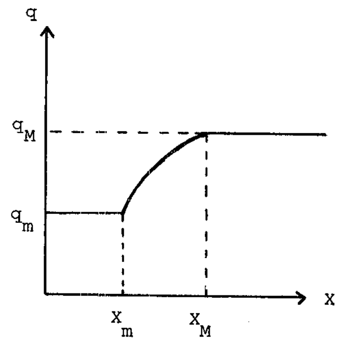

Fig. 10. The generic shape of the function $q(x)$ in region II of fig. 9; $q(x)$ is a constant in the two regions $x < x_m$ and $x > x_M$.

A long argument shows that eq. (6.19) implies that

$$\chi = 1 - \mathrm{O}(h^{4/3}).$$

When we cross the AT line $x_m \to x_M \neq 0$. The following semiempirical rules are exact, or well satisfied, in region II of fig. 8, (in region I $q(x)$ is obviously a constant) [34]:

$$q_m(\beta, h) = q_m(h),$$
$$q_M(\beta, h) = q_M(\beta),$$
$$q(x, \beta, h) = q(x\beta), \quad x_m < x < x_M. \tag{6.20}$$

Numerical investigations support the hypothesis that $S(0) = 0$ in this scheme; the ground state energy estimated is $U(0) \simeq -0.7633 \pm 10^{-4}$. Apart from the region of very small temperature, the results are very similar to those obtained for $k = 1$.

Before discussing in the next section the physical implications of the replica symmetry breaking we notice that (let us consider $h = 0$ for simplicity) eqs. (6.15) and (6.17) may be simplified by introducing the inverse function $x(q)$. We obtain:

$$A[x] = -\frac{\beta^2}{4}\left\{(1+q_{\mathrm{M}})^2 - \frac{2}{\beta}\int_0^{q_{\mathrm{M}}} \chi_A(q) q \, \mathrm{d}q\right\} - a[x],$$

$$a[x] = g(q,0), \quad g(q_{\mathrm{M}}, y) = \ln[2\mathrm{ch}(\beta y)],$$

$$\frac{\partial g}{\partial q} = -\frac{1}{2}\left[-\frac{\partial^2 g}{\partial y^2} + \chi_A(q)\left(\frac{\partial g}{\partial y}\right)^2\right],$$

$$\chi_A = \beta x(q), \quad \chi = \beta(1-q_{\mathrm{M}}) + \int_0^{q_{\mathrm{M}}} \chi_A(q)\mathrm{d}q,$$

$$U = -\frac{1}{2}\left\{\beta(1-q_{\mathrm{M}}) + \int_0^{q_{\mathrm{M}}} \chi_A(q)\mathrm{d}q\right\}. \tag{6.21}$$

Eq. (6.21) for the function g can be associated to a stochastic differential equation [33, 35]:

$$\frac{\mathrm{d}w}{\mathrm{d}q} = \eta(q) - \tfrac{1}{2}\chi_A(q)\mathrm{th}(\beta w),$$

$$\overline{\eta(q_1)\eta(q_2)} = \delta(q_1 - q_2),$$

$$g(0,0) = \overline{g(q_{\mathrm{EA}}, w(q_{\mathrm{EA}}))} = \overline{\ln[\mathrm{ch}(2\beta w(q_{\mathrm{EA}}))]},$$

$$w(0) = 0, \tag{6.22}$$

where η is a white noise and $w(q)$ is an η dependent random function and the bar denotes the average over η.

It is satisfactory that we end up with a stochastic differential equation to control a random system; we must admit however that the derivation is not very direct. Let us make a short speculative comment on eq. (6.22). The relation

$$\chi = \chi_{\mathrm{Fisher}} + \int_0^1 \chi_A(q)\mathrm{d}q$$

suggests that $\chi_A(q)$ parametrizes the anomalous part of the susceptibility. If $\chi_A(q) = 0$, the replica symmetry and the linear response theory would both be exact. We now try to interpret pictorially eq. (6.22). Suppose we add a new spin to a system of N spins. Let us now turn on the interaction of the new spin with the old spins by coupling it to only

$N(q) \equiv Nq/q_{EA}$ spins. The force $h_{ef}(q)$ acting on this spin is

$$h_{ef}(q) = \sum_{k=1}^{N(q)} J_{N+1,k}\langle\sigma_k\rangle. \tag{6.23}$$

We would like to identify $w(q)$ with $h_{ef}(q)$. If the expectation values $\langle\sigma_k\rangle$ are not changed by the addition of the new spin, $h_{ef}(q)$ is a random gaussian variable with covariance q ($dh_{ef}/d(q)$ is white noise) and we recover eq. (6.22) for $\chi_A = 0$. If the addition of a new spin at $N+1$ changes discontinously the values of $\langle\sigma_k\rangle$ we should write [36]

$$dh_{ef}/dq = \eta(q) + \Delta h_{ef}(q), \tag{6.24}$$

where $\Delta h_{ef}(q)$ is the variation of h_{ef} due to the process of jumping between different thermodynamical ground states. It is reasonable to suppose that

$$\Delta h_{ef} = \langle\sigma_{N+1}\rangle\chi_A(q), \tag{6.25}$$

where $\langle\sigma_{N+1}\rangle$ is given by $\text{th}(\beta h_{ef})$ and $\chi_A(q)$ parametrizes the violations of the linear response theory. This derivation of the eq. (6.22) is rather suggestive but might not be very sound. It is however tempting to identify the probability distribution of $w(q_{EA})$ with that of the equilibrium

$$h_{ef}^{e} \equiv \sum_{1}^{N} J_{N+1,k}\langle\sigma_k\rangle.$$

It is amusing to note that eq. (6.20) implies at $T = 0$

$$\chi(q) \sim 1/(1-q)^{1/2}. \tag{6.26}$$

An easy computation shows that, if $w(q_{EA})$ and $h_{ef}(q_{EA})$ are identified:

$$P(h_{ef} \simeq |h_{ef}| \quad \text{for } h_{ef} \sim 0,\ T = 0, \tag{6.27}$$

as is expected from theoretical and numerical arguments [20, 37, 38].

We close this section recalling that very recently all the eigenvalues of the hessian matrix have been computed [39, 40]. Their number is infinite: zero is an accumulation point. Some eigenvalues are zero (χ_R is likely to be infinite) but none are negative, confirming the correctness of the ansatz (6.9) and opening the road to the computation of the corrections to the saddle point approximation, also for the original EA model in d dimensions.

