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DIRAC BRACKETS FOR GALILEAN GAUGE THEORIES

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ABSTRACT

The constraints appearing in the quantization of Galilean gauge theories are redundant. They are replaced by independent constraints, which allow a direct evoluation of Dirac brackets. The canonical variables are then determined.

1. INTRODUCTION

In a previous paper, to be referred to as I, Galilean gauge theories have been quantized according to Dirac theory of constrained systems(1). Second class constraints and first class constraints with their gauge fixing have been solved for the color group SU(2) and the independent variables have been determined. In order to identify among these variables the canonical ones, however, one has to evaluate the Dirac brackets, and this cannot be done directly because the constraints are redundant. In this paper we replace the original constraints by equivalent but independet constraints. We then evaluate the Dirac brackets and determine the canonical variables.

We report for convenience the basic equations concerning the original constraints, but to avoid repetitions we do not write a selfcontained paper and assume a preliminary reading of I.

2. INDEPENDENT CONSTRAINTS

The constraints appearing in the quantization of Galilean gauge theories are

$$\phi^a = \mathscr{D}_k^{ab} E_k^b = 0 \ , \tag{1}$$

$$\phi_i^a = \varepsilon_{ijk} \ F_{jk}^a = 0 \qquad ; \chi_i^a = \varepsilon_{ijk} \mathscr{D}_j^{ab} E_k^b = 0 \ . \tag{2}$$

In the above equations F_{jk}^a are the stress tensors, $\mathscr{D}_k^{ab}$ the covariant derivatives in the adjoint representation and E_i^a the momenta canonically conjugate to the gauge fields A_i^a.

Since the quantization has been performed in a cubic box of volume L^3 with periodic boundary conditions, fields and constraints have been expanded in Fourier series. The general notation adopted is

$$f(x) = \frac{1}{L^{3/2}} \sum_{\vec{n}} f_{\vec{n}} e^{i \frac{2\pi}{L} \vec{n} \cdot \vec{x}} \ .$$

The constraints (1) are first class, so that it is possible, and it has been found convenient in I to select the zero-momentum component ϕ_o^a as a condition on the physical states. The components $\phi_{\vec{n}}^a$ with $\vec{n} \neq 0$ have instead been supplementd by appropriate gauge fixing constraints which do not affetc A_{io}^a

$$A_{3\vec{n}}^a = 0 \ , \ n_3 \neq 0$$

$$A_{2\vec{n}}^a = 0 \ , \ n_3 = 0 \ , n_2 \neq 0 \tag{3}$$

$$A_{1\vec{n}}^a = 0 \ , n_3 = n_2 = 0 \ , n_1 \neq 0 \ .$$

It I it has been shown that the whole set of constraints, with the exception of ϕ_o^a, imply

$$A^a_{i\vec{n}} = E^a_{i\vec{n}} = 0, \quad \vec{n} \neq 0 \, . \qquad (4)$$

$$\begin{aligned} &\varepsilon_{ijk} \, \varepsilon^{abc} \, A^b_{jo} \, A^c_{ko} = 0 \\ &\varepsilon_{ijk} \, \varepsilon^{abc} \, E^b_{jo} \, A^c_{ko} = 0 \, . \end{aligned} \qquad (5)$$

Equations (5) admit the solution

$$\begin{aligned} &A^a_{io} = q_i \, \hat{v}^a \, , \; \hat{v}^a = \frac{v^a}{v} \\ &E^a_{io} = p_i \, \hat{v}^a + \frac{1}{q^2} \, q_i \, l^a_\perp \, , \; \hat{v}^a \, l^a_\perp = 0 \end{aligned} \qquad (6)$$

as already noted in I.

We will now show that the formulae (6) actually represent the most general solution of Eq. (5). This will anable us to replace the original redundant constraints by equivalent, independent ones.

