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Two loop corrections to the Λ -parameters

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Abstract

We calculate the relationship between the coupling constants of different $SU(N)$ lattice actions in two loop weak coupling perturbation theory. We perform a complete calculation for arbitrary one plaquette lattice actions using the background field method. Our final result is presented in analytic form. Numerical results are also given for various actions (Wilson,Manton,Heat Kernel and mixed Fundamental-Adjoint).

Monte Carlo simulation of lattice QCD has established itself as an important method of investigating the physics of strong interactions. Many physically interesting results have already been obtained and with the advent of ever more powerful computers we can look forward to quantative predictions for low energy hadronic physics.

To obtain reliable results for continuum physics we must take proper account of the conditions under which the Monte Carlo simulation is performed. The continuum limit of the theory corresponds to the limit of zero lattice coupling, but computer experiments are performed at finite values of the coupling constant g. Two modifications of the theory arise from the use of finite g,

1) Effects of finite lattice spacing a.

2) Perturbative effects due to finite bare coupling. Since the values of g are rather large ($g^2 \approx 1{,}2$) these effects may be numerically significant.

In refs.[1-6] a systematic procedure is used to construct actions for which the effects of finite lattice spacing are reduced.

In this paper we investigate some effects due to finite bare coupling and the influence which they have on the tests of the universality of lattice actions. Lang et al. [7] have reported sizeable discrepancies (between the experimental results and one loop theoretical predictions) for the ratio of Λ parameters derived from the string tension of different $SU(2)$ lattice actions. Similar problems have been found using an action with an admixture of fundamental and adjoint representations by Bhanot and Dashen [8].The universality of lattice actions has been further investigated by Gavai et al.[9] using $SU(2)$ at finite temperature. Comparison of these results make it plausible that the observed deviations from universality are in fact due to higher order corrections in the bare coupling constant. Since universality provides a strong check that a sensible continuum limit has been reached, it is important to establish whether or not this is the case.

In this paper we give a complete calculation of the two loop corrections needed to compare the results of different lattice actions. Our formula allows the comparison of the results from any single plaquette action (Wilson,Manton,Heat Kernel,etc.) at two loop level. Our result supersedes a partial computation by Sharatchandra and Weisz [10]. Apart from a minor difference, we are in agreement with the results of these authors for the parts of the calculation performed in ref. [10]. To our great surprise many of the results of our complete calculation are numerically indistinguishable from the results of ref. [10], where only a subset of the contributing diagrams was considered. We also compare our results with those of Jurkiewicz et al.[11] obtained in the context of the $1/N$ expansion.

1. The Λ parameter and the weak coupling expansion.

In the limit as the lattice spacing a tends to zero, the bare coupling $g(a)$ of an asymptotically free theory varies with the lattice spacing according to the renormalisation group equation.

$$a\frac{dg(a)}{da} = -\beta(g) = b_0 g^3(a) + b_1 g^5(a) + b_2 g^7(a) + O(g^9(a)) \tag{1.1}$$

The solution of this equation defines the scale parameter Λ ,

$$\Lambda^2 a^2 = \left(1 + \frac{1}{b_0^3}(b_1^2 - b_2 b_0)g^2\right) exp\left(-\frac{1}{b_0 g^2} - \frac{b_1}{b_0^2} ln(b_0 g^2)\right) \tag{1.2}$$

The first two coefficients of the beta function $\beta(g)$ are universal and are given in pure $SU(N)$ gauge theory by,

$$b_0 = \frac{11}{3}\frac{N}{16\pi^2} \qquad b_1 = \frac{34}{3}\left(\frac{N}{16\pi^2}\right)^2 \tag{1.3}$$

The coefficient b_2 is dependent on the regularization scheme and is known only for the dimensionally regularized continuum theory [12]. Since it is unknown for a lattice regulated theory, eq.(1.2) including the correction term of order g^2 can not be used directly to determine Λ_L .

On the lattice any physical quantity such as a mass,string tension or deconfinement temperature is proportional to Λ_L. Under a change of the lattice action Λ_L changes, but physical quantities must remain the same. Thus a mass calculated with two different lattice actions is given in the continuum limit by

$$m = k\Lambda_L = k'\Lambda_{L'} \tag{1.4}$$

The measurement of the ratio k'/ k provides an estimate of $\Lambda_L/\Lambda_{L'}$. The ratio of Λ parameters for different lattice actions is calculable in weak coupling perturbation theory, so eq.(1.4) can be used to check universality in the continuum limit.

$$\frac{k'}{k} = \left(\frac{\Lambda_L}{\Lambda_{L'}}\right)_{exp} \tag{1.5}$$

It has been pointed out by many authors [7,9,10,13] that the perturbative corrections of order g^2 in eq.(1.2) could jeopardize the correct determination of ratio $\Lambda_L/\Lambda_{L'}$, because the Monte Carlo measurements are carried out at finite bare coupling constant.

In this paper we calculate the relationship between the coefficients b_2 of different lattice actions. This is sufficient to estimate the O(g^2) corrections to the experimental determination of the ratio of Λ parameters. Consider two lattice actions whose coupling constants are related in the weak coupling region by,

$$\frac{1}{g'^2(a')} = \frac{1}{g^2(a)}(1 + g^2(a)(b_0 L + c_0) + g^4(a)(b_1 L + c_1)) \tag{1.6}$$

where the coefficients of the logarithms

$$L = ln\left(\frac{a'^2}{a^2}\right)$$

are fixed by the renormalization group equation. If the coupling constants $g(a)$ satisfies eq.(1.1) with coefficients b_0, b_1 and b_2 then $g'(a)$ satisfies the same equation with the same coefficients b_0, b_1, but with b_2' given by,

$$b_2' = b_2 + (b_1 c_0 - b_0 c_1) \tag{1.7}$$

Combining eqs.(1.2) , (1.6) and (1.7) we find that the ratio of Λ parameters is fixed by a one loop calculation,

$$\frac{\Lambda_L}{\Lambda_{L'}} = exp(\frac{c_0}{2b_0}) \tag{1.8}$$

In practice since b_2 is not known on the lattice, an experimental Λ is determined from the Monte Carlo data using the one loop expression,

$$\Lambda^2_{exp} = \frac{1}{a^2} exp\left(-\frac{1}{b_0 g^2} - \frac{b_1}{b_0^2} ln(b_0 g^2)\right) \tag{1.9}$$

Assuming that $g^2(a)$ and $g'^2(a)$ are approximatively equal, and given by $\bar{g}^2$ in the scaling region of the Monte Carlo data, we find that [10],

$$\left(\frac{\Lambda_L}{\Lambda_{L'}}\right) \simeq (1 - \bar{g}^2 \delta_{L,L'})\left(\frac{\Lambda_L}{\Lambda_{L'}}\right)_{exp} \tag{1.10}$$

where,

$$\delta_{L,L'} = \frac{1}{2b_0^2}(b_0c_1 - b_1c_0)$$

The correction to the experimentally measured ratio of Λ' s therefore reduces to the calculation of the constants c_0 and c_1 . This can most easily be done by calculating the effective action $\Gamma(F)$ in the limit of weak coupling. In this limit we find that,

$$\Gamma(F) = \frac{1}{2}\sum_{\mu\nu}\int d^4x Tr F_{\mu\nu}F_{\mu\nu}\left(\frac{1}{g^2} + a_0 ln(a^2\lambda^2) + d_0 + g^2(a_1 ln(a^2\lambda^2) + d_1)\right) \qquad (1.11)$$

where λ is an infrared cutoff. The coefficients a_0 and a_1 are the same for all actions if calculated in the same gauge. Imposing the condition that two lattice actions give the same effective action in the continuum limit,we obtain eq.(1.6) with

$$c_0 = d_0 - d_0' \ , c_1 = d_1 - d_1' \qquad (1.12)$$

We will calculate the effective action using the background field method.

2. The background field method on the lattice.

In this section we review briefly the background field method and its implementation on the lattice. This serves both to establish our notation and to outline our strategy for the calculation of the effective action on the lattice. The basic idea of the background field method [14] can be paraphrased as follows,(for the case of the continuum theory). The gauge field in the action is written as the sum of the quantum field Q_μ and a background field B_μ. The gauge fixing term, which breaks the gauge invariance with respect to transformations of the quantum field, is chosen in such a way that the invariance of the action under gauge transformations of the background field is preserved. The gauge invariant effective action is just the background field effective action considered as a functional of B and evaluated with vanishing quantum field. This effective action can be obtained from the calculation of the one particle irreducible two point function of the background field. For a lucid treatment of this problem in the continuum at two loop level we refer the reader to ref. [15].

The extension of the background field method to discrete Euclidean space time has been given in refs. [16,17,18]. Consider the usual hypercubic lattice with sites labelled by an integer-valued four vector x, and define e_μ to be a vector of length one lattice spacing in the direction μ ($\mu = 1, 2, 3, 4$). The variable $U(l) = U_\mu(x)$ associated with the link l connecting the points x and $(x + e_\mu)$ is an element of the group $SU(N)$ in the fundamental representation. The plaquette variable is defined as the ordered product of four link variables around an elementary plaquette.

$$U(P) = U_\mu(x)U_\nu(x+e_\mu)U^\dagger_\mu(x+e_\nu)U^\dagger_\nu(x) \tag{2.1}$$

Gauge transformations on the the link variable $U_\mu(x)$ are defined by,

$$U^\Omega_\mu(x) = \Omega(x)U_\mu(x)\Omega^{-1}(x+e_\mu) \tag{2.2}$$

The lattice actions which we consider are formed from the sum over plaquettes of gauge invariant combinations of the one plaquette variable.

$$S = \sum_P S(U_P) \tag{2.3}$$

For example, for the Wilson action the combination used is,

$$S_W(P) = \beta_F\Big(1 - \frac{1}{2N}Tr(U(P) + U^\dagger(P)\Big) \tag{2.4}$$

The background field is introduced by modifying the link variable by the replacement,

$$U_\mu(x) = V_\mu(x)U_{c\mu}(x) \tag{2.5}$$

where $V_\mu(x)$ is the link variable of the quantum field, and $U_{c\mu}(x)$ is the corresponding quantity for the background field. These link variables may be written as,

$$V_\mu(x) = e^{igQ_\mu(x)} \tag{2.6}$$

$$U_{c\mu}(x) = e^{iaB_\mu(x)} \tag{2.7}$$

where Q_μ and B_μ are hermitean N $\times$ N matrices such that,

$$Q_\mu(x) = \sum_A t^A Q^A_\mu(x) \qquad Trt^At^B = \frac{1}{2}\delta^{AB} \tag{2.8}$$

In the presence of the background field the gauge transformation on the link variable can be viewed in two ways. The first of these treats the quantum field as a matter field which transforms purely locally, whilst the background field transforms as a true gauge field.

$$U^{\Omega}_{\mu}(x) = [\Omega(x)V_{\mu}(x)\Omega^{-1}(x)][\Omega(x)U_{c\mu}(x)\Omega^{-1}(x+e_{\mu})] \tag{2.9}$$

Lattice derivatives which transform covariantly with respect to this gauge invariance can be defined,

$$D^{+}_{\mu}(U_c)Q_{\nu}(x) = U_{c\mu}(x)Q_{\nu}(x+e_{\mu})U^{-1}_{c\mu}(x) - Q_{\nu}(x) \tag{2.10}$$

$$D^{-}_{\mu}(U_c)Q_{\nu}(x) = U_{c\mu}(x-e_{\mu})Q_{\nu}(x-e_{\mu})U^{-1}_{c\mu}(x-e_{\mu}) - Q_{\nu}(x) \tag{2.11}$$

where Q_{ν} is a matrix valued function defined on lattice sites. Under the gauge transformation eq.(2.2) the matter field Q_{ν} transforms as,

$$Q^{\Omega}_{\nu}(x) = \Omega(x)Q_{\nu}(x)\Omega^{-1}(x) \tag{2.12}$$

and hence,

$$\left[D^{\pm}_{\mu}(U_c)Q_{\nu}(x)\right]^{\Omega} = \Omega(x)D^{\pm}_{\mu}(U_c)Q_{\nu}(x)\Omega^{-1}(x) \tag{2.13}$$

The second interpretation of the gauge transformation on the link variable,

$$U^{\Omega}_{\mu}(x) = [\Omega(x)V_{\mu}(x)U_{c\mu}(x)\Omega^{-1}(x+e_{\mu})U^{-1}_{c\mu}(x)]U_{c\mu}(x) \tag{2.14}$$

treats the background field as invariant. It is this second manifestation of gauge invariance which is removed by the gauge fixing term. In the presence of the background field the plaquette variable becomes,

$$U(P) = V_{\mu}(x)U_{c\mu}(x)V_{\nu}(x+e_{\mu})U_{c\nu}(x+e_{\mu})U^{\dagger}_{c\mu}(x+e_{\nu})V^{\dagger}_{\mu}(x+e_{\nu})U^{\dagger}_{c\nu}(x)V^{\dagger}_{\nu}(x) \tag{2.15}$$

Using standard manipulations [16] we may rewrite the plaquette variable as,

$$U(P) = e^{igQ_{\mu}(x)}e^{ig(D^{+}_{\mu}Q_{\nu}+Q_{\nu}(x))}U_c(P)e^{-ig(D^{+}_{\nu}Q_{\mu}+Q_{\mu}(x))}e^{-igQ_{\nu}(x)} \tag{2.16}$$

where for notational convenience here (and in the following) we have dropped the background field dependence of the covariant derivative. This expression for the plaquette variable allows us to define field strengths.

$$U_c(P) = U_{c\mu}(x)U_{c\nu}(x+e_\mu)U^\dagger_{c\mu}(x+e_\nu)U^\dagger_{c\nu}(x) = e^{ia^2 f_{\mu\nu}} \tag{2.17}$$

where,

$$f_{\mu\nu} = \frac{1}{a}(\Delta^+_\mu B_\nu(x) - \Delta^+_\nu B_\mu(x)) + i[B_\mu(x), B_\nu(x)] + O(a) \tag{2.18}$$

and Δ^+_ν is the normal lattice difference operator,

$$\Delta^+_\nu(x)B_\mu(x) = B_\mu(x+e_\nu) - B_\mu(x) \tag{2.19}$$

For calculational convenience we choose the background field to have a wavelength much greater than the lattice spacing a, so eq.(2.18) has a finite limit as a tends to zero.

Analogously for the quantum field we define,

$$e^{igF_{\mu\nu}} = e^{-ig(D^+_\nu Q_\mu + Q_\mu(x))}e^{-igQ_\nu(x)}e^{igQ_\mu(x)}e^{ig(D^+_\mu Q_\nu + Q_\nu(x))} \tag{2.20}$$

By use of the Baker Campbell Hausdorff identity,

$$exp(gA)exp(gB) = exp\Big(g(A+B) + \frac{g^2}{2}[A,B] + \frac{g^3}{12}([A,[A,B]] + [[A,B],B])\Big) + O(g^4) \tag{2.21}$$

we find that,

$$F_{\mu\nu} = F_{\mu\nu}(1) + gF_{\mu\nu}(2) + g^2F_{\mu\nu}(3) + \ldots \tag{2.22}$$

where,

$$\begin{aligned} F_{\mu\nu}(1) &= D^+_\mu Q_\nu - D^+_\nu Q_\mu \\ F_{\mu\nu}(2) &= i\Big([Q_\mu, Q_\nu] + \frac{1}{2}[D^+_\mu Q_\nu, D^+_\nu Q_\mu] + \frac{1}{2}[D^+_\mu Q_\nu, Q_\nu] - \frac{1}{2}[D^+_\nu Q_\mu, Q_\mu]\Big) \\ F_{\mu\nu}(3) &= (\frac{1}{2}[D^+_\mu Q_\nu + Q_\mu, [Q_\mu, Q_\nu]] + \frac{1}{12}[D^+_\mu Q_\nu, [D^+_\mu Q_\nu, D^+_\nu Q_\mu + Q_\nu]] \\ &+ \frac{1}{4}[D^+_\nu Q_\mu, [D^+_\mu Q_\nu, Q_\nu]] + \frac{1}{6}[Q_\nu, [D^+_\mu Q_\nu, Q_\nu]]) - (\mu \leftrightarrow \nu) \end{aligned} \tag{2.23}$$

Knowledge of further terms in the expansion of $F_{\mu\nu}$ is unnecessary for the calculation of b_2 on the lattice.