7. Spin glasses (the physical interpretation)

The key result of the previous section was the failure of the linear response theory (i.e. $\chi \neq \beta(1-q_{EA})$).

The only possible explanation is the following: due to the large number of ground states, the direction in which the system is magnetized changes completely when the magnetic field is changed, in a finite volume system we expect this abrupt change on steps of $N^{-1/2}$ in the magnetic field [25, 30, 41, 43]. We expect that the dependence of the magnetization on the field is the one shown in fig. 11. It is clear why, when $N \to \infty$, the linear response theory gives the wrong result. A very nice verification of this picture has been given in ref. 38: they have computed at $T = 0$ the magnetization (for H positive) both in the true ground state and in a metastable state near the $H = 0$ ground state; they find $m \sim H$ and $m \sim H^2$ respectively, see fig. 11. The second result agrees with the linear response theory which predicts

$$\chi_{LR} = (1-q_{EA})/T \sim T.$$

In order to clarify the situation it is better to come back to the precise

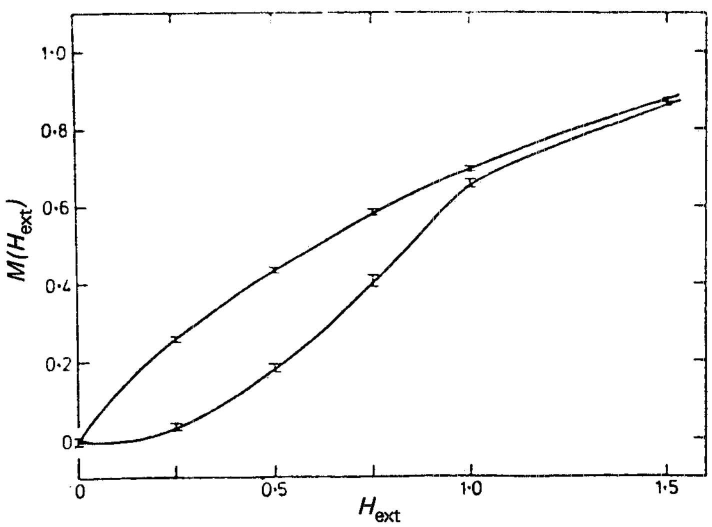

Fig. 11. Magnetization versus external field for $N = 200$. The upper curve is for equilibrium states; the lower curve shows the effect of applying a field to the zero-field ground state and then descending to a nearly local energy minimum.

definition of $q_{\mathrm{EA}} = \overline{\langle\sigma_i\rangle^2}$. Indeed we must define the magnetization $m_i \equiv \langle\sigma_i\rangle$ for a given choice of the J's.

If we consider a real system (or a computer simulation) the magnetizations m_i are defined [41] by

$$m_i = \frac{1}{t}\int_0^t \mathrm{d}\tau\,\sigma_i(\tau), \tag{7.1}$$

where $\sigma_i(\tau)$ is the value that σ_i takes at the time τ and t is a large (macroscopic) but not too large observation time. For example in a d-dimensional ferromagnetic system of size L, t must satisfy the conditions

$$t_m \ll t \ll t_{\mathrm{M}} \approx t_m \exp(L^{d-1}), \tag{7.2}$$

where t_m is the microscopic relaxation time, e.g. one Monte Carlo step. When we change the initial conditions (e.g. $\sigma_i(\tau)$ at $\tau = 0$) we may obtain different results: in the Ising ferromagnetic case below T_c there are essentially two possibilities ($m_i > 0$ or $m_i < 0$); it is important to note that, if the initial state is disordered, the approach to equilibrium for quantities invariant under the global Z_2 is slow (there are corrections proportional to powers of t) while one needs a time t at least proportional to the volume ($t > L^d$) in order to establish a translationally invariant state.

What do we expect for a spin glass? There will be many minima of the thermodynamic potential which are separated by very high walls. At relative short times the system will remain near one minimum, later on at a very large (macroscopic?) time the system will start jumping from one minimum to the other one, by thermodynamic tunnel effects [42, 43].

Let us denote by $m_i^{[\alpha]}$ the expectation value of σ_i in the state labelled by α. We have approximately

$$q(t) \equiv \frac{1}{V}\sum_{i\in V}\left[\frac{1}{t}\int_0^t \sigma_i(t')\mathrm{d}t'\right]^2 = \frac{1}{V}\sum_{i\in V}\left[\sum_{\alpha=1}^{M[t/t_{\mathrm{M}}]} (m_i^{[\alpha]})^2/M[t/t_{\mathrm{M}}]\right], \tag{7.3}$$

where t_{M} is a macroscopic time $M[1] = 1$ and when $t \to \infty$ the sum runs over all possible states of the system. In this language $q(t_{\mathrm{M}}) = q_{\mathrm{EA}}$ while $q(\infty)$ is obviously much smaller ($q(\infty)$ is zero at $h = 0$). In the same spirit we can define a time dependent susceptibility:

$$\chi(t) = [m_{h+\Delta h}(t) - m_h(t)]/\Delta h,$$

$$m(t) = \frac{1}{tV}\sum_{i\in V}\int_0^t \mathrm{d}\tau\,\sigma_i(\tau), \tag{7.4}$$

where the two systems with different magnetic fields coincide at $t = 0$.

$\chi(t)$ will be an increasing function of t, because the jumping from one state to another tends to increase the difference between $m_{h+\Delta h}$ and m_h ($\chi(t_{\mathrm{M}}) = \beta(1-q_{\mathrm{EA}})$ at $h = 0$).

Now it has been found that it is possible to recover the results of the replica approach in this framework [35, 44]: as far as $q(t)$ is a monotonic decreasing function of t one can invert it and obtain t as function of q. One is led to consider the function $\chi(q)$. The final result is at $h = 0$:

$$\chi(q) = \beta(1-q_{\mathrm{EA}}) + \int_q^{q_{\mathrm{EA}}} \chi_A(q)\mathrm{d}q, \tag{7.5}$$

where the function $\chi_A(q)$ coincides with the one defined in the previous section.