Eqs. (5) can be written in compact form

$$\begin{aligned} &M\,(A, A) = 0 \\ &M\,(A, E\,) = 0, \end{aligned} \qquad (7)$$

where M(A,E) is the 3x3 matrix whose elements are

$$\left[M\,(A, E)\right]^a_i = \varepsilon_{ijk} \, \varepsilon^{abc} \, A^b_{jo} \, E^c_{ko} \, . \qquad (8)$$

It is easy to verify that for arbitrary nonsingular 3x3 matrices R, S

$$M\,(RAS, RES) = (R^{-1})^T \, M\,(A,E)\,(S^{-1})^T \det R \det S \, . \qquad (9)$$

Let A be a solution of M (A,A) = 0. It is then possible to determine a pair of nonsingular matrices R,S in such a way that

$$A = R\bar{A}S\ ,\ \bar{A}^a_{io} = \delta^a_i \lambda_a\ ,\ \lambda_a = 0,1\ . \tag{10}$$

It follows from Eq. (9) that $M(\bar{A}\ ,\ \bar{A}) = 0$, i.e. $\varepsilon^{abc}\ \varepsilon_{ibc}\lambda_b\ \lambda_c = 0$. Therefore either $\lambda_a = 0$ for every a, or there is at most one value of a for which $\lambda_a \neq 0$.

For a nonvanishing solution we can assume

$$\bar{A}^a_{io} = \delta^{a3}\ \delta_{i3}\ , \tag{11}$$

and therefore we derive from Eq. (10) that the most general solution is

$$A^a_{io} = (R\bar{A}S)^a_i = R^a_3\ S^T_{i3} \stackrel{\text{def}}{=} q_i\ v^a\ . \tag{12}$$

We want now to solve the other equation M(A,E)=0. Eq. (9) with R and S satisfying Eq. (10) yields

$$\left[\phi\,(\bar{A},\,\bar{E})\right]^a_i = \varepsilon^{a3c}\ \varepsilon_{i3k}\ \bar{E}^c_{ko} = 0\ ,\ E = R\bar{E}\,S\ , \tag{13}$$

which requires $\bar{E}^a_{io} = \delta^{a3}\ \bar{\xi}_i + \delta_{i3}\ \bar{\eta}^a$, with $\bar{\xi}_i$, $\bar{\eta}^a$ arbitrary.

It follows that

$$E^a_{io} = (R\bar{E}\,S)^a_i \stackrel{\text{def}}{=} v^a\ p_i + l^a\ q_i\ . \tag{14}$$

The variables v^a, l^a, q_i, p_i are not all essential. The replacement $v^a \rightarrow \gamma v^a$, $l^a \rightarrow \gamma l^a$, $q_i \rightarrow \frac{1}{\gamma}\ q_i$, $p_i \rightarrow\ \frac{1}{\gamma}\ p_i$, leaves A^a_{io} and E^a_{io} unchanged. This redundance is eliminated by replacing v^a by $\hat{v}^a$ and l^a by $l^a_\perp$ according to Eqs. (5). We have thus proved that Eqs. (5) provide the most general solution to Eqs. (7).

We can now proceed to replace the constraints (7) by independent ones.

Let us denote by $\mathcal{N}$ the set (A,E) of the solutions (5) with $A \neq 0$, and by $\mathcal{N}^{\bar{a}}_{\bar{i}}$ the subset of $\mathcal{N}$ with $A^{\bar{a}}_{\bar{i}o} \neq 0$. It is clear that $\mathcal{N} = \bigcup_{\bar{a}\,\bar{i}} \mathcal{N}^{\bar{a}}_{\bar{i}}$. It is now easy to produce a family of independent constraints which locally (in each of the regions $\mathcal{N}^{\bar{a}}_{\bar{i}}$) describe the set $\mathcal{N}$.

In $\mathcal{N}^{\bar{a}}_{\bar{i}}$, $q_{\bar{i}}\ \ v^{\bar{a}} \neq 0$, so that for $(A,E) \in \mathcal{N}^{\bar{a}}_{\bar{i}}$ we can write

$$A^a_{io} = \frac{\hat{v}^a\ q_{\bar{i}}\ \ \hat{v}^{\bar{a}}\ q_i}{\hat{v}^{\bar{a}}\ q_{\bar{i}}} = \frac{A^a_{\bar{i}}\ \ A^{\bar{a}}_i}{A^{\bar{a}}_{\bar{i}}}\ . \tag{15}$$