3. The one plaquette lattice action.

The general lattice action in the presence of the background field is derived from the product of four link variables around a single plaquette is presented in this section. This action contains all the terms necessary for the calculation of b_2 on the lattice. We define the hermitean $N \times N$ matrix ϕ such that,

$$exp(i\phi) = exp(igF_{\mu\nu})exp(ia^2 f_{\mu\nu}) \tag{3.1}$$

where for compactness of notation we have suppressed the μ, ν indices of ϕ. In forming lattice actions from the plaquette variable, eq.(2.15),we retain all possible gauge invariant terms which can contribute in two loop order in the weak coupling expansion. The resultant one plaquette lattice action can be expressed in terms of ϕ. It is the sum of two terms,

$$S = S_2 + S_I \tag{3.2}$$

where,

$$S_2 = \frac{1}{g^2} \sum_x \sum_{\mu,\nu} Tr \frac{\phi^2}{2} \tag{3.3}$$

and

$$S_I = \frac{1}{g^2} \sum_x \sum_{\mu,\nu} s_4(Tr\phi^4) + s_6(Tr\phi^6) + t_4(Tr\phi^2)^2 + t_6(Tr\phi^2)^3 + u_6(Tr\phi^2 Tr\phi^4) + v_6(Tr\phi^3)^2 \tag{3.4}$$

The term S_2 is common to all lattice actions; its normalisation is fixed because it contains the only term which survives in naïve continuum limit.

$$S_2 \to \frac{1}{2} \sum_x \sum_{\mu,\nu} Tr F_{\mu\nu} F_{\mu\nu} \tag{3.5}$$

S_I is the action of irrelevant operators. The coefficients s_4, s_6 etc. determine the particular form of the lattice action. For example, the Wilson action eq.(2.4) has $s_4 = -\frac{1}{4!}, s_6 = \frac{1}{6!}$, and all other coefficients zero.

Since the action S_I distinguishes different actions on the lattice, we separate the generating functional into two pieces.

$$Z[B, J] = Z_2[B, J] + Z_I[B, J] \tag{3.6}$$

where

$$Z_2[B,J] = \int \prod_l [dV(l)]\, e^{-S_2(\phi)+\sum_x \sum_\mu J_\mu(x)Q_\mu(x)} \tag{3.7}$$

$$Z_I[B,J] = \int \prod_l [dV(l)]\, e^{-S_2(\phi)+\sum_x \sum_\mu J_\mu(x)Q_\mu(x)}(e^{-S_I}-1) \tag{3.8}$$

In a similar way the effective action derived from these two generating functionals can be divided into two contributions.

$$\Gamma(B) = \Gamma_2(B) + \Gamma_I(B) \tag{3.9}$$

Only the contribution to the effective action $\Gamma_I(B)$, which changes from one form of the lattice action to another will be calculated in this paper.

The perturbative evaluation of eqs.(3.7,3.8) requires a choice of gauge. Introducing the gauge fixing function G^A, eq.(3.7) may be written as,

$$Z_2[B,J] = \int [dQ] det|\frac{\delta G^A}{\delta \omega^B}| exp\Big(-S_2(\phi) - \frac{1}{2\alpha_0}\sum_x\sum_A G^A(x)G^A(x) - S_m + \sum_x\sum_\mu J_\mu(x)Q_\mu(x)\Big) \tag{3.10}$$

The particular choice of gauge fixing which we make is

$$G^A(x) = 2\sum_\mu Tr t^A D_\mu^- Q_\mu(x) \tag{3.11}$$

The gauge fixing term may be written explicitly as,

$$S_{gf} = \frac{1}{\alpha_0}\sum_{\mu,\nu}\sum_x Tr D_\mu^- Q_\mu D_\nu^- Q_\nu \tag{3.12}$$

Since D_μ^- is the lattice covariant derivative eq.(2.11), the gauge fixing term is invariant under the transformation eq.(2.13). α_0 is the bare gauge fixing parameter. $\delta G^A/\delta\omega^B$ is the derivative of the gauge fixing term under an infinitesimal gauge transformation

$$\Omega(x) \to 1 + ig\omega(x) \quad \omega(x) = \sum_A t^A \omega^A(x) \tag{3.13}$$

under this transformation the quantum field transforms as

$$\delta Q_\mu(x) = -D_\mu^+\omega(x) - ig[Q_\mu(x),\omega(x)] - \frac{ig}{2}[Q_\mu(x), D_\mu^+\omega(x)] + \frac{g^2}{12}[Q_\mu(x),[Q_\mu(x),D_\mu^+\omega(x)]] \tag{3.14}$$

Useful formulae for the derivation of eq.(3.14) can be found in ref.[19]. Using this equation the determinant in eq.(3.10) may be written in terms of anticommuting ghost fields ω and $\omega^\dagger$

$$S_{gh} = 2\sum_x\sum_\mu Tr(D_\mu^+\omega)^\dagger\Big(D_\mu^+\omega + ig[Q_\mu,\omega] + \frac{ig}{2}[Q_\mu, D_\mu^+\omega] - \frac{g^2}{12}[Q_\mu,[Q_\mu,D_\mu^+\omega]]\Big) \quad (3.15)$$

S_m is the term coming from the change of measure in eq.(3.7).

$$\int\prod_l [dV(l)] = \int [dQ]\, e^{-S_m} \quad (3.16)$$

and S_m is given by,

$$S_m = \frac{Ng^2}{12}\sum_x\sum_\mu TrQ_\mu Q_\mu + O(g^4) \quad (3.17)$$

After some manipulation we find that,

$$Z_2[B,J] = \int [dQ][d\omega][d\omega^\dagger] e^{-S_2^{eff}(\phi)+\sum_x\sum_\mu J_\mu(x)Q_\mu(x)} \quad (3.18)$$

where,

$$\begin{aligned} S_2^{eff} = Tr\sum_{\mu,\nu}\sum_x\ & D_\mu^+Q_\nu D_\mu^+Q_\nu + (\frac{1}{\alpha_0}-1)D_\nu^+Q_\mu D_\mu^+Q_\nu + \frac{1}{2}(F_{\mu\nu}F_{\mu\nu} - F_{\mu\nu}(1)F_{\mu\nu}(1)) \\ & + \frac{a^2}{g}F_{\mu\nu}f_{\mu\nu} + \frac{a^4}{2g^2}f_{\mu\nu}f_{\mu\nu} + \frac{a^4}{2\alpha_0}[D_\nu^+Q_\mu + Q_\mu, f_{\mu\nu}][D_\mu^+Q_\nu + Q_\nu, f_{\mu\nu}] \\ & + i\frac{a^2}{\alpha_0}([Q_\mu,Q_\nu] + [D_\nu^+Q_\mu, D_\mu^+Q_\nu] + [D_\nu^+Q_\mu, Q_\nu] + [Q_\mu, D_\mu^+Q_\nu])f_{\mu\nu} \\ & + \frac{a^4}{24}[f_{\mu\nu},F_{\mu\nu}][f_{\mu\nu},F_{\mu\nu}] - \frac{a^4g^2}{720}F_{\mu\nu}F_{\mu\nu}F_{\mu\nu}F_{\mu\nu}f_{\mu\nu}f_{\mu\nu} \\ & + \frac{a^4g^2}{180}F_{\mu\nu}f_{\mu\nu}F_{\mu\nu}F_{\mu\nu}F_{\mu\nu}f_{\mu\nu} - \frac{a^4g^2}{240}f_{\mu\nu}F_{\mu\nu}F_{\mu\nu}f_{\mu\nu}F_{\mu\nu}F_{\mu\nu} \\ & + S_{gh} + S_m \end{aligned} \quad (3.19)$$

S_{gh} and S_m are given by eqs.(3.15) and (3.17) respectively. In analogy with eq.(3.18) the expression for Z_I becomes,

$$Z_I[B,J] = \int [dQ][d\omega][d\omega^\dagger] e^{-S_2^{eff}(\phi)+\sum_x\sum_\mu J_\mu(x)Q_\mu(x)}(e^{-S_I}-1) \quad (3.20)$$

4. Calculation of Feynman graphs

To calculate the effective action $\Gamma(B)$ we consider all one particle irreducible Feynman graphs, including at least one insertion of S_I, which contribute to the two point function of the background field. Because we work in the two loop order we will have to consider at most two insertions of the operator S_I . The expansion of the terms in S_I in terms of the background field strength $f_{\mu\nu}$ and the quantum field $F_{\mu\nu}$ has been given in Table(1).

For the purposes of explanation we shall consider only the insertion of one operator,

$$S_4 = \frac{1}{g^2} Tr\phi^4$$

Some remarks about the insertion of other operators will be made at the end. We will consider separately the insertions of the various component pieces of S_4 given in Table(1) .We first consider four groups of terms in which S_4 is inserted once.

a) Terms bilinear in both the background and the quantum field strength ,

$$S_{42} + S_{43} = 4a^4 TrF_{\mu\nu}F_{\mu\nu}f_{\mu\nu}f_{\mu\nu} \; + \; 2a^4 TrF_{\mu\nu}f_{\mu\nu}F_{\mu\nu}f_{\mu\nu} \tag{4.1}$$

Because these terms are already bilinear in the background field all further dependence on the background field (implicit in the covariant derivatives in $F_{\mu\nu}$) can be discarded. The contribution of these terms is thus proportional to the expectation value of the quantum field $\langle F_{\mu\nu}F_{\mu\nu}\rangle$. The result for this quantity can be extracted from the result for the plaquette operator of the Wilson action calculated in ref[20] . These authors calculate,

$$w(g^2) = \langle Tr\frac{U(P)}{N}\rangle = \langle 1 - \frac{g^2}{2!\,N} TrF_{\mu\nu}F_{\mu\nu} + \frac{g^4}{4!\,N} TrF_{\mu\nu}F_{\mu\nu}F_{\mu\nu}F_{\mu\nu} + ...\rangle \tag{4.2}$$

where the functional average is taken with respect to the full Wilson action (i.e. including terms quartic in the field strength, c.f. S_{41}) Their result for this quantity written in our notation is,

$$w(g^2) = 1 - \frac{N^2-1}{8N} g^2 - \frac{N^2-1}{8N} g^4 \left(Na_R + \frac{2N^2-3}{48N} \right) + O(g^6) \tag{4.3}$$

a_R is a pure number which is estimated [20] to be

$$a_R = 2(0.0203 \pm .0001 - \frac{1}{48}) = -.0011 \pm .0002 \tag{4.4}$$

S_2	$\frac{1}{g^2}Tr\phi^2$	S_4	$\frac{1}{g^2}Tr\phi^4$	T_4	$\frac{1}{g^2}(Tr\phi^2)^2$
S_{21}	$TrF_{\mu\nu}F_{\mu\nu}$	S_{41}	$g^2TrF_{\mu\nu}F_{\mu\nu}F_{\mu\nu}F_{\mu\nu}$	T_{41}	$g^2TrF_{\mu\nu}F_{\mu\nu}TrF_{\mu\nu}F_{\mu\nu}$
S_{22}	$-\frac{1}{6}a^4TrF_{\mu\nu}F_{\mu\nu}f_{\mu\nu}f_{\mu\nu}$	S_{42}	$4a^4TrF_{\mu\nu}F_{\mu\nu}f_{\mu\nu}f_{\mu\nu}$	T_{42}	$4a^4TrF_{\mu\nu}f_{\mu\nu}TrF_{\mu\nu}f_{\mu\nu}$
S_{23}	$\frac{1}{6}a^4TrF_{\mu\nu}f_{\mu\nu}F_{\mu\nu}f_{\mu\nu}$	S_{43}	$2a^4TrF_{\mu\nu}f_{\mu\nu}F_{\mu\nu}f_{\mu\nu}$	T_{43}	$2a^4TrF_{\mu\nu}F_{\mu\nu}Trf_{\mu\nu}f_{\mu\nu}$
S_{24}	$\frac{2}{g}a^2TrF_{\mu\nu}f_{\mu\nu}$	S_{44}	$4ga^2TrF_{\mu\nu}F_{\mu\nu}F_{\mu\nu}f_{\mu\nu}$	T_{44}	$4ga^2TrF_{\mu\nu}F_{\mu\nu}TrF_{\mu\nu}f_{\mu\nu}$
S_{25}	$-\frac{1}{360}g^2a^4TrF_{\mu\nu}F_{\mu\nu}F_{\mu\nu}F_{\mu\nu}f_{\mu\nu}f_{\mu\nu}$	S_{45}	$-\frac{1}{3}g^2a^4TrF_{\mu\nu}F_{\mu\nu}F_{\mu\nu}F_{\mu\nu}f_{\mu\nu}f_{\mu\nu}$	T_{45}	$-\frac{1}{3}g^2a^4TrF_{\mu\nu}F_{\mu\nu}TrF_{\mu\nu}F_{\mu\nu}f_{\mu\nu}f_{\mu\nu}$
S_{26}	$\frac{1}{90}g^2a^4TrF_{\mu\nu}f_{\mu\nu}F_{\mu\nu}F_{\mu\nu}F_{\mu\nu}f_{\mu\nu}$	S_{46}	$\frac{1}{3}g^2a^4TrF_{\mu\nu}f_{\mu\nu}F_{\mu\nu}F_{\mu\nu}F_{\mu\nu}f_{\mu\nu}$	T_{46}	$\frac{1}{3}g^2a^4TrF_{\mu\nu}F_{\mu\nu}TrF_{\mu\nu}f_{\mu\nu}F_{\mu\nu}f_{\mu\nu}$
S_{27}	$-\frac{1}{120}g^2a^4TrF_{\mu\nu}F_{\mu\nu}f_{\mu\nu}F_{\mu\nu}F_{\mu\nu}f_{\mu\nu}$	U_6	$\frac{1}{g^2}Tr\phi^2Tr\phi^4$	V_6	$\frac{1}{g^2}(Tr\phi^3)^2$
S_{28}	$\frac{1}{g^2}a^4Trf_{\mu\nu}f_{\mu\nu}$	U_{61}	$g^2a^4TrF_{\mu\nu}F_{\mu\nu}F_{\mu\nu}F_{\mu\nu}Trf_{\mu\nu}f_{\mu\nu}$	V_{61}	$9g^2a^4TrF_{\mu\nu}F_{\mu\nu}f_{\mu\nu}TrF_{\mu\nu}F_{\mu\nu}f_{\mu\nu}$
S_6	$\frac{1}{g^2}Tr\phi^6$	U_{62}	$4g^2a^4TrF_{\mu\nu}F_{\mu\nu}TrF_{\mu\nu}F_{\mu\nu}f_{\mu\nu}f_{\mu\nu}$	V_{62}	$6g^2a^4TrF_{\mu\nu}F_{\mu\nu}F_{\mu\nu}TrF_{\mu\nu}f_{\mu\nu}f_{\mu\nu}$
S_{61}	$6g^2a^4TrF_{\mu\nu}F_{\mu\nu}F_{\mu\nu}F_{\mu\nu}f_{\mu\nu}f_{\mu\nu}$	U_{63}	$2g^2a^4TrF_{\mu\nu}F_{\mu\nu}TrF_{\mu\nu}f_{\mu\nu}F_{\mu\nu}f_{\mu\nu}$	T_6	$\frac{1}{g^2}(Tr\phi^2)^3$
S_{62}	$6g^2a^4TrF_{\mu\nu}f_{\mu\nu}F_{\mu\nu}F_{\mu\nu}F_{\mu\nu}f_{\mu\nu}$	U_{64}	$8g^2a^4TrF_{\mu\nu}f_{\mu\nu}TrF_{\mu\nu}F_{\mu\nu}F_{\mu\nu}f_{\mu\nu}$	T_{61}	$3g^2a^4TrF_{\mu\nu}F_{\mu\nu}TrF_{\mu\nu}F_{\mu\nu}Trf_{\mu\nu}f_{\mu\nu}$
S_{63}	$3g^2a^4TrF_{\mu\nu}F_{\mu\nu}f_{\mu\nu}F_{\mu\nu}F_{\mu\nu}f_{\mu\nu}$			T_{62}	$12g^2a^4TrF_{\mu\nu}F_{\mu\nu}TrF_{\mu\nu}f_{\mu\nu}TrF_{\mu\nu}f_{\mu\nu}$

TABLE 1 - The decomposition of the traces in eqs. (3.4, 3.4) in terms of the background field strength f and the quantum field strength F. Terms which have more than two powers of f or four powers of F do not contribute and have been omitted.

Subtracting the two terms dependent on the fourth power of the field strength we find that,

$$\langle F^A_{\mu\nu} F^B_{\mu\nu} \rangle = \frac{1}{2}(1 - \delta_{\mu\nu})\delta^{AB}(1 + g^2 N a_R + O(g^4)) \tag{4.5}$$

where the functional average is now taken with respect to the action S_2^{eff} eq.(3.19). With this result it is simple to show that the contribution to the effective action of the diagrams of Fig.(1) is,

$$S_{42} + S_{43} \to s_4\left(\frac{2N^2 - 3}{N}\right)(1 + g^2 N a_R)\Gamma_0(B) \tag{4.6}$$

where the zeroth order effective action is,

$$\Gamma_0(B) = \frac{a^4}{2g^2} \sum_{\mu\nu} Tr \int d^4x f_{\mu\nu} f_{\mu\nu} \tag{4.7}$$

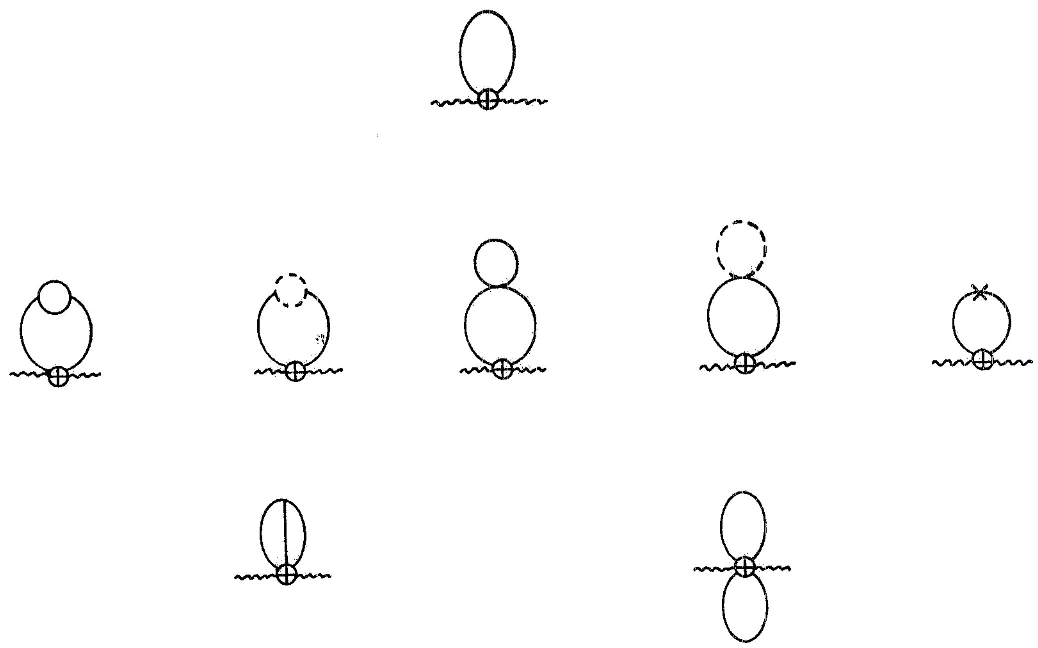


FIG. 1 - Diagrams generated by the insertion of S_{42} or S_{43}. Background fields are denoted by wavy lines, quantum fields by solid lines and ghost fields by dashed lines. The crossed circle ($\oplus$) indicates the insertion of S_{42} or S_{43}. Other vertices are those of S_2^{eff}. The cross on the quantum field line indicates the contribution of the measure.

b) Terms of order g^2 bilinear in the background field strength.

$$S_{45} + S_{46} = \frac{a^4 g^2}{3} Tr F_{\mu\nu} F_{\mu\nu} F_{\mu\nu} [f_{\mu\nu}, F_{\mu\nu}] f_{\mu\nu} \tag{4.8}$$

The contribution of these terms is readily estimated by direct application of Wick's theorem,yielding the diagrams of the form shown in Fig.(2). As before these terms are already bilinear in the background field strength so we obtain using eq.(4.5).

$$S_{45} + S_{46} \rightarrow -s_4 g^2 \left(\frac{2N^2 - 3}{24} \right) \Gamma_0(B) \tag{4.9}$$

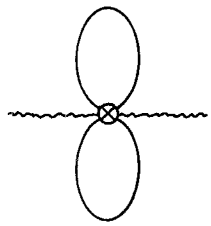


FIG. 2 - Diagram generated by the insertion of S_{45} or S_{46}.

c) The single insertions of S_{41} and S_{44}

$$S_{41} = g^2 Tr F_{\mu\nu} F_{\mu\nu} F_{\mu\nu} F_{\mu\nu} \tag{4.10}$$

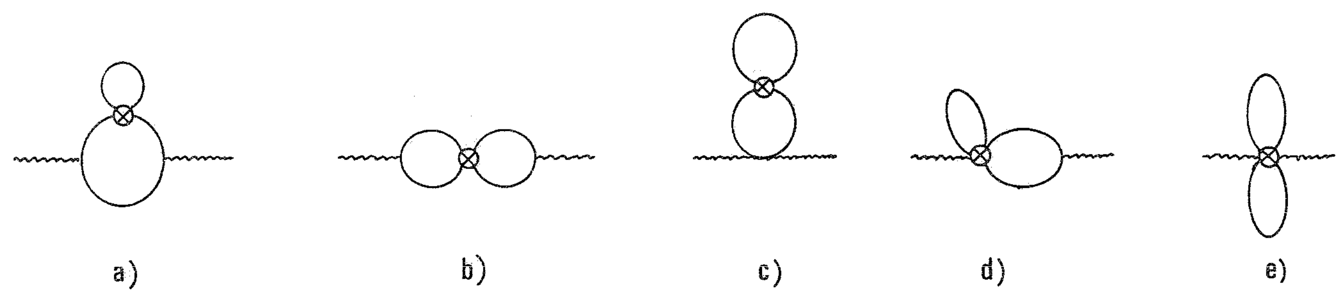
$$S_{44} = 4a^2 g Tr F_{\mu\nu} F_{\mu\nu} F_{\mu\nu} f_{\mu\nu} \tag{4.11}$$

The evaluation of these diagrams is the most challenging since it requires the extraction of the background field from the covariant derivatives contained in the quantum field strength. These diagrams were not calculated in ref.[10]. We calculated these graphs after deriving the Feynman rules generated by these vertices (and by S_2^{eff} eq(3.19)) using the algebraic manipulation programme *Schoonschip* [21,22]. The graphs of fig.(3b) and figs.(4 a,b,c) are zero for reasons of symmetry in the colour indices. The remaining diagrams are individually non - Lorentz invariant, non gauge invariant and divergent. After summing all the contributions of fig.(3) we obtain,

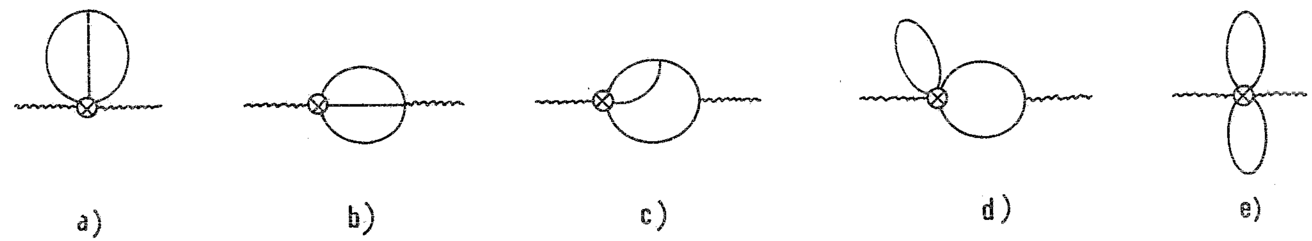
$$S_{41} \rightarrow s_4 g^2 \left(\frac{2N^2 - 3}{12} \right) \left[7Z_{0000} - 1 + \frac{9}{2\pi^2} - \frac{3}{\pi^2} L(\lambda^2) \right] \Gamma_0(B) \tag{4.12}$$

where $L(\lambda^2)$ diverges in the limit $\lambda^2 \rightarrow 0$

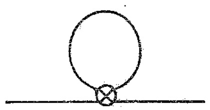
$$L(\lambda^2) = ln(a^2\lambda^2) + \gamma_E - F_{0000} \tag{4.13}$$



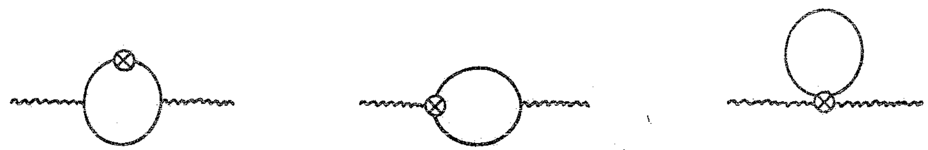
<u>FIG. 3</u> - Diagrams generated by the insertion of S_{41} ($\otimes$).



<u>FIG. 4</u> - Diagrams generated by the single insertion of S_{44} ($\otimes$).



<u>FIG. 5</u> - Corrections to the two point function of the quantum field due to the insertion of S_{41}.



<u>FIG. 6</u> - Diagrams due to the insertion of S_{ct} ($\otimes$).

The quantities Z_{0000}, F_{0000} are constants related to integrals performed on the lattice whose numerical value is of minor interest since they cancel in the sum of all diagrams. For the convenience of the reader their definitions and numerical values are given in the Appendix. The contribution of the diagrams of fig.(4) is,

$$S_{44} \rightarrow s_4 g^2 \left(\frac{2N^2 - 3}{12} \right) \left[\frac{3}{2} - 7Z_{0000} - \frac{6}{\pi^2} + \frac{3}{\pi^2} L(\lambda^2) \right] \Gamma_0(B) \tag{4.14}$$

d) Counterterms

$$S_{ct} = \frac{1}{\alpha_R} (Z_Q^{-1} - 1) \sum_{\mu\nu} Tr D_\mu^- Q_\mu D_\nu^- Q_\nu \tag{4.15}$$

The diagram shown in fig.(5) renormalises the gauge parameter in a way dependent on the particular form of the action S_I. Renormalisation of the quantum and ghost fields is not necessary since they appear only on internal lines [15]. Thus the two point function of the quantum field is changed by the insertion of the vertex S_{41} as shown in fig.(5)

$$\Gamma_{\mu\nu}^Q(p) = (\delta_{\mu\nu} \hat{p}^2 - \hat{p}_\mu \hat{p}_\nu) \left[1 + s_4 g^2 \frac{2N^2 - 3}{N} \right] + \hat{p}_\mu \hat{p}_\nu \frac{1}{Z_Q \alpha_R} \tag{4.16}$$

where,

$$\hat{p}_\mu = 2sin\left(\frac{p_\mu}{2} \right), \hat{p}^2 = \sum_\mu \hat{p}_\mu \hat{p}_\mu \tag{4.17}$$

Imposing the condition that the renormalised gauge parameter is equal to one, we obtain from eq.(4.16) (and from the insertion of the term T_{41}).

$$(Z_Q^{-1} - 1) = g^2 \left(s_4 \frac{2N^2 - 3}{N} + t_4 (N^2 + 1) + ... \right) \tag{4.18}$$

In this equation we have included the contributions proportional to s_4 and t_4 but neglected terms common to all actions coming from S_2^{eff}. Inserting the counterterm in the diagrams of fig.(6), we obtain in the Feynman gauge $\alpha_R = 1$, (dropping the term proportional to t_4 for the purposes of our present explanation).

$$S_{ct} \rightarrow s_4 g^2 \left[\frac{2N^2 - 3}{8\pi^2} \right] \Gamma_0(B) \tag{4.19}$$

Adding up all the terms coming from a single insertion of S_4 all divergent terms cancel and we obtain,

$$S_4 \to s_4\left[\frac{2N^2-3}{N}(1+g^2Na_R)\right]\Gamma_0(B) \tag{4.20}$$

The extremely simple form of eq.(4.20) reflects the fact that the contribution due to a single insertion of a subset of the terms in S_4 vanishes.

$$S_{41}+S_{42}+S_{45}+S_{46}+S_{ct} \to 0 \tag{4.21}$$

That the sum of the diagrams of the type $S_{41}+S_{42}+S_{45}+S_{46}$ inserted once gives a contribution equivalent to a renormalisation of the gauge parameter has already been noted by the authors of ref.[11]. The assertion that these diagrams exactly cancel the gauge parameter renormalisation counterterm is new, since the renormalisation of the gauge parameter was overlooked in ref [11].

Lastly we turn to the double insertion diagrams of fig.(7). The result for the top part of fig.(7a) has already been given in eq.(4.16). The result for the full diagram is

$$S_{41}\times(S_{42}+S_{43}) \to -s_4^2g^2\left[\frac{2N^2-3}{N}\right]^2\Gamma_0(B) \tag{4.22}$$

a) b)

FIG. 7 - Double insertion diagrams: a) insertion of S_{41} (denoted by ⊗) and of S_{42} or S_{43} (denoted by ⊕); b) double insertion of S_{44} (denoted by ⊗).

The double insertion of S_{44} gives fig.(7b). This diagram is calculated using integral eq.(A.3),

$$S_{44}\times S_{44} \to -s_4^2g^2\left[\frac{N^4-6N^2+18}{2N^2}\right]\Gamma_0(B) \tag{4.23}$$

This concludes our discussion of the insertions of S_4. The calculation of the contributions of the other terms in eq.(3.4) presents no new difficulties. The insertions of T_4 exactly parallel the calculation outlined for S_4. Indeed a useful check on our formula is provided by the formula valid only for $N=2$.

$$Tr\phi^4 \to \frac{1}{2}Tr\phi^2 Tr\phi^2 \tag{4.24}$$

All other contributions are obtained simply from diagrams of the type shown in fig.(2) using eq.(4.5) in lowest order.

The final result of our graphical calculation is the relationship between the coupling constant g of an arbitrary action, specified by the constants s_4, s_6, t_4 etc (cf eq.(3.4)) and the coupling constant g_M of the Manton action in which all s_4, s_6, t_4 etc. are equal to zero.

$$\frac{1}{g_M^2} = \frac{1}{g^2} + s_4\frac{2N^2-3}{N} + t_4(N^2+1) + g^2 a_R\Big(s_4(2N^2-3) + t_4 N(N^2+1)\Big)$$
$$+g^2\Big(s_6\frac{15}{8}\frac{(N^4-3N^2+3)}{N^2} + v_6\frac{9}{8}\frac{(N^2-4)}{N} + u_6\frac{3}{8}\frac{(2N^2-3)(N^2+3)}{N} + t_6\frac{3}{8}(N^2+1)(N^2+3)\Big)$$
$$-g^2\Big(s_4^2\frac{9N^4-30N^2+36}{2N^2} + 2s_4t_4\frac{(2N^2-3)(N^2+2)}{N} + t_4^2(N^2+1)(N^2+2)\Big) \tag{4.25}$$

5. Results for different lattice actions.

In this section we discuss the application of our results to different forms of the one plaquette action. The first action which we consider is the one originally proposed by Wilson,

$$S_W(P) = \beta_F\Big(1 - \frac{1}{2N}Tr(U(P) + U^\dagger(P)\Big) \tag{5.1}$$

Writing the plaquette variable $U(P) = e^{i\phi}$ and summing over plaquettes we find that in the weak coupling limit the action can be written as,

$$S_W = \frac{\beta_F}{2N}\sum_{\mu,\nu} Tr(\frac{\phi^2}{2!} - \frac{\phi^4}{4!} + \frac{\phi^6}{6!}) \qquad \frac{1}{g^2} = \frac{\beta_F}{2N} \tag{5.2}$$

We may modify this action by including also an adjoint representation term [23] leading to an action which can be written as [8],

$$S_{FA}(P) = \beta_F\Big(1 - \frac{1}{2N}Tr(U(P) + U^\dagger(P))\Big) + \beta_A\Big(1 - \frac{1}{N^2}|TrU(P)|^2\Big) \tag{5.3}$$

In the weak coupling limit this action can be written as,

$$S_{FA} = (\frac{\beta_F}{2N}+\frac{\beta_A}{N})\sum_{\mu,\nu}\Big(Tr(\frac{\phi^2}{2!}-\frac{\phi^4}{4!}+\frac{\phi^6}{6!})+(\frac{r}{8N}Tr\phi^2Tr\phi^2-\frac{r}{48N}Tr\phi^2Tr\phi^4+\frac{r}{72N}Tr\phi^3Tr\phi^3)\Big) \tag{5.4}$$

where the parameter r controls the mixture of the two forms of the action,

$$r = \frac{-2\beta_A}{(\beta_F+2\beta_A)} \qquad \frac{1}{g^2} = (\frac{\beta_F}{2N}+\frac{\beta_A}{N}) \tag{5.5}$$

A third possible form of the action is given by the Villain or Heat Kernel action[24,25] . We write the plaquette variable $U(P)$ as $U = SU_DS^\dagger$ where $U_D = diag[e^{i\phi_1},...e^{i\phi_N}]$ and define the quantity,

$$K(U(P),g_{HK}) = \mathcal{N}\sum_{l=-\infty}^{\infty}\prod_{i<j}\frac{\phi_i-\phi_j+2\pi(l_i-l_j)}{2sin[(\phi_i-\phi_j+2\pi(l_i-l_j))/2]}exp(-\frac{1}{g^2_{HK}}\sum_j(\phi_j+2\pi l_j)^2) \tag{5.6}$$

where the overall normalization $\mathcal{N}$ is independent of ϕ and the sum on l runs over all integers $(l_1,.....l_N)$, subject to the constraint $\sum l_j = 0$ for $SU(N)$. In terms of K the plaquette action is,

$$S_{HK} = -ln(\frac{K(U(P),g_{HK})}{K(1,g_{HK})}) \tag{5.7}$$

so that in the weak coupling limit we obtain,

$$S_{HK} = \sum_{\mu,\nu}[(\frac{1}{g^2_{HK}}-\frac{N}{24})Tr\frac{\phi^2}{2}-\frac{1}{5760}(NTr\phi^4+3Tr\phi^2Tr\phi^2) \tag{5.8}$$

Lastly we consider the action suggested by Manton [26],

$$S_M(P) = \frac{1}{g^2_M}\Big(d(U(P),1)\Big)^2 \tag{5.9}$$

where $d(U(P),1)$ is the geodesic distance of U from the identity with respect to the invariant metric on the group. Summed over plaquettes the action becomes,

$$S_M = \frac{1}{g^2_M}\sum_{\mu,\nu}Tr\frac{\phi^2}{2} \tag{5.10}$$

For SU(3) it may be more convenient to construct an action having the same form as eq(5.10) in the limit of weak coupling.

$$S_M(P) = -\frac{3}{g_M^2} Tr\left(U(P) - \frac{1}{10}U^2(P) + \frac{1}{135}U^3(P) - \frac{49}{54}\right) \tag{5.11}$$

Because eq.(5.11) resembles eq.(5.10) in the limit of weak coupling, we refer to them both as the Manton action. The parameters in eq.(3.4) which characterize all these lattice actions are given in Table(2).

	S_4	S_6	T_4	T_6	U_6	V_6
WILSON	$-\frac{1}{24}$	$\frac{1}{720}$	0	0	0	0
FUNDAMENTAL + ADJOINT	$-\frac{1}{24}$	$\frac{1}{720}$	$\frac{r}{8N}$	0	$-\frac{r}{48N}$	$\frac{r}{72N}$
MANTON	0	0	0	0	0	0
HEAT KERNEL	$-\frac{Ng^2}{5760}$	0	$-\frac{3g^2}{5760}$	0	0	0

TABLE 2 - The coefficients s_4, s_6, t_4, t_6, u_6, v_6 which determine the contribution of different actions to the general one plaquette form eq. (3.4). The ratio r is given in eq. (5.5).

The size of the corrections associated with any pair of actions can be extracted from Table (3). The coefficient δ changes the experimental ratio of Λ parameters according to,

$$\left(\frac{\Lambda_L}{\Lambda_{L'}}\right) \simeq (1 - \bar{g}^2\delta_{L,L'})\left(\frac{\Lambda_L}{\Lambda_{L'}}\right)_{exp} \tag{5.12}$$

$\delta_{L,L'}$	$SU(2)$	$SU(3)$
$\delta_{M,W}$	$4.48\ 10^{-2}$	0.132
$\delta_{M,FA}$	$4.48\ 10^{-2} + 3.37\ 10^{-2} r + 1.26 r^2$	$0.132 - 1.26 r + 1.37 r^2$
$\delta_{M,HK}$	$2.26\ 10^{-3}$	$3.39\ 10^{-3}$

TABLE 3 - The correction parameter $_{L,L'}$ for various pairs of actions in SU(2) and SU(3).

A comparison with $SU(2)$ data is shown in Table (4). The corrections to the experimentally determined Λ ratios have been estimated using eq.(5.12) with $\bar{g}^2 = 2$, corresponding roughly to the mean of the values of g^2 at which the various measurements were made. The corrections go in the right direction but appear too small to completely remove the discrepancy between theory and experiment. The size of the corrections to Λ_M/Λ_W and Λ_M/Λ_{HK} suggests that the higher order terms in g^2 can safely be ignored. The remaining discrepancy should be attributed to order a^2 corrections [5,6] which may be large at presently investigated values of the coupling constant. Note however that the experimental errors are large and that the evidence for a remaining discrepancy is not overwhelming.

	Theory	*String tension data*	*String tension data (corrected)*	*T_c data*	*T_c data (corrected)*
$\frac{\Lambda_M}{\Lambda_W}$	3.07	5.1 ± 1.0	4.7	4.08	3.71
$\frac{\Lambda_M}{\Lambda_{HK}}$	2.45	3.0 ± 0.3	3.0	2.60	2.59

TABLE 4 - Comparison of the theoretical ratio of Λ parameters with SU(2) data with and without our correction included. The string tension data is taken from refs. (7, 30). The data on the deconfinement temperature T_c comes from ref. (9).

We are in agreement with ref.[10] for the part of the calculation which they performed. The only exception to the above statement is Λ_M/Λ_{HK} . We believe that their calculation of this quantity is in error. Our result for the relationship between the coupling constants is,

$$\frac{1}{g_M^2} = \frac{1}{g_{HK}^2} - \frac{N}{24} - \frac{g_{HK}^2 N^2}{1152} \tag{5.13}$$

In all other cases the results of the complete calculation are very similar to the results already found in ref.[10] . For example, the estimate given in ref. [10] for $\delta_{M,FA}$ in $SU(2)$ is

$$\delta_{M,FA} = 4.72 \times 10^{-2} + 2.68 \times 10^{-2} r + 1.26 r^2 \tag{5.14}$$

Comparison with our exact result shown in Table(3), shows that $\delta_{M,FA}$ for the pure Wilson term ($r = 0$) differs only by a few percent, the term linear in r differs by about 20%and the term quadratic in r is identical. The reason for this agreement is that many of the extra diagrams which we calculated cancel one another as explained in eq.(4.21) and the discussion which follows. Our final result for the relationship between Manton and Wilson coupling constants is,

$$\frac{1}{g_M^2} = \frac{1}{g_W^2} - \frac{2N^2-3}{24N} - g_W^2\left(\frac{2N^4-7N^2+9}{384N^2} + \frac{2N^2-3}{24}a_R\right) \tag{5.15}$$

The result given in ref. [10] for this quantity is identical but with $a_R = 0$. Since a_R is a small number the approximate numerical equality follows.

In accordance with the results obtained in ref. [10], the correction $\delta_{W,FA}$ which relates the Wilson and Mixed fundamental adjoint actions is quite large. This is due to the large coefficient of r^2 as shown in Table (3). Although the correction goes in the right direction it does not bring the data into agreement with theory for $\beta_A > 0$,where the perturbative approximation should work best.

Our results for the fundamental and adjoint action may also be compared with the results of the large N expansion [11,27,28,29]. The action eq.(5.3) may be rewritten as

$$S_{FA}(P) = (\beta_F + 2\beta_A w)\left(1 - \frac{1}{2N}Tr(U(P) + U^\dagger(P))\right) - \beta_A|Tr\frac{U(P)}{N} - w|^2 \tag{5.16}$$

where $w = w(g_S^2)$ is the average of the plaquette variable for a Wilson action with coupling

$$g_S^2 = \frac{2N}{(\beta_F + 2\beta_A w(g_S^2))} \tag{5.17}$$

$$w(g_S^2) = \langle\frac{TrU(P)}{N}\rangle \tag{5.18}$$

Using the two loop expression for w eq.(4.3) together with our results g_S^2 may be written in terms of an effective coupling g_W^2 for the pure Wilson theory eq.(5.15).

$$\frac{1}{g_W^2} = \frac{1}{g_S^2} + \frac{r}{4N} + rg_S^2\left(\frac{1}{4}a_R + \frac{1}{64} - \frac{1}{16N^2}\right) - \frac{3}{64}r^2g_S^2\frac{N^2+1}{N^2} + O(g_S^4) \tag{5.19}$$

Following ref [11] , eq. (5.19) re-expressed in terms of w can be used to derive a recursive relation for the effective coupling of the particular Wilson theory which is equivalent to eq.(5.16) up to the order in which we calculate.

$$B_W = B_F + B_A w(\frac{1}{NB_W})$$

$$-\frac{B_A}{N^2-1}\left(1 - w^2(\frac{1}{NB_W})\right) - \frac{B_A}{B_W^2}\frac{1}{48N^2}(1 - \frac{3}{N^2}) - \frac{B_A^2}{B_W^3}\frac{N^2+1}{64N^4} \tag{5.20}$$

The following changes of variables have been made,

$$B_W = \frac{1}{N g_W^2}\, , B_F = \frac{\beta_F}{2N^2}\, , B_A = \frac{\beta_A}{N^2} \tag{5.21}$$

to ensure the correct behaviour in the limit $N \to \infty$. Since the function w,

$$w(\frac{1}{NB_W}) = 1 - \frac{N^2-1}{8N^2}\frac{1}{B_W} - \frac{N^2-1}{8N^2}\left(a_R + \frac{2N^2-3}{48N^2}\right)\frac{1}{B_W^2} \tag{5.22}$$

approaches a constant in this limit, all corrections to eq.(5.20) are explicitly of order $1/N^2$. Eq.(5.20) differs from the corresponding result in ref.[11]. This is because thay did not include the contributions from the renormalisation of the gauge parameter. The difference between the two results is numerically insignificant.

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Appendix.

The integrals required for the evaluation of the Feynman graphs on the lattice are given in this appendix. The one loop integrals are of the form:

$$\int_{-\pi}^{\pi} \frac{d^4r}{(2\pi)^4} I = A \qquad (A.1)$$

Table 5 gives the integrand I and thc corresponding answer A. The denominators of the integrands are defined in terms of:

$$\hat{r}^2 = 4\sum_{\mu=1}^{4} sin^2\frac{r_\mu}{2} \qquad (A.2)$$

The singular function $L(p^2)$ is

$$L(p^2) = lnp^2a^2 + \gamma_E - F_{0000}$$

γ_E is the Euler-Mascheroni constant and $F_{0000} \simeq 4.369$ has been given in ref.[17]. The only two loop integral required is [11] :

$$I = \int_{-\pi}^{\pi} \frac{d^4r}{(2\pi)^4}\frac{d^4s}{(2\pi)^4} Y_{12}(r)Y_{12}(s)Y_{12}(r+s) \; = \frac{1}{8} \qquad (A.3)$$

where

$$Y_{\mu\nu}(r) = (sin^2\frac{r_\mu}{2} + sin^2\frac{r_\nu}{2})/(\sum_\rho sin^2\frac{r_\rho}{2}) \qquad (A.4)$$

I		A
$\frac{1}{\hat{r}^2}$		$Z_{0000} \approx 0.1549$
$(\sin^4\frac{r_\mu}{2})\,/\,(\hat{r}^2)^2$		$\frac{1}{64} - \frac{Z_{0000}}{16}$
$(\sin^2\frac{r_\mu}{2}\sin^2\frac{r_\nu}{2})\,/\,(\hat{r}^2)^2$	$(\mu\neq\nu)$	$\frac{Z_{0000}}{48}$
$(\sin^4\frac{r_\mu}{2})\,/\,(\hat{r}^2)^3$		$\frac{Z_{0000}}{128} - \frac{1}{512\pi^2}$
$(\sin^2\frac{r_\mu}{2}\sin^2\frac{r_\nu}{2})\,/\,(\hat{r}^2)^3$	$(\mu\neq\nu)$	$\frac{Z_{0000}}{384} + \frac{1}{1536\pi^2}$
$(\sin^6\frac{r_\mu}{2})\,/\,(\hat{r}^2)^3$		$\frac{1}{256} - \frac{5Z_{0000}}{256} - \frac{1}{512\pi^2}$
$(\sin^2\frac{r_\mu}{2}\sin^4\frac{r_\nu}{2})\,/\,(\hat{r}^2)^3$	$(\mu\neq\nu)$	$\frac{Z_{0000}}{768} + \frac{1}{1536\pi^2}$
$(\sin^2\frac{r_\mu}{2}\sin^2\frac{r_\nu}{2}\sin^2\frac{r_\sigma}{2})\,/\,(\hat{r}^2)^3$	$(\mu\neq\nu\neq\sigma)$	$\frac{Z_{0000}}{768} - \frac{1}{1536\pi^2}$
$\frac{1}{\hat{r}^2\,\hat{k}^2}$	$p \to 0$	$\frac{1}{16\pi^2}(2 - L(p^2))$
$(\sin\frac{r_\mu}{2}\cos\frac{r_\mu}{2})\,/\,(\hat{r}^2\hat{k}^2)$	$p \to 0$	$\frac{1}{16\pi^2}\frac{p_\mu}{4}(L(p^2) - 2) + \frac{p_\mu Z_{0000}}{32}$
$(\sin\frac{r_\mu}{2}\cos\frac{r_\mu}{2}\sin\frac{r_\nu}{2}\cos\frac{r_\nu}{2})\,/\,[(\hat{r}^2)^2\hat{k}^2]$	$p \to 0$	$\frac{1}{16\pi^2}[\frac{\delta_{\mu\nu}}{16}(2 - L(p^2)) + \frac{p_\mu p_\nu}{8p^2}] - \frac{\delta_{\mu\nu}Z_{0000}}{128}$

TABLE 5 - Table of lattice integrals. The momentum k is defined to be $k = r + p$.

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