Therefore

$$q_m = \lim_{t\to\infty} q(t)$$

and this explains why $q(0)$ is zero at $h = 0$. In this approach the interval of x ($x_m \leqslant x \leqslant x_{\mathrm{M}}$) of the replica space corresponds to the region of very large times. The relation between the internal space of replicas and the real time is most mysterious. We try now to define $q(x)$, the framework of equilibrium, time independent, statistical mechanics. The only reasonable way to define $m_i^{[\alpha]}$ is to use the bootstrap equation [41]:

$$m_i^{[\alpha]} = \lim_{\varepsilon\to 0^+} \langle\sigma_i\rangle_{\varepsilon h^{[\alpha]}},$$
$$h_i^{[\alpha]} \propto m_i^{[\alpha]}, \tag{7.6}$$

where $\langle\ \rangle_{\varepsilon h^{[\alpha]}}$ stands for the expectation value in presence of an extra term in the hamiltonian

$$H = H_0 - \sum_{i\in V} \varepsilon h_i^{[\alpha]}\sigma_i. \tag{7.7}$$

Unfortunately in spin glasses we do not know the direction in which the magnetization points: eq. (7.7) seems to be useless. The way out can be found by introducing two real identical weakly coupled replicas of the same system. The global hamiltonian is

$$H^{(2)} = \sum_{i,k} J_{i,k}(\sigma_i^1\sigma_k^1 + \sigma_i^2\sigma_k^2) - 2\varepsilon\sum_i \sigma_i^1\sigma_i^2.$$

For $\varepsilon = 0$ the two replicas are decoupled, for $\varepsilon > 0$ each of the two replicas acts as an external magnetic field on the other one: both must be locked in the same state. In the limit $\varepsilon \to 0$ we find that $\langle \sigma_i^1 \sigma_i^2 \rangle$ is the average over α of $(m_i^{[\alpha]})^2$. Therefore:

$$q_{\mathrm{EA}} = \langle \sigma_i^1 \sigma_i^2 \rangle = -\frac{\mathrm{d}}{\mathrm{d}\varepsilon} F^{(2)}(\varepsilon)|_{\varepsilon=0},$$

$$F^{(2)}(\varepsilon) = \lim_{V \to \infty} \frac{-1}{2\beta V} \ln \left\{ \sum_{\{\sigma\}} \exp[-\beta H^{(2)}(\varepsilon)] \right\}. \tag{7.8}$$

Now we are very happy because replicas have been introduced in a natural way. It is easy to introduce r replicas; $(a = 1, \ldots, r)$

$$H^{(r)}(\varepsilon) = \sum_{i,k} J_{i,k} \left(\sum_{a=1}^{r} \sigma_i^a \sigma_k^a \right) - \varepsilon \sum_i \left(\sum_{a,b} \sigma_i^a \sigma_i^b - r \right),$$

$$F^{(r)}(\varepsilon) = -\frac{1}{r\beta V} \ln \left[\sum_{\{\sigma\}} \exp(-\beta H^{(r)}(\varepsilon)) \right]. \tag{7.9}$$

The same arguments tell us:

$$Q(r) \equiv -\frac{\mathrm{d}}{\mathrm{d}\varepsilon} F^{(r)}(\varepsilon)|_{\varepsilon=0} = (r-1) q_{\mathrm{EA}}. \tag{7.10}$$

In this way we define the function $Q(r)$ for integer r: the definition can be extended to non integer values of r in a constructive way:

$$F^{(r)}(\varepsilon) = -\frac{1}{\beta r V} \ln \int \mathrm{d}h_i \exp \left\{ -\frac{\beta}{4} \sum_i h_i^2 - r\beta V F[\varepsilon^{1/2} h] \right\},$$

$$rF[\varepsilon^{1/2} h] = -\frac{1}{\beta V} \ln \left\{ \sum_{\{\sigma\}} \exp[-\beta H(\varepsilon^{1/2} h)] \right\},$$

$$H(\varepsilon^{1/2} h) = \sum_{i,k} J_{i,k} \sigma_i \sigma_k + \sum_i (\varepsilon - \varepsilon^{1/2} h_i \sigma_i),$$

$$Q(r) = -\frac{\mathrm{d}}{\mathrm{d}\varepsilon} F^{(r)}(\varepsilon)|_{\varepsilon=0}. \tag{7.11}$$

It is easy to check that for integer r eqs. (7.9) and (7.11) coincide. The function $Q(r)$ tells us how the system behaves in the presence of an external random magnetic field. If the linear response theory is assumed to be valid, we get

$$Q(r) = -1+\chi_{LR}/\beta+rq_{EA}, \tag{7.12}$$

by expanding eq. (7.11) in powers of ε.

The relation $Q(1)$ implies the Fischer result

$$\chi_{LR} = \beta(1-q_{EA}). \tag{7.13}$$

On the other hand under reasonable hypotheses we can show that

$$\chi = \beta(1-Q(0)). \tag{7.14}$$

If we stay in the glassy phase the linear response theory is no longer valid and the function cannot be linear. We want to argue that

$$Q(r) = \int_1^r q(x)\mathrm{d}x, \tag{7.15}$$

where $q(x)$ coincides with the order parameter defined in the previous section for $x < 1$, and it is equal to $q(1)$ foı $x > 1$. If the identification (7.15) is correct, it obviously implies that

$$q_{EA} = q(1) = \max_x q(x)$$

(the function $q(x)$ being monotonic).

Indeed for integer r, we should compute the quenched average of $F^{(r)}$ defined by eq. (7.10). Using the standard formalism we can write:

$$F^{(r)} = \lim_{n\to 0} F_n^{(r)},$$

$$F_n^{(r)} = \frac{1}{rn\beta V}\ln\left\{\sum_{\{\sigma\}}\exp[-\beta H_n^{(r)}(\sigma)]\right\},$$

$$H_n^{(r)}(\sigma) = \sum_{\alpha=1}^{r}\sum_{a=1}^{n}\sum_{i,k}\sigma_i^{\alpha,a}\sigma_k^{\alpha,a}J_{i,k} - \sum_i\left\{\sum_{a=1}^{n}\left[\sum_{\alpha=1}^{r}\sum_{\beta=1}^{r}\varepsilon^{1/2}\sigma_i^{\alpha,a}\sigma_i^{\alpha,\beta}-r\right]\right\}. \tag{7.16}$$

In other words we couple the nr replicas in groups of r; we have introduced an explicit symmetry breaking in one of the directions in which the symmetry was spontaneously broken. This leads to the result for $Q(x)$. It would be rather interesting to find the consequences of this result on the shape of $Q(x)$ and to measure $Q(x)$ in numerical simulations. So far not much work in this direction has been done.

Although there are still some obscure points we now understand reasonably well the static properties of the SK model. There are many things that must be understood in the dynamics, especially as far as the hysteresis curves are concerned.

After the key work of De Dominicis and Kondor [40] the study of the EA model in finite dimensions is open. Unfortunately as in the previous case, the free energy would cease being analytic at a temperature $T_G > T_c$. In the whole region $T_G > T > T_c$ (T_c being the real critical temperature) the free energy is likely to be C^∞ but not analytic.

If we stick to the infinite range model, there are still many applications of this formalism which have not been mentioned (e.g. the extension of these results to the Heisenberg spins [45] and to the random anisotropy case [46]; it would also be rather interesting to compute quantities like

$$q(h_1, h_2) = \overline{\langle \sigma_i \rangle_{h_1} \langle \sigma_i \rangle_{h_2}}, \tag{7.17}$$

they can be easily measured in Monte Carlo simulations and quite likely have a peculiar behavior for $h_1 \sim h_2$ in the glassy phase; e.g.

$$q(h_1, h_2) \simeq q(h_1, h_1) - |h_1 - h_2|^\alpha.$$

The most important field we have neglected are the TAP equations [47] for the local magnetizations m_i; in a first approximation the TAP equations say that:

$$\langle \sigma_i \rangle \equiv m_i = \operatorname{th}\left(\beta \sum_k J_{ik} m_k\right). \tag{7.18}$$

At $T = 0$, they imply for the ground state:

$$\sigma_i = \operatorname{sign}\left(\sum_k J_{ik} \sigma_k\right). \tag{7.19}$$

A configuration satisfying eq. (7.19) is stable against the flipping of a single spin, but it is not necessarily stable with respect to the flipping of $N^{1/2}$ spins. Eq. (7.19) tells us that the configuration is a local minimum of $-H$, (a metastable state!) not a true minimum. It is interesting to note that an explicit computation tells us that the number of solutions of the TAP equation grows like [47, 48]:

$$\exp[\mu(h, T) N], \tag{7.20}$$

where $\mu(h, T)$ is different from zero below the AT line.

$$\mu(0, T) \sim O(T - T_c)^6, \quad T < T_c,$$

$$\mu(0, 0) \sim 0.19.$$

In other words the direct study of the TAP equations confirms the existence of a very large number of metastable states.

Let us now try to compare the theory (in the present rudimentary state) with real experiments; let me select arbitrarily two of the most favourable cases. The main prediction of the theory is the existence of a line in the T, h plane below which the linear response susceptibility χ_{LR} is different from the equilibrium susceptibility χ. Both susceptibilities can be measured: χ_{LR} by applying a magnetic field for a short time (10^{-3}–10^{2} seconds), χ_e by cooling the system in presence of the magnetic field. A typical experiment result is shown in fig. 12 [49].

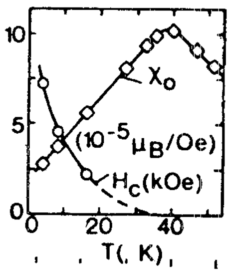

Fig. 12. H_c is the higher field (for a given T) below which irreversible phenomena are present; χ_0 is the short time ($t \simeq O(1)$ s) susceptibility. The sample is CuMn 9%.

At the quantitative level the mean field approximation predicts that below the AT line $d\chi_e/dT = 0$. Experimental results are shown in fig. 13 [50]; we are somewhat at a loss to explain why the corrections to the mean field approximation are so small. It may be that this is due to the relatively long range nature of the interaction between spins:

$$J(x, y) \simeq \frac{\sin|x-y|}{(x-y)^3}.$$

However there is another fact that impresses me more than the experimental success of the broken replica theory of spin glasses: it is the intrinsic strength of the formalism which compells us first to do manipulations whose meaning is obscure from both the mathematical and the physical point of view, and finally, just as the last formulae, gives us a theory (eqs. (6.21) and (6.22)) which explains very complicated phenomena in a simple and direct way. It is the strangest thing I have yet seen.

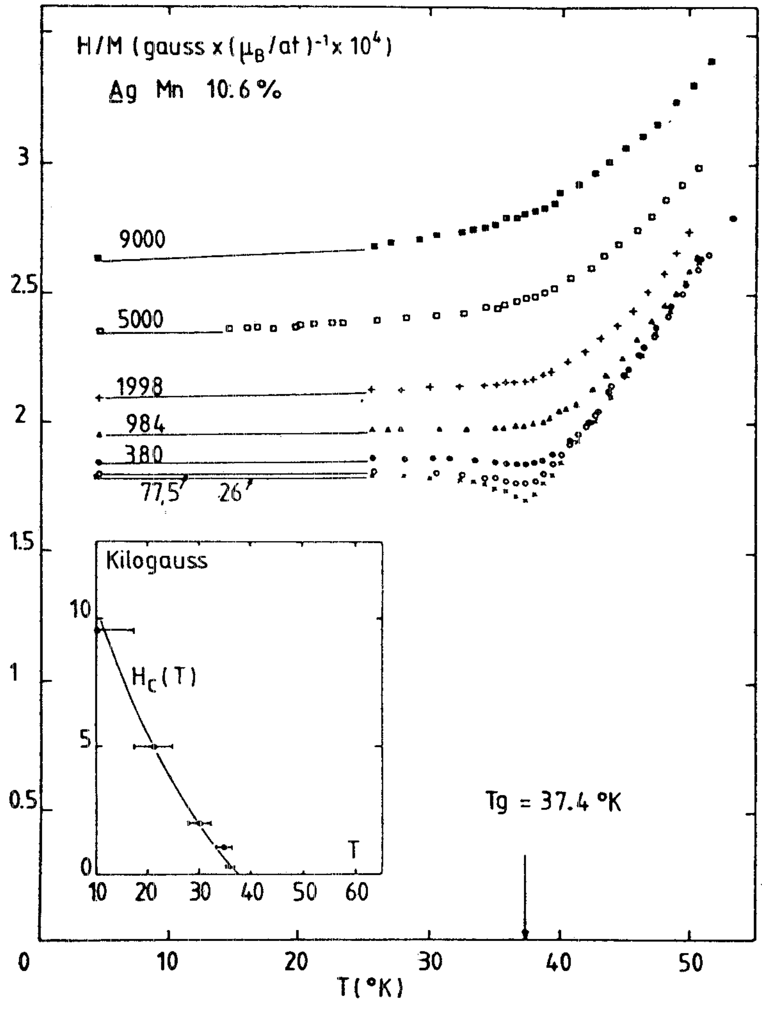

Fig. 13. The ratio H/M for AgMn 10.6% as a function of the magnetic field in Gauss, obtained by cooling the sample in the field. For $H < H_c$, $T(\mathrm{d}/\mathrm{d}T)(H/M)$ is zero with a good approximation.

8. Random potential model (the density of states)

Since the first paper by Anderson [51] on localization of electrons a lot of work has been done on the electronic structure of amorphous materials (for a review see refs. 1, 52–54). In recent years sophisticated field theoretical techniques have been used [55–60]. Here we want to present an introduction to current problems. If one neglects the electron–electron interaction, the problem is reduced to the computation of the levels and the eigenvectors of the Schrödinger operator:

$$H = -\Delta + V(x), \tag{8.1}$$

Here $V(x)$ is a random potential dependent on the atomic structure of the material and Δ is the d-dimensional Laplacian. In the simplest case $V(x)$ is white noise, i.e. it has a Gaussian distribution with covariance:

$$\overline{V(x)V(y)} = \lambda\delta^d(x-y); \quad \overline{V(x)} = 0,$$

where λ plays the rôle of the coupling. In these lectures we will consider only this case.

The goal consists in computing the average over V of the Green function of the operator H. From this knowledge, by filling the levels of H with electrons up to the Fermi energy, one can extract physically interesting quantities such as the density of states, the conductivity of the system and the nature of the electronic states (extended or localized).

A possible approach to this problem consists of mapping it onto the problem of computing the Green function of an appropriate field theory. Let us start by computing the density of states. We define

$$\begin{aligned} G_E(x, y|V) &= \langle x|(H-E)^{-1}|y\rangle, \\ G_E(x-y) &= \overline{G(x, y|V)}. \end{aligned} \tag{8.2}$$

We do not indicate in an explicit way the dependence of G on λ. The density of states is given by

$$\rho(E) = \frac{1}{V}\mathrm{Tr}[\delta(H-E)] = \lim_{\varepsilon\to 0+} \frac{1}{\pi}\mathrm{Im}[G_{E+i\varepsilon}(0)]. \tag{8.3}$$

In order to compute ρ we must set up a computational scheme for $G_E(x)$ at all x. The simplest way to set up a diagrammatical expansion would be to use the identity [61]

$$\frac{1}{H-E} = \sum_{n=0}^{\infty} \frac{1}{-\Delta-E}\left(V(x)\frac{1}{-\Delta-E}\right)^n. \tag{8.4}$$

We prefer however to reduce the problem to a field theory. Let us consider the O(n) invariant action [55, 60]:

$$A[\phi] = \int \mathrm{d}^d x \left\{ \sum_{a=1}^{n} \left[\tfrac{1}{2}(\partial_\mu \phi_a)^2 + \tfrac{1}{2} M^2 \phi_a^2\right] + \tfrac{1}{8} g \left[\sum_{a=1}^{n} \phi_a^2\right]^2 \right\}, \tag{8.5}$$

where ϕ_a are n component fields. The two point correlation function of the field ϕ is given by

$$\langle \phi_a^{(x)} \phi_b^{(0)} \rangle = C(x)\delta_{ab} = \frac{\int \mathrm{d}[\phi]\phi_a(x)\phi_b(0)\exp[-A[\phi]]}{\int \mathrm{d}[\phi]\exp[-A[\phi]]},$$

$$\rho(E) = \lim_{n\to 0} \frac{1}{n} \sum_{a=1}^{n} \mathrm{Im}\langle(\phi_a(0))^2\rangle. \tag{8.6}$$

We shall now prove that $C(x) = G_E(x)$ if $M^2 = -E$, $g = -\lambda$ in the limit $n \to 0$. The proof is straightforward: for $g < 0$ eq. (8.6) can be written as

$$C(x)\delta_{ab} = \frac{\int \mathrm{d}[\phi]\mathrm{d}[V]\exp\{-A[\phi, V]\}\phi_a(x)\phi_b(0)}{\int \mathrm{d}[\phi]\mathrm{d}[V]\exp\{-A[\phi, V]\}},$$

$$A[\phi, V] = \int \mathrm{d}^d x \left\{ \frac{1}{2} \sum_{1}^{n} a[(\partial_\mu \phi_a)^2 + [M^2 - V(x)]\phi_a^2(x)] - \frac{V^2(x)}{2g} \right\}. \tag{8.7}$$

This can be easily proved by integrating over the V field. Now in eq. (8.7) the ϕ integration is gaussian and the final result is:

$$C(x) = \int \mathrm{d}\mu_n[V] G_E(x, 0|V),$$

$$\mathrm{d}\mu_n[V] \propto \mathrm{d}[V]\exp\left[-\int \mathrm{d}^d x\, V^2(x)/2\lambda\right] \det{}^{-n/2}[-\Delta + V - E],$$

$$\int \mathrm{d}\mu_n[V] = 1. \tag{8.8}$$

It is clear that eq. (8.8), for $n = 0$, coincides with eq. (8.2). Before going on let us make two remarks: if λ is positive, g is negative and eq. (8.6) is not defined for any n because the integral over ϕ is divergent at large ϕ because of the ϕ^4 term. Moreover at $g = 0$, the integral is not defined in the interesting region $E > 0$ (i.e. $M^2 < 0$). Of course this difficulty may be removed by declaring that the functional integral is computed in the region of M^2 and g positive and is afterwards analytically continued into the region M^2 and g negative. This problem may be bypassed by doing an explicit rotation of $\exp(-i\pi/4)$ on the fields if $\mathrm{Im}\,E < 0$ and of $\exp(i\pi/4)$ if $\mathrm{Im}\,E > 0$ [61].

The final formula is

$$C(x)\delta_{ab} = \pm i \frac{\int d[\phi]\phi_a(x)\phi_b(0)\exp[-A_\pm[\phi]]}{\int d[\phi]\exp[-A_\pm[\phi]]},$$

$$A_\pm[\phi] = \int d^D x \left\{ \frac{\pm i}{2} \sum_{a=1}^{n} [(\partial_\mu \phi_a)^2 + M^2 \phi_a^2] + \frac{g}{2}\left(\sum_{a=1}^{n} \phi_a^2\right)^2 \right\}, \tag{8.9}$$

where the sign $\pm$ is chosen depending on the sign of $\mathrm{Im}\,E = -\mathrm{Im}M^2$. The existence of two functional representations for $\mathrm{Im}\,E$, positive or negative, explains the presence of a discontinuity at $\mathrm{Im}\,E = 0$:

$$\rho(E) = \frac{1}{2\pi}\mathrm{Disc}[G(E)].$$

This observation will be very useful in the next section.

Those who do not like the $n \to 0$ limit, but insist on using a field theory representation, will be happy to see that in this case eq. (8.6) can be replaced by [62]:

$$G(x) = \frac{\int d[\phi_1]d[\phi_2]d[\bar\psi]d[\psi]\phi_1(x)\phi_1(0)\exp[-A]}{\int d[\phi_1]d[\phi_2]d[\bar\psi]d[\psi]\exp[-A]},$$

$$A = \frac{1}{2}\int d^D x \left\{ \sum_{a=1}^{2} [(\partial_\mu \phi_a)^2 + M^2\phi_a^2] + \partial_\mu\bar\psi\partial_\mu\psi + M^2\bar\psi\psi + g(\phi_1^2 + \phi_2^2 + \bar\psi\psi)^2 \right\}, \tag{8.10}$$

where the ψ are spin zero fermions.

Indeed if we use the usual gaussian integration to reduce the quartic term to a quadratic one, we see that the integration over the fermions produces a $\det[-\Delta + M^2 + V]$ which compensates the $\det^{-1}$ coming from the integration over the ϕ fields. (We are in the presence of a trivial supersymmetry.)

Let us go on and see what happens in the free case ($\lambda = 0$). We easily obtain

$$\rho(E) \sim \int d^D K\, \delta(K^2 - E) \sim E^{(D-2)/2}. \tag{8.11}$$

The density of levels is therefore zero for negative E, as it should be. Let us now make the so-called coherent potential approximation (CPA). In this approximation one retains only the self energy diagram shown in fig. 14 while the full propagator is used in the blob. We finally get:

$$M^2 = m^2 + \lambda \int_0^{\Lambda} \frac{d^D K}{m^2 + K^2} \simeq m^2 - \lambda(m^{D-2} - \Lambda^{D-2}), \tag{8.12}$$

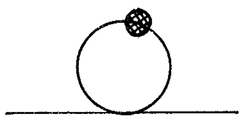

Fig. 14. The self-energy diagrams of the CPA approximation. The blob denotes the full propagator.

Λ is as ultraviolet cut-off. The CPA approximation is exact for $n \to \infty$, but let us use it at $n = 0$ for pedagogical reasons. In a normal field theory λ is negative; eq. (8.12) would imply therefore

$$\begin{aligned} &(M^2 - M_c^2) \simeq m^2, && D > 4, \\ &m^2 \sim (M^2 - M_c^2)^{2/(D-2)}, && D < 4, \\ &M_c^2 < 0. \end{aligned} \tag{8.13}$$

However in this case λ is positive, the right hand side of eq. (8.12) has the behavior shown in fig. 15 for $D < 4$. This means that one obtains:

$$\begin{aligned} &m^2 = m_c^2 + (M^2 - M_c^2)^{1/2}, \\ &m_c^2 > 0, \quad M_c^2 > 0, \\ &\rho(E) \propto (E + M_c^2)^{1/2}\theta(E + M_c^2), \quad E \sim -M_c^2. \end{aligned} \tag{8.14}$$

In fig. 16 we show the $\rho(E)$ of the free theory and of the CPA approximation.

It is rather interesting that m^2 does not become zero at the transition. One might think that the result of the CPA approximation is stable against higher order corrections; infrared divergences should be absent since the renormalized mass m^2 is non-zero at the edge of the band ($E = E_c \equiv -M_c^2$). Unfortunately the ϕ^2 propagator is a source of infrared divergences. Indeed let us define:

$$\begin{aligned} &\langle \phi_a(x)\phi_b(x)\phi_c(0)\phi_d(0)\rangle - \langle \phi_a(x)\phi_b(x)\rangle\langle \phi_c(0)\phi_d(0)\rangle \\ &= \delta_{ab}\delta_{cd}R(x) + (\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\chi(x). \end{aligned} \tag{8.15}$$

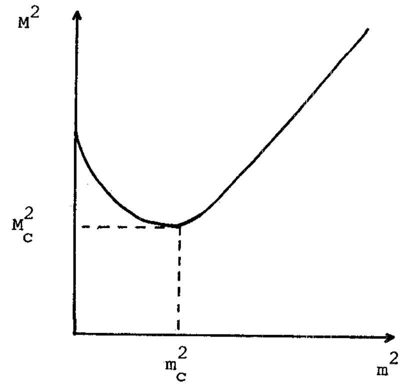

Fig. 15. M^2 versus m^2 in the CPA approximation.

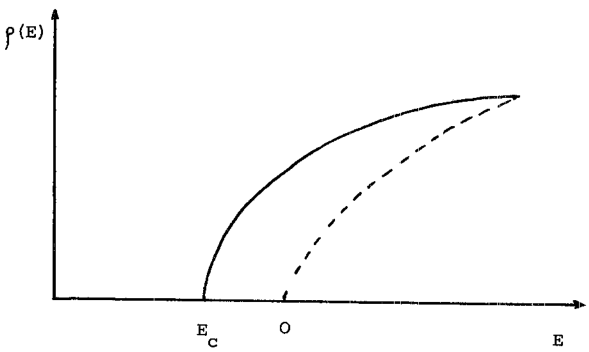

Fig. 16. The density of states in the free case (dashed line) and in the CPA approximation (full line).

It is easy to prove that for $n = 0$

$$R(x) = \overline{G_E(x, x|V)G_E(0, 0|V)} - \overline{G_E(x, x|V)}\;\overline{G_E(0, 0|V)},$$
$$\chi(x) = \overline{G_E^2(x, 0|V)}. \tag{8.16}$$

If one computes R and G in momentum space, in the CPA approximation, by summing all the diagrams shown in fig. 17 one finds

$$R(p) \sim [(M^2 - M_c^2)^{1/2} + p^2 + O(p^4)]^{-2};$$
$$\chi(p) \sim [(M^2 - M_c^2)^{1/2} + p^2 + O(p^4)]^{-1}. \tag{8.17}$$

Fig. 17. Diagrams for G (a) and χ (b) in the CPA.

This result may be checked by noticing that

$$\chi(p)|_{p=0} = -\frac{\mathrm{d}}{\mathrm{d}M^2}\lim_{n\to 0}\langle\phi^2\rangle = -\frac{\mathrm{d}m^2}{\mathrm{d}M^2}\frac{\mathrm{d}}{\mathrm{d}m^2}\int\frac{\mathrm{d}^D K}{(K^2+m^2)}$$

$$= \frac{\mathrm{d}m^2}{\mathrm{d}M^2}\int\frac{\mathrm{d}^D K}{(K^2+m^2)^2}, \tag{8.18}$$

and that $\mathrm{d}m^2/\mathrm{d}M^2$ goes like $(M^2-M_c^2)^{-1/2}$ according to eq. (8.14).

In field theory language the ϕ^4 interaction is attractive and not repulsive as usual: a two-particle bound state is produced [59] and by decreasing the mass of the ϕ particle (m) the mass of the bound state becomes zero at $m \neq 0$. The self interaction of this bound state produces infrared divergences e.g. the diagram of fig. 18. These infrared divergences can be easily studied using the standard machinery [63, 64]: one introduces a field $q_{ab} = \phi_a\phi_b$, and derives an effective lagrangian for the field q.

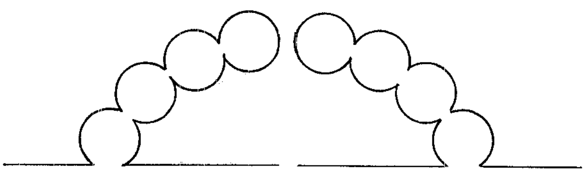

Fig. 18. Leading infrared divergent diagram for the corrections of the CPA.

An interaction proportional to q^3 is present suggesting that the corrections to the critical behavior vanish for $D > D_c = 6$ and corrections to the exponent $\frac{1}{2}$ in eq. (8.14) can be computed in powers of $\varepsilon = D_c - D$ for $D < D_c$. An explicit computation shows that as in the polymer case D_c is shifted from 6 to 8 (indeed the two theories seem to belong to the same universality class [64]).

We will not enter into details because the problem is purely academic [59]; let us summarize the situation: we are in the presence of a theory with a negative coupling constant and we suppose that the imaginary part of the Green functions starts when the mass becomes sufficiently small that a massless two-particle bound state is produced. However naive arguments tell us (as remarked by Stone) that the mass of an N particle state is:

$$M_N = Nm - \lambda(N^3 - N)/m^{3-D}. \tag{8.19}$$

In other words we expect that a three-particle bound state would become massless before the two-particle bound state. And so on. Moreover since the two-particle bound state may decay into two three-particle bound states, it becomes a resonance, its mass acquires an imaginary part and it does not become zero for real E. If we take care of only the three and two particle bound states the situation is the one shown in fig. 19. The two particle cut is shifted onto the second sheet.

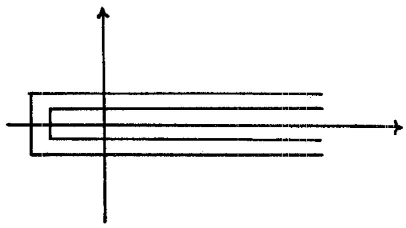

Fig. 19. The cut structure in the complex energy plane, considering only two- and three-particle bound states.

The final effect is to shift all the singularities onto the second sheet leaving a regular $\rho(E)$. The behaviour of $\rho(E)$ for $E \to -\infty$ is dominated by the condensation into the vacuum of $N(N \to \infty)$ particles, and this phenomenon is just the instanton in a different guise [65].

Indeed as shown in the lectures of Zinn-Justin in this Volume one can find the imaginary part of the Green function by looking for localized solutions of the classical equation. In our case we assume that

$$\phi_1(x) \equiv \phi(x) \neq 0;$$

we must solve the equation:

$$-\Delta\phi + M^2\phi = \lambda\phi^3. \tag{8.20}$$

The imaginary part of the Green function behaves for $\lambda \to 0$ in less than four dimensions as

$$\rho(E) \propto \frac{1}{\pi} \operatorname{Im} G_E(0) \simeq \exp[-cM^{2(4-D/2}/\lambda] \sim \exp[-c(-E)^{(4-D)/2}/\lambda]. \tag{8.21}$$

Here we have neglected the prefactor, which is computed in ref. 66.

Eq. (8.20) can be understood directly: we want to know the probability of finding a solution of the Schrödinger equation:

$$[-\Delta + V(x)]\psi = E\psi, \tag{8.22}$$

for a given E. We must therefore minimize $\int d^D x\, V^2(x)$ with the constraint (8.22). An easy computation shows that the constrained minimum problem is equivalent to the pair of equations [67]:

$$(-\Delta + V(x) - E)\psi = 0,$$
$$V(x) \propto \psi^2(x), \tag{8.23}$$

hence eq. (8.20).

The final conclusion is that $\rho(E)$ is an analytic function which can be computed in the standard perturbative expansion when $E \to +\infty$ and in the perturbative expansion around the instanton when $E \to -\infty$. I am convinced that using appropriate resummation techniques one can

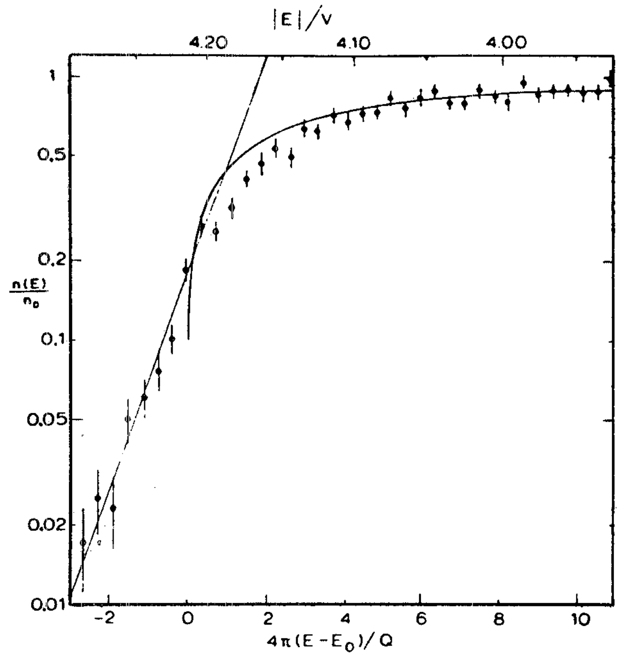

Fig. 20. Numerical results for the density of states in $D = 2$. The curve is the CPA approximation and the straight line is an exponential fit. The one instanton computation agrees with the data for $E - E_0 \sim -(2\text{–}1)$. E_0 is the edge of the band in the CPA.

obtain reasonable approximations in the whole E range. Of course $\rho(E)$ may be evaluated in two dimensions by solving the Schrödinger equation numerically on a large lattice; the results are shown in fig. 20 [68].

9. Random potential model (localisation)

Up to now we have discussed only the spectral density. It is also very important to know if the spectrum is discrete (localisation) or continuous (extended states). The "eigenstates" of the continuum spectrum may be functions that do not go to zero at infinity (Ex) (like plane waves) or functions which decay like a power, but are not L^2, in the last case the states are said to be quasi-extended (QE) or power localized.

It is very important to consider the following quantity:

$$\chi_E(x) = |G_E(x, 0|V)|^2. \tag{9.1}$$

The simple identity [69]

$$1/(H-\bar{E}) - 1/(H-E) = (\bar{E}-E)/[(H-\bar{E})(H-E)] \tag{9.2}$$

tells us that:

$$\chi_E(p)|_{p=0} = \frac{1}{\varepsilon}\rho(E_R), \quad E = E_R + \mathrm{i}\varepsilon. \tag{9.3}$$

We expect that in the localized region and in the extended region, respectively,

$$\chi_E(p) \sim \frac{1}{\varepsilon} f(p) \qquad \text{(L)},$$

$$\chi_E(p) \sim \frac{\rho(E_R)}{\varepsilon + \sigma p^2 + \mathrm{O}(p^4)} \quad \text{(Ex)}. \tag{9.4}$$

The quantity σ can be identified with the conductivity which is given by

$$\lim_{\varepsilon \to 0}\left[-\varepsilon^2 \frac{\mathrm{d}}{\mathrm{d}p^2}\chi_{E_R+\mathrm{i}\varepsilon}(p^2)|_{p^2=0}\right]\Big/\rho(E_R).$$

If we transcribe the problem in a field theory language, using the representation (8.9) we find in the limit $n \to 0$:

$$\chi_{E_R+i\varepsilon}(x) = \frac{\int d[\phi_+]d[\phi_-]\,\phi_+^1(x)\phi_-^1(x)\phi_+^1(0)\phi_-^1(0)\exp(-A[\phi_+,\phi_-])}{\int d[\phi_+]d[\phi_-]\exp(-A[\phi_+,\phi_-])},$$

$$A[\phi_+,\phi_-] = \int d^Dx\,\frac{1}{2}\sum_{a=1}^{n}\{i(\partial_\mu\phi_+^a)^2 - i(\partial_\mu\phi_-^a)^2 + iE_R[(\phi_+^a)^2-(\phi_-^a)^2] + \varepsilon[(\phi_+^a)^2+(\phi_-^a)^2]\} + \lambda\left\{\sum_{a=1}^{n}[(\phi_+^a)^2-(\phi_-^a)^2]\right\}^2. \tag{9.5}$$

Now the action in eq. (9.5) is invariant under the non-compact $O(n,n)$ group in the limit $n \to 0$, if $\varepsilon = 0$ [59]. If $\varepsilon \neq 0$, the group is broken to $O(n) \otimes O(n)$ in an explicit way. Now since

$$\langle(\phi_+^1)^2\rangle = i\overline{G_{E_R+i\varepsilon}(x,x|V)},$$
$$\langle(\phi_-^1)^2\rangle = i\overline{G_{E_R-i\varepsilon}(x,x|V)}, \tag{9.6}$$

we get:

$$\langle(\phi_+^1)^2\rangle - \langle(\phi_-^1)^2\rangle \simeq 2\pi\rho(E_R), \quad \varepsilon \to 0. \tag{9.7}$$

Therefore if $\rho(E) \neq 0$, the $O(n,n)$ symmetry is kinematically broken: that also happens in the free theory! In some sense in the localized phase the symmetry is restored as far as the symmetry breaking part of the Green functions becomes of order ε with respect to the symmetry invariant part which goes to infinity. This peculiar way of restoring the symmetry is strictly connected with the non-compactness of the model [59]. An easy exercise shows that eq. (9.3) is a Ward identity of the $O(n,n)$ symmetry and the singularity at $p^2 = 0$ in $\chi(p)$ is a Goldstone singularity [58–60]. Now as shown in Fradkin's lectures in this Volume the infrared singularities of the Goldstone bosons are well represented by a non-compact σ-model which is asymptotic and free in two dimensions. Now the instanton computation tells us that for $E \to -\infty$ the states are localized while the perturbative expansion in $D > 2$ tells us that for $E > 2$ the states are extended. A transition must happen somewhere: the simplest scenario is shown in fig. 21 (left) where we consider the $\lambda - D$ plane for a fixed positive energy E: indeed the asymptotic freedom of the model in two dimensions tells us that the extended region is unstable. However the states, if they are not extended, can be Quasi-Extended and the other scenario of fig. 21 may be possible [70]. Also if fig. 21 (left) is correct it is

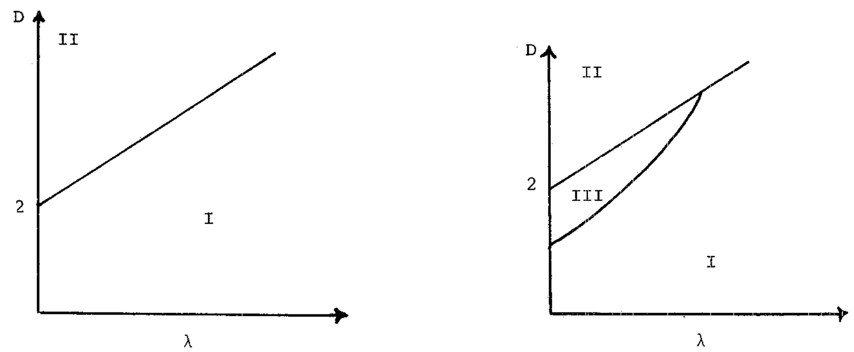

Fig. 21. On the left a possible phase diagram for a fixed positive E in the D–λ plane. In region I all states are localized, in region II all states are extended. On the right an alternative phase diagram for a fixed positive E in the D–λ plane. In region I all states are localized, in region II al states are extended and in region III all states are quasi-extended. The shape of the region III is a pure guess.

not clear if the transition in $D > 2$ from localized to extended states is second order, σ vanishing at the transition ($\sigma = 0$ in the localized region), or if it jumps at the transition, (Mott minimum metallic conductivity [71]). In the same way what happens to gauge theories in $4+\varepsilon$ dimensions (first or second order transition) is completely unknown.

This situation is somewhat frustrating: the localization transition cannot be studied in $6-\varepsilon$ or $8-\varepsilon$ dimensions because the existence of localized states is a non-perturbative phenomenon; the σ-model representation for the infrared divergences tells us that extended states do not exist in $D = 2$ but it does not tell us what exists in their place. In my opinion the situation may be clarified only by working directly in the localized phase using either the original hopping parameter expansion [51] or a model of a dense gas of interacting instantons. It is quite possible that the correct phase diagram has been already guessed. Unfortunately up to now, numerical simulations have not reached an agreement among different groups.

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