$$
\begin{aligned}
E^{a}_{io} &= \vartheta^{a}\, p_i + l^{a}_{\perp}\, q_i \\
&= \frac{1}{\vartheta^{\bar a}\, q_{\bar i}} \left\{ \vartheta^{a}\, q_{\bar i}\, \vartheta^{\bar a}\, p_i + l^{a}_{\perp}\, q_{\bar i}\, \vartheta^{\bar a}\, q_i \right\} \\
&= \frac{1}{A^{\bar a}_{\bar i o}} \Big\{ \vartheta^{a}\, q_{\bar i}\, (\vartheta^{\bar a}\, p_i + l^{\bar a}_{\perp}\, q_i) + \vartheta^{\bar a} q_i\, (\vartheta^{a}\, p_{\bar i} + l^{a}_{\perp}\, q_{\bar i}) \\
&\quad - \vartheta^{a}\, q_{\bar i}\, l^{\bar a}_{\perp}\, q_i - \vartheta^{\bar a}\, q_i\, \vartheta^{a}\, p_{\bar i} \Big\} = \\
&= \frac{1}{A^{\bar a}_{\bar i o}} \left\{ A^{a}_{\bar i o}\, E^{\bar a}_{io} + A^{\bar a}_{io}\, E^{a}_{\bar i o} - A^{a}_{io}\, E^{\bar a}_{\bar i o} \right\}
\end{aligned}
\qquad (16)
$$

Equations (15) and (16) are identities for $a = \bar a$ or $i = \bar i$. For $a \neq \bar a$, $i \neq \bar i$ they constituite a set of 8 independent constraints which express A^{a}_{io}, E^{a}_{io} as functions of 10 independent quantities $A^{\bar a}_{io}$, $A^{a}_{\bar i o}$, $E^{\bar a}_{io}$, $E^{a}_{\bar i o}$. It should be observed that the form of the equations (15) and (16) is the same in all the regions $\mathcal{N}^{\bar a}_{\bar i}$. As a consequence, the calculation of the dirac brackets does not depend on the choice of the indeces $\bar a$, $\bar i$. In the following for the sake of definiteness we will set $\bar a = \bar i = 3$.

3. DIRAC BRACKETS AND CANONICAL VARIABLES

The Dirac brackets of two arbitrary functions C, D, of the canonical variable A^{a}_{i}, E^{b}_{i} are defined by

$$
\left[C, D\right]^{*} = \left[C, D\right] - \left[C, \Xi_{\sigma}\right] (\Delta^{-1})_{\sigma\sigma'} \left[\Xi_{\sigma'}, D\right], \qquad (17)
$$

where Ξ_{σ} are the contraints and $\Delta_{\sigma\sigma'}$ is their Poisson matrix

$$
\Delta_{\sigma\sigma'} = \left[\Xi_{\sigma}, \; \Xi_{\sigma'}\right] . \qquad (18)
$$

We must evaluate the Dirac brackets only for A^{a}_{io}, E^{a}_{io}, and we have the two sets of constraints (15), (16)

$$\Xi^a_{1i} = A^a_{io} - \frac{A^3_{io}\, A^a_{3o}}{A^3_{3o}} \quad ; \qquad i, a = 1,2 \tag{19}$$

$$\Xi^a_{2i} = E^a_{io} - \frac{1}{A^3_{3o}} (A^3_{io}\, E^a_{3o} + A^a_{3o}\, E^3_{io} - A^a_{io}\, E^3_{3o}), \qquad i,a = 1,2$$

Using the canonical commutation rules

$$\left[A^a_{io}, E^b_{jo}\right] = \delta_{ij}\, \delta^{ab} \tag{20}$$

we find

$$\Delta^{ab}_{11ij} = \left[\Xi^a_{1i}, \Xi^b_{1j}\right] = 0 \tag{21}$$

$$\Delta^{ab}_{12ij} = -\Delta^{ba}_{21ji} = \left[\Xi^a_{1i}, \Xi^b_{2j}\right] = \tag{22}$$

$$=(\delta_{ij} + \frac{1}{(A^3_{3o})^2}\, A^3_{io}\, A^3_{jo})(\delta^{ab} + \frac{1}{(A^3_{3o})^2}\, A^a_{3o}\, A^b_{3o})$$

$$\Delta^{ab}_{22ij} = \left[\Xi^a_{2i}, \Xi^b_{2j}\right] = + \frac{1}{(A^3_{3o})^2} (E^a_{3o}\, A^b_{3o} - E^b_{3o}\, A^a_{3o}) \,.$$

$$. (\delta_{ij} + \frac{A^3_{io} A^3_{jo}}{(A^3_{3o})^2}) + \frac{1}{(A^3_{3o})^2} (E^3_{io}\, A^3_{jo} - E^3_{jo}\, A^3_{io}) \,. \tag{23}$$

$$. (\delta^{ab} + \frac{A^a_{3o}\, A^b_{3o}}{(A^3_{3o})^2}) \,.$$

The inverse Poisson matrix is

$$\Delta^{-1} = \begin{pmatrix} \Delta^{-1}_{12} \quad \Delta_{22} \quad \Delta^{-1}_{12} & -\Delta^{-1}_{12} \\ \Delta^{-1}_{12} & 0 \end{pmatrix} \tag{24}$$

We still need the commutation relations of the canonical variables with the constraints

$$
\begin{aligned}
\left[A^a_{io}, \Xi^b_{1j}\right] &= 0 \\
\left[A^a_{io}, \Xi^b_{2j}\right] &= \frac{1}{A^3_{3o}}\Big[\delta^{ab}(\delta_{ij}\, A^3_{3o} - \delta_{i3}\, A^3_{jo}) \\
&\quad - \delta^{a3}(\delta_{ij}\, A^b_{3o} - \delta_{i3}\, A^b_{jo})\Big] \\
\left[E^a_{io}, \Xi^b_{1j}\right] &= \frac{1}{A^3_{3o}}\Big[- \delta^{ab}\,(\delta_{ij}\, A^3_{3o} - \delta_{i3}\, A^3_{jo}) + \\
&\quad + \delta^{a3}\,(\delta_{ij}\, A^b_{3o} - \delta_{i3}\, \frac{A^b_{3o}\, A^3_{jo}}{A^3_{3o}}\,)\Big] \\
\left[E^a_{io}, \Xi^b_{2j}\right] &= \frac{1}{A^3_{3o}}\Big[- \delta^{ab}(\delta_{ij}\, E^3_{3o} - \delta_{i3}\, E^3_{jo}\,) + \\
&\quad + \delta^{a3}(\delta_{ij}\, E^b_{3o} - \delta_{i3}\, E^b_{jo})\Big].
\end{aligned}
\tag{25}
$$

Inserting Eqs. (21) to (25) into Eq. (17) and using in the resulting expressions Eqs. (6) we finally obtain

$$
\begin{aligned}
\left[A^a_{io}, A^b_{jo}\right]^* &= \left[q_i\, \hat{v}^a, q_j\, \hat{v}^b\right] = 0 \\
\left[A^a_{io}, E^b_{jo}\right]^* &= \left[q_i\, \hat{v}^a, p_j \hat{v}^b + \frac{1}{q^2}\, q_j\, l^b_\perp\right] = \delta_{ij}\, \delta^{ab} - \mathcal{P}^{ab}\, \mathcal{P}_{ij} \\
\left[E^a_{io}, E^b_{jo}\right]^* &= \left[p_i\, \hat{v}^a + \frac{1}{q^2}\, q_i\, l^a_\perp, p_j\, \hat{v}^b + \frac{1}{q^2}\, q_j l^b_\perp\right] = \\
&= \mathcal{P}_{ij}\,(l^a_\perp\, \hat{v}^b - l^b_\perp\, \hat{v}^a) + \frac{1}{q^2} \mathcal{P}^{ab}(p_i\, q_j - p_j q_i)\,,
\end{aligned}
\tag{26}
$$

where

$$
\begin{aligned}
\mathcal{P}^{ab} &= \delta^{ab} - \hat{v}^a\, \hat{v}^b \\
\mathcal{P}_{ij} &= \delta_{ij} - \hat{q}_i\, \hat{q}_j\,.
\end{aligned}
\tag{27}
$$

Eqs. (26) are satisfied if

$$
l^a_\perp \overset{\text{def}}{=} v\, \mathcal{P}^{ab} \pi^b
$$

and the nonvanoshing commutators are

$$\left[q_i, p_j\right] = \delta_{ij}$$
$$\left[v^a, \pi^b\right] = \delta^{ab} . \tag{28}$$

REFERENCE

(1) F. Palumbo, Nuclear Phys. B, submitted to.