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QUANTIZATION OF GALILEAN GAUGE THEORIES

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ABSTRACT

Galilean gauge theories are quantized according to Dirac's theory of canonical quantization of constrained systems. Only the zero-momentum term in the Fourier expansion of the gauge fields is compatible with the constraints, and it is different from zero for periodic boundary conditions (b.c.), while it is zero if the fields are required to vanish on the surface of the quantization box. Such term has physical effects which therefore depend on b.c..

The effect of the zero-momentum term of the electric potential is to forbid charged states. This constraint holds both in the abelian and nonabelian case and it is true also in the relativistic theory.

The zero-momentum term of the magnetic potential in the abelian case gives only rise to radiative corrections (which are the $c \to \infty$ limit of the relativistic ones), while in the nonabelian case also affects the matter field interaction.

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1. - INTRODUCTION

Looking for the features of nonabelian gauge theories which could possibly provide the confinement of color, it would be desirable to introduce the most drastic approximations which could respect such features. We have considered[1] the approximation obtained by letting the velocity of light $c \to \infty$. This limit is not unique in general, but it is strongly constrained by requiring that Poincarè invariance should contract into Galilei invariance while other symmetries (gauge invariance, chiral invariance, charge symmetry) should be conserved. If the limit is done in this way, the low energy behaviour of relativistic theories (for which such behaviour is known) is reproduced in the limiting theories. This has been shown for the Goldstone and Higgs models, the Schwinger mechanism for spontaneous mass generation, the massless Wess-Zumino and Fayet Iliopoulos models of supersymmetry[2] and for QED. Concerning the latter, which is most relevant to QCD, we have shown[1] that in Galilean QED there are radiative corrections which factorize as in the relativistic case.

The above results give some confidence that the $c \to \infty$ limit should conserve the infrared features of nonabelian gauge theories as well. The study of such features requires these theories in the quantum form. This has already been formally obtained by performing the limit of the relativistic theory in the path integral formulation[3]. The resulting Galilean theory contains constraints which, in the nonabelian case, we have been able to solve only in a gauge in which the canonical quantization of the relativistic theory is not known. In addition the choice of the gauge proves to be crucial, because as we will see the Faddev-Popov determinant for the Coulomb, Landau, and $A_3=0$ gauges vanishes identically in the Galilean limit. We prefer therefore as a first step to quantize directly classical Galilean gauge theories, using Dirac's formalism[4] of canonical quantization of constrained systems. In the abelian case we recover the result obtained by performing the limit in QED.

A general feature of Galilean quantum gauge theories is that only the zero-momentum term of the Fourier expansion of the gauge fields survives consistently with the fact that a field propagating with infinite velocity must be constant over space. Such a constant term is compatible with periodic boundary conditions (p.b.c.), but must be zero if the fields are required to vanish on the surface of the quantization volume. We will refer to the latter b.c. by the elliptic expression: Vanishing boundary conditions (v.b.c.).

We will consider in this paper only p.b.c. and v.b.c.. Constant gauge fields are shown to have physical effects which therefore depend on b.c.. We will show that comparison with experiment requires v.b.c. in QED. This conclusion is made possible by the exact solution of the infrared sector in QED. We do not try to solve the infrared sector of Galilean QCD in the present paper, and therefore we cannot draw a definite conclusion concerning b.c., but we argue that in this case they should be periodic.

The paper is organized in the following way. In Sect. 2. we prove the identical vanishing of the Faddeev-Popov determinant for the Coulomb gauge in Galilean gauge theories and in Sect. 3. we perform the canonical quantization. In Sects. 4. and 5. we solve the constraints for the abelian and nonabelian case respectively. In Sect. 6. we solve the infrared sector of Galilean QED. In Sect. 7 we discuss the effects of b.c. and in Sect. 8. we present our conclusions. Some of the results of this paper have been anticipated in Ref. (5).

2. - IDENTICAL VANISHING OF THE FADDEEV-POPOV DETERMINANT FOR THE COULOMB GAUGE IN THE $c \to \infty$ LIMIT

A necessary condition for the quantization of gauge theories in the Coulomb gauge

$$\partial_k A_k = 0 \ , \tag{2.1}$$

is that the determinant of the Faddeev-Popov operator $\mathscr{D}_k \partial_k$ be different from zero.

This condition is necessary both in the canonical quantization à la Dirac and in the formalism of longitudinal and transverse fields[6].

We will show in this section that in the Galilean limit only field configurations survive for which such determinant is identically zero. As a consequence we cannot perform the $c \to \infty$ limit on the quantum theory in the Coulomb or the Landau gauges which coincide in the limit.

In order to establish the notation let us write the Galilean expressions of the stress tensors

$$F^a_{ij} = \partial_i A^a_j - \partial_j A^a_i + g f^{abc} A^b_i A^c_j$$
$$F^a_{oi} = \partial_t A^a_i + \partial_i V^a + g f^{abc} A^b_i V^c, \tag{2.2}$$

and the covariant derivatives

$$\mathscr{D}_t = \partial_t + i g t^a V^a$$
$$\mathscr{D}_k = \partial_k - i g t^a A^a_k . \tag{2.3}$$

In the above equations A^a_i and V^a are the gauge fields, f^{abc} are the structure constants and t^a the generators of the color group in the appropriate representation. In the regular real representation

$$(t^a)^{bc} = i f^{abc}. \tag{2.4}$$

We will often write A_k instead of $f_a A^a_k$ and so on.

The gauge-field Lagrangian density is[1]

$$\mathscr{L}_G = \frac{1}{2} F^a_{oi} F^a_{oi} - \Lambda^a_i \phi^a_i, \tag{2.5}$$

where

$$\phi^a_i = \frac{1}{2} \varepsilon_{ijk} F^a_{jk}. \tag{2.6}$$

The Λ^a_i are Lagrange multipliers, whose variation generates the constraints

$$\phi^a_i = 0 . \tag{2.7}$$

These constraints can be rewritten

$$\mathscr{D}_j A_k = \partial_k A_j. \tag{2.8}$$

By taking the $\mathscr{D}_k$- derivative of the above equation and using the commutativity of the covariant derivatives which follows from the constraint (2.7) we get

$$\mathscr{D}_k \partial_k A_j = \mathscr{D}_j \mathscr{D}_k A_k = \mathscr{D}_j \partial_k A_k = 0, \tag{2.9}$$

showing that the Faddeev-Popov operator $\mathscr{D}_k \partial_k$ has at least one eigenstate with vanishing eigenvalue, so that its determinant must vanish.

3. - CANONICAL QUANTIZATION OF GALILEAN GAUGE THEORIES

Galilean gauge theories contain the constraints (2.7) in addition to the constraints of the relativistic theory. The theory of canonical quantization of constrained systems has been developed by Dirac[4]. This theory is well known and will not be reviewed here. In order to establish the notation, however, we summarize the main points.

One requires that the original constraints ϕ , called primary constraints, be valid at all the times, by imposing the vanishing of their commutator with the Hamiltonian. This condition generates new constraints χ_σ, called secondary constraints, for which again commutativity with the Hamiltonian is required and so on. If the Lagrangian is consistent this process goes to an end. From the whole set of ϕ_σ's and χ_σ's a maximal subset is chosen for which the determinant of the Poisson brackets does not vanish. The constraints of this set are called second class. If we denote second class constraints by Ξ , by definition

$$\det \Delta_{\sigma\tau} \neq 0; \quad \Delta_{\sigma\tau} = \left[\Xi_\sigma , \Xi_\tau \right]. \tag{3.1}$$

Second class constraints allow the elimination of pairs of conjugate variables. The remaining variables no longer satisfy, in general, canonical Poisson brackets. Their commutation relations can be deduced from the commutation relations obeyed by the old variables once the constraints are taken into account. Such commutation relations are called Dirac's brackets and read

$$\left[C, D\right]^{*} = \left[C,D\right] - \left[C, \Xi_{\sigma}\right] \Delta^{-1}_{\sigma\tau} \left[\Xi_{\tau}, D\right] \ . \tag{3.2}$$

An essential point to be recalled is that the constraints should not be used until all the commutation relations have been worked out.

The remaining constraints are called first class. They can either be used as constraints on physical states, or can be supplemented by gauge fixing constraints chosen in such a way that first class constraints plus gauge fixing be second class.

We will adopt an intermediate procedure, namely we will use a subset of first class constraints on the physical states and we will add to the remaining first class constraints appropriate gauge fixings.

The matter field Lagrangian density is[2]

$$\mathscr{L}_M = \psi^{*} i \mathscr{D}_t \psi + \bar{\psi}^{*} i \mathscr{D}^{*}_t \bar{\psi} + \psi^{*} \frac{\mathscr{D}^2}{2m} \psi + \bar{\psi}^{*} \frac{\mathscr{D}^{*2}}{2m} \psi - m c^2(\psi^{*}\psi + \bar{\psi}^{*}\bar{\psi}), \tag{3.3}$$

where $\psi, \bar{\psi}$ are the matter, antimatter fields respectively. The above expression holds both for bosonic (commuting) and fermion (anticommuting) fields.

For the gauge field Lagrangian it is convenient to use the first order formulation

$$\mathscr{L}_G = E^a_i \, \partial_t \, A^a_i - \frac{1}{2} E^a_i \, E^a_i - V^a \, \mathscr{D}^{ab}_i E^b_i - \Lambda^a_i \, \phi^a_i \ . \tag{3.4}$$

From the total Lagrangian we find that the canonical variables are A^a_i, ψ and $\bar{\psi}$ with canonical momenta

$$\pi = \frac{\partial \mathscr{L}}{\partial \psi} = i\,\psi^{*}$$
$$\bar{\pi} = \frac{\partial \mathscr{L}}{\partial \bar{\psi}} = i\,\bar{\psi}^{*} \qquad (3.5)$$
$$E_i^a = \frac{\partial \mathscr{L}}{\partial \dot{A}_i^a}$$

The nonvanishing Poisson brackets are

$$\left\{\psi_\alpha^r(x), \pi_\beta^s(y)\right\} = \delta_{\alpha\beta}\,\delta^{rs}\,\delta^3(x-y)$$
$$\left\{\psi_\alpha^r(x), \pi_\beta^s(y)\right\} = \delta_{\alpha\beta}\,\delta^{rs}\,\delta^3(x-y) \qquad (3.6)$$
$$\left[A_i^a(x), E_j^b(y)\right] = \delta^{ab}\delta_{ij}\,\delta^3(x-y)$$

where r,s are color indices and α, β spinor indices. The fields Λ^a and V^a are Lagrange multipliers for the primary constraints (2.7) and

$$\phi^a(x) = \mathscr{D}_k^{ab}(x)\,E_k^b(x) + g\,\varrho^a(x) = 0, \qquad (3.7)$$

with

$$\varrho^a(x) = \psi^{*}\,t^a\,\psi - \bar{\psi}^{*}\,t^a\,\psi. \qquad (3.8)$$

The constraint (3.7) is the same as in the relativistic theory.

Following Dirac's theory we assume the Hamiltonian density

$$\mathscr{H} = E_i^a\,\partial_t\,A_i^a + \psi^{*}\,i\,\partial_t\psi + \bar{\psi}^{*}\,i\,\partial_t\bar{\Psi} - (\mathscr{L}_M + \mathscr{L}_G), \qquad (3.9)$$

and require the vanishing of the time derivative of the primary constraints

$$\left[\phi^a, H\right] = 0,$$
$$\left[\phi_i^a, H\right] = 0. \qquad (3.10)$$

The first of Eqs. (3.10) is automatically satisfied as in the relativistic case, while the second one gives

$$\left[\phi_i^a(y), H\right] = g\,f^{abc}\,V^b(y)\,\phi_i^c(y) + \chi_i^a(y) = 0\,, \qquad (3.11)$$

generating the secondary constraints

$$\chi_k^a(y) = \varepsilon_{kij} \mathscr{D}_i^{ab}(y)\, E_j^b(y) = 0 \,. \tag{3.12}$$

These are the only secondary constraints, because the vanishing of $[\chi_i^a(y), H]$ yields only conditions on Λ_i^a.

We quantize in a cubic box of volume L^3 with p.b.c.. Fields and constraints can therefore be expanded in Fourier series

$$A_i^a(x) = \frac{1}{L^{3/2}} \sum_{\vec{n}} A_{i\vec{n}}^a \, e^{i\frac{2\pi}{L} n\cdot x}, \qquad A_{i,\vec{n}}^a = A_{i,-\vec{n}}^{a*}$$

$$E_i^a(x) = \frac{1}{L^{3/2}} \sum_{\vec{n}} E_{i\vec{n}}^a \, e^{-i\frac{2\pi}{L} n\cdot x}, \qquad E_{i,\vec{n}}^a = E_{i,-\vec{n}}^{a*}, \tag{3.13}$$

$$\phi^a(x) = \frac{1}{L^{3/2}} \sum_{\vec{n}} \phi_{\vec{n}}^a \, e^{i\frac{2\pi}{L} n\cdot x}, \qquad \phi_{\vec{n}}^a = \phi_{-\vec{n}}^{a*} \tag{3.14}$$

and so on.

The nonvanishing Poisson brackets among the Fourier coefficients are

$$\left[A_{i\vec{m}}^a, E_{j\vec{n}}^b\right] = \delta_{ij}\, \delta^{ab}\, \delta_{\vec{m},\vec{n}} \,. \tag{3.16}$$

It is easy to check that the constraints (3.7) are first class. We will use their constant components ϕ_0^a as constraints on the physical states, and we will add a gauge fixing in such a way that the whole of gauge fixing plus secondary and primary constraints (with the exception of ϕ_0^a) be second class.

The reason to single out the constraint ϕ_0^a will be explained in Sect. 7.

4. - SOLUTION OF THE CONSTRAINTS IN THE ABELIAN CASE

4.1. - The Coulomb Gauge

Let us write down the full set of constraints, including the gauge fixing one

$$\begin{aligned}
\phi_{i\vec{n}} &= \frac{1}{2}\,\varepsilon_{ijk}\; i\,\frac{2\pi}{L}\, n_j\, A_{k\vec{n}} = 0 \\
\chi_{i\vec{n}} &= -\,\varepsilon_{ijk}\; i\,\frac{2\pi}{L}\, n_j\, E_{k\vec{n}} = 0 \\
\phi_{\vec{n}} &= -i\,\frac{2\pi}{L}\, n_i\, E_{i\vec{n}} + e\,\varrho_{\vec{n}} = 0 \qquad ,\vec{n}\neq 0 \\
\chi_{\vec{n}} &= i\,\frac{2\pi}{L}\, n_i\, A_{i\vec{n}} = 0.
\end{aligned} \tag{4.1}$$

These constraints act only on the Fourier coefficients with $\vec{n}\neq 0$. The associated homogeneous equations have the unique solution $A_{i\vec{n}}=E_{i\vec{n}}=0$, $\vec{n}\neq 0$, ensuring that the determinant of their Jacobian matrix is different from zero. The full solution is therefore

$$\begin{aligned}
A_{i\vec{n}} &= q_i\;\delta_{\vec{n},0} \\
E_{i\vec{n}} &= p_i\;\delta_{\vec{n},o} - e\,(\Delta^{-1}\,\partial_i\varrho\,)_{\vec{n}}(1-\,\delta_{\vec{n},o}).
\end{aligned} \tag{4.2}$$

Since there is no constraint for the constant terms, they satisfy their original Poisson brackets

$$\left[q_i, p_j\right] = \delta_{ij}. \tag{4.3}$$

The Hamiltonian is

$$H = H_o + H_c\,, \tag{4.4}$$

where H_c is Coulomb interaction and

$$H_o = \frac{1}{2}p^2 + \frac{1}{2}\,\omega^2 q^2 - \omega q_i\, I_i - \frac{1}{2m}\int d^3x\,(\psi^*\Delta\psi + \bar{\psi}^*\Delta\bar{\psi}) \tag{4.5}$$

$$+ mc^2\int d^3x\,(\psi^*\psi + \bar{\psi}^*\bar{\psi})\,.$$

$$I_i = \sqrt{\frac{m}{N}}\int d^3x\ \frac{1}{m\,i}\ (\psi^*\,\partial_i\psi\ - \bar{\psi}^*\,\partial_i\,\bar{\psi}) \tag{4.6}$$

$$\omega^2 = \frac{g^2 N}{mL^3} \tag{4.7}$$

$$N = \int d^3x\,(\psi^*\psi\ + \bar{\psi}^*\bar{\psi})\,. \tag{4.8}$$

For v.b.c., $q_i = p_i = 0$. We have put the terms involving q_i, p_i in H_o because their effect is to change the definition of in- and out- states for matter fields, rather than to give rise to a real interaction. We will see this in detail in Sect. 6 where we will solve the infrared sector of the Hamiltonian (4.4).

4.2. - The Gauge $A_3=0$

The gauge fixing $A_3(x)=0$ is not consistent with p.b.c. because

$$\det\left[A_{3\,\vec{n}},\, n'_k E_{k\,\vec{n}'}\right] = \det n_3\ \delta_{\vec{n},\vec{n}'} = \prod_{\vec{n}} n_3 = 0.$$

We therefore assume the gauge fixing

$$\begin{aligned} &A_{3\,\vec{n}} = 0, \quad n_3 \neq 0 \\ &A_{2\,\vec{n}} = 0,\ n_3=0,\ n_2 \neq 0 \\ &A_{1\,\vec{n}}=0, \quad n_3 = n_2 = 0,\ n_1 \neq 0. \end{aligned} \tag{4.9}$$

It is very easy to check that the determinant of the Poisson matrix does not vanish.

Let us introduce the the definitions

$$B_i = \frac{1}{L^{3/2}} \sum_{\vec{n}} A_{i n_1, n_2, n_3 \neq 0} \, e^{i \frac{2\pi}{L} n \cdot x}$$

$$C_i = \frac{1}{L^{3/2}} \sum_{\vec{n}} A_{i n_1, n_2 \neq 0, n_3 = 0} \, e^{i \frac{2\pi}{L} n \cdot x} \tag{4.10}$$

$$Q_i = \frac{1}{L^{3/2}} \sum_{\vec{n}} A_{i n_1, n_2 = n_3 = 0} \, e^{i \frac{2\pi}{L} n \cdot x} ,$$

so that

$$A_i = B_i + C_i + D_i \; . \tag{4.11}$$

Eqs. (4.9) imply

$$A_3 = C_3(x_1, x_2, t) + D_3(x_1, t)$$

$$A_2 = B_2(x_1, x_2, x_3, t) + D_2(x_1, t) \tag{4.12}$$

$$A_1 = B_1(x_1, x_2, x_3, t) + C_1(x_1, x_2, t) + D_1(t)$$

Inserting the above expressions into the homogeneous equations associated to the constraints we again find $A_{i\vec{n}} = E_{i\vec{n}} = 0$ for $\vec{n} \neq 0$, so that the full solution is again given by Eq. (4.2).

5. - SOLUTION OF THE CONSTRAINTS FOR SU(2)

We have shown in Sect. 2. that the Coulomb gauge is not accetable in the nonabelian case. We will see here that the gauge of Eq. (4.9) is.

The constraint (2.7) is equivalent to

$$G \, \partial_k \, G^{-1} = - g \, A_k \; , \tag{5.1}$$

where G is a unitary operator and

$$(\varepsilon_a)_{bc} = \varepsilon_{abc} = (\varepsilon^a)^{bc}. \tag{5.2}$$

Since according to Eq. (4.12) A_3 does not depend on x_3, Eq. (5.1) can be easily integrated for k=3 with the result

$$G = e^{g A_3 x_3} T(x_1 x_2) , \tag{5.3}$$

$T(x_1, x_2)$ being an arbitrary unitary operator.

Eq. (5.1) can now be rewritten

$$- g A_k = - g e^{g A_3 x_3} \tilde{A}_k e^{-g A_3 x_3} + e^{g A_3 x_3} \partial_k e^{-g A_3 x_3} \tag{5.4}$$

where

$$- g \tilde{A}_k = T \partial_k T^{-1} . \tag{5.5}$$

We now impose p.b.c. with respect to x_3 for k=1,2

$$- g \tilde{A}_k = - g e^{g A_3 L} \tilde{A}_k e^{-g A_3 L} + g e^{g A_3 L} \partial_k e^{-g A_3 L} . \tag{5.6}$$

If we put

$$g A_3^a L = \hat{v}^a v, \qquad \hat{v}^a \hat{v}^a = 1 , \tag{5.7}$$

Eq. (5.6) becomes

$$(\delta_{ab} - G_{ab}) g \tilde{A}_k^b = \hat{v}^a \partial_k v + \sin v \partial_k \hat{v}^a - (1 - \cos v) \varepsilon^{abc} \hat{v}^b \partial_k \hat{v}^c . \tag{5.8}$$

Since $\hat{v}^a (\delta_{ab} - G_{ab}) = 0$, the necessary condition for the above equation to be solvable is

$$\hat{v}^a \left[\hat{v}^a \partial_k v + \sin v \partial_k \hat{v}^a - (1 - \cos v) \varepsilon^{abc} \hat{v}^b \partial_k \hat{v}^c \right] = \partial_k v = 0 , \tag{5.9}$$

i.e.

$$v = v(t). \tag{5.10}$$

Eq. (5.8) can now be solved for $\tilde{A}_k^a$

$$g \tilde{A}_k^a = q_k \hat{v}^a + \left[(1-G)^{-1} \right]_{ab} \left[(1 - \cos v) \varepsilon^{bcd} \hat{v}^c \partial_k \hat{v}^d + \sin v \partial_k \hat{v}^b \right] , \tag{5.11}$$

where q_k is an arbitrary function of x_1 and x_2.

The above expression of $\tilde{A}_k^a$ is only a consequence of p.b.c.. We must still require that $\tilde{A}_k^a$ be a pure gauge field according to Eq. (5.5). It is however convenient to impose first the gauge fixing conditions on $\tilde{A}_k^a$, which can be obtained using Eq. (5.11) in Eq. (5.4)

$$A_k^a = q_k \vartheta^a + \varepsilon^{abc} \vartheta^b \partial_k \vartheta^c \left[\cos v\,(1-\frac{x_3}{L}) - 1 + \cos v \frac{x_3}{L}\right]$$
$$- \partial_k \vartheta^a \left[\sin v\,(1 - \frac{x_3}{L}) + \sin v \frac{x_3}{L}\right] . \tag{5.12}$$

According to Eqs. (4.12) we must impose $C_2=0$, $\partial_1 D_1=0$. To this end we evaluate

$$C_i^a + D_i^a = \frac{1}{L}\int_o^L dx_3\, A_i^a = q_i \vartheta^a + (2\,\frac{\sin v}{v} - 1)\,\varepsilon^{abc} \vartheta^b \partial_i \vartheta^c$$
$$- \frac{2}{v}(1 - \cos v)\,\partial_i \vartheta^a . \tag{5.13}$$

For i=2, since $C_2=0$ and $D_2=D_2(x_1)$,

$$\partial_2 \frac{1}{L}\int_o^L dx_3\, A_2^a = \partial_2\; q_2 \vartheta^a + q_2\, \partial_2 \vartheta^a + (2\,\frac{\sin v}{v} - 1)\,\varepsilon^{abc} \vartheta^b \partial_2 \vartheta^c$$
$$- \frac{2}{v}(1 - \cos v)\,\partial_2^2 \vartheta^a = 0 . \tag{5.14}$$

Multiplying by ϑ^a and integrating over x_2 we obtain

$$\int_o^L dx_2 \left[\partial_2 q_2 - \frac{2}{v}(1-\cos v)\,\vartheta^a\, \partial_2^2 \vartheta^a\right] = \frac{2}{v}(1-\cos v)\int_o^L dx_2\, (\partial_2 \vartheta^a)^2 = 0 .$$

It follows

$$\partial_2 \vartheta^a = 0, \quad \vartheta^a = \vartheta^a(x_1), \quad q_2 = q_2(x_1) . \tag{5.15}$$

Now since D_1 must not depend on x_1, ϑ^a must be a constant, $\vartheta^a = \vartheta^a(t)$, and q_1 must either depend on both x_1 and x_2 or be a constant.

It is now very easy to impose the condition that $\tilde{A}_k$ be a pure gauge. This gives

$$q_i = \partial_i Q \,, \quad i = 1,2 \ . \tag{5.16}$$

Integrating the above Equation for i=2, and taking into account Eq. (5.15) we get

$$Q = q_2 (x_1)\, x_2 + f\,(x_1), \tag{5.17}$$

where $f(x_1)$ is an arbitrary function. P.b.c. on q_1 require q_2 to be a constant and the ambiguity that q_1 must either depend on both x_1 and x_2 or be a constant must therefore be solved in favor of the second possibility.

In conclusion we have our main result

$$A_i^a = \frac{1}{L^{3/2}}\, q_i\,(t)\, \hat{v}^a\,(t) \ . \tag{5.18}$$

We have thus far imposed the constraints (2.7) and (4.9). It remains to impose the constraints (3.7) and (3.12), whose Fourier components are

$$\mathscr{D}_i(\vec{n})\, E_{j\,\vec{n}} - \mathscr{D}_j(\vec{n})\, E_{i\,\vec{n}} = 0 \tag{5.19}$$

$$\mathscr{D}_k(\vec{n})\, E_{k\,\vec{n}} + g\,\varrho_{\vec{n}} = 0, \ \vec{n} \neq 0.$$

where

$$\mathscr{D}_i(\vec{n}) = -\,i\, \frac{2\pi}{L}\, \vec{n} - g\,\hat{v}\,\vec{q}\,. \tag{5.20}$$

Let us call $\tilde{E}_i$ a special solution to Eqs. (5.19) and P_i the general solution to the associated homogeneous equations, so that

$$E_i^a = P_i^a + \tilde{E}_i^a . \tag{5.21}$$

From Eqs. (5.19) for $\vec{n} \neq 0$ we have

$$\mathscr{D}^2(\vec{n})\, P_{k,\,\vec{n}} = 0, \quad \vec{n} \neq 0 \tag{5.22}$$

$$\mathscr{D}^2(\vec{n})\, \tilde{E}_{k\,\vec{n}} = -\,g\,\mathscr{D}_k(\vec{n})\,\varrho_{\vec{n}}\,, \ \vec{n} \neq 0,$$

whose solution is

$$E_{i\vec{n}} = P_{io}\,\delta_{\vec{n}o} - (1 - \delta_{\vec{n}o})\, g\, \mathscr{D}^{-2}(\vec{n})\, \mathscr{D}_i(\vec{n})\, \varrho_{\vec{n}} \tag{5.23}$$

with P_{io} satisfying the first of Eqs. (5.19), whose solution is

$$P^a_{io} = p_i\, \hat{v}^a + \frac{1}{q^2}\, l^a_\perp\, q_i, \tag{5.24}$$

with

$$\hat{v}^a\, l^a_\perp = 0\,. \tag{5.25}$$

In Eq. (5.24) p_i and $l^a_\perp$ are arbitrary apart from condition (5.25) and the factor $1/q^2$ has been introduced for convenience.

The Dirac brackets are evaluated in a separate paper(7), where it is shown that the determinant of the Poisson matrix does not vanish and the following relations hold

$$\left[q_h, p_k\right] = \delta_{hk} \tag{5.26}$$

$$l^a_\perp = (\pi^a - \hat{v}^a\, \hat{v}\,.\,\pi)\, v \tag{5.27}$$

$$\left[v^a, \pi^b\right] = \delta^{ab} \tag{5.28}$$

To go from classical to quantum commutation relations we must solve the ambiguity in Eq. (5.27) arising from the noncommutativity of $\hat{v}$ and π. This will be solved in the standard way by assuming the symmetric form $\frac{1}{2}(\hat{v}.\,\pi + \pi\,.\hat{v})$.

It follows from the above analysis that for v.b.c.

$$A^a_i = E^a_i = 0\,.$$

The Hamiltonian resulting from the Lagrangian (3.3) and (3.4) with p.b.c. is

$$H = \frac{1}{2} E^a_{io} E^a_{io} - \frac{1}{2m}\int d^3x(\psi^*\Delta\psi + \bar{\psi}^*\Delta\bar{\psi}) + m c^2 \int d^3x(\psi^*\psi + \bar{\psi}^*\bar{\psi}) - \omega q_k\, \hat{v}^a\, l^a_k\,, \tag{5.29}$$

where

$$\omega^2 = \frac{g^2 N}{4mL^3}$$

$$I_k^a = \sqrt{\frac{m}{N}} \int d^3x \; \frac{1}{mi} \left(\psi^* \sigma^a \, \partial_k \psi - \bar{\psi}^* \sigma^a \, \partial_k \bar{\psi}\right) \tag{5.30}$$

Using Eq. (5.25) and (5.27)

$$\frac{1}{2} E^a_{io} E^a_{io} = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 q^2 + \frac{1}{2} \frac{l_\perp^2}{q^2} - \frac{1}{2} g^2 \int d^3x \int d^3y$$

$$\varrho^a (x) \left[\mathscr{D}^{-2}(x-y)\right]^{ab} \varrho^b(y) \, ,$$

where

$$\left[\mathscr{D}^{-2}(x)\right]^{ab} = - \frac{1}{4\pi} \frac{1}{|x|} \left[\exp \frac{g}{L^{3/2}} \hat{v} \, \vec{q} \cdot \vec{x} \right]^{ab} =$$

$$= - \frac{1}{4\pi} \frac{1}{|x|} \Big\{ \delta^{ab} - \left[(1-\cos \frac{g}{L^{3/2}} \vec{q} \cdot \vec{x}\right] \left[\delta^{ab} - \hat{v}^a \hat{v}^b\right] \tag{5.31}$$

$$+ \varepsilon^{abc} \hat{v}^c \sin \frac{g}{L^{3/2}} \vec{q} \cdot \vec{x} \Big\}$$

Using Eq. (5.27) we have

$$l_\perp^2 = (v \wedge \pi)^2 , \tag{5.32}$$

i.e. l^2 is the square of angular momentum in color space.

6. - THE INFRARED SECTOR IN GALILEAN QED

Introducing creation and destruction operators:

$$a_k^+ = (2\omega)^{-1/2} (p_k + i \; \omega \, q_k) \tag{6.1}$$

$$a_k = (2\omega)^{-1/2} \, (p_k - \omega q_k) \, ,$$

we can rewrite H_o of Eq. (4.5) as

$$H_o = \frac{3}{2} \; \omega + \omega \; a_k^+ a_k + (a_k^+ - a_k) \, i \sqrt{\frac{\omega}{2}} \; I_k - \frac{1}{2m} \int d^3x (\psi^* \Delta \psi + \bar{\psi}^* \Delta \bar{\psi})$$

$$+ mc^2 \int dx^3 (\psi^* \psi + \bar{\psi}^* \bar{\psi}) \tag{6.2}$$

Let us denote by $|i\rangle$, $|f\rangle$, the normalized eigenstates of the kinetic energy operator. Such states are also eigenstates of the operator I_k with eigenvalues $I_k(i)$, $I_k(f)$, ... H_o can then be put in diagonal form by introducing the shifted operatos

$$b_k(f) = a_k - \frac{1}{\sqrt{2\omega}} \; I_k(f) , \tag{6.3}$$

$$H_o(f) = \frac{3}{2}(\omega^2 + I_k^2(f)) + \omega b_k^+(f)\, b_k(f) - \frac{1}{2m}\int d^3x\, (\psi^* \Delta\psi + \overline{\psi}^* \Delta\overline{\psi}) + mc^2 \int dx^3 (\psi^*\psi + \overline{\psi}^*\overline{\psi}) \tag{6.4}$$

The normalized eigenstates of H_o are

$$|f, n_1, n_2\, n_3\rangle = \prod_{k=1}^{3} \frac{1}{\sqrt{n_k!}} \left[a_k^+ + \frac{i}{\sqrt{2\omega}} \; I_k(f)\right]^{n_k} . \;\; \circ \; e^{\frac{i}{\sqrt{2\omega}}(a_k^+ + a_k)\, I_k(f)} \; |f\rangle . \tag{6.5}$$

To evaluate cross-sections we need the matrix elements

$$\langle i, 0, 0, 0, \, | S | \, f, n_1, n_2, n_3 \rangle = \langle 0 | \; e^{-\frac{i}{\sqrt{2\omega}}(a_k^+ + a_k)\, I_k(i)} \prod_{h=1}^{3} \left[a_h^+ + \frac{i}{\sqrt{2\omega}} \; I_h(f)\right]^{n_h} e^{\frac{i}{\sqrt{2\omega}}(a_\ell^+ + a_\ell)\, I_\ell(f)} \; |0\rangle . \tag{6.6}$$

$$\langle i | \, S\left[(n_1 + n_2 + n_3)\omega\right] | f \rangle$$

The factorization in the above Equation is due to the fact that H_c of Eq. (4.4) contains only matter fields. The S-matrix is however a function of the radiated energy $(n_1, + n_2 + n_3)\,\omega$. By choosing the axes in the appropriate way we can put

$$\Delta I_h = I_h(f) - I_h(i) = \Delta I \; \delta_{h3} , \tag{6.7}$$

so that

$$\left| \langle i, 0, 0, 0, \mid S \mid f, n_1, n_2, n_3 \rangle \right|^2 = \delta_{on_1} \delta_{on_2} \left| \langle i \mid S(n_3 \omega) \mid f \rangle \right|^2 \tag{6.8}$$

$$\frac{1}{n_3!} \left[\frac{(\Delta I)^2}{2\omega} \right]^{n_3} e^{-\frac{(\Delta I)^2}{2\omega}} ,$$

which is proportional to the probability for a classical source ΔI to radiate n_3 photons[1].

The cross section for a process measured with infinite energy resolution in the initial state $\mid i \rangle$ and energy resolution ΔE in the final state $\mid f \rangle$ is[8]

$$\sigma \sim \lim_{L \to \infty} \sum_{n_3=0}^{\infty} \left| \langle i \mid S(n_3 \omega) \; f \rangle \right|^2 \frac{1}{n_3!} \left[\frac{(\Delta I)^2}{2\omega} \right]^{n_3}$$

$$e^{-\frac{(\Delta I)^2}{2\omega}} \theta(\Delta E - n_3 \omega) = \left| \langle i \left| S \left[\tfrac{1}{2} (\Delta I)^2 \right] \right| f \rangle \right|^2 \tag{6.9}$$

$$\cdot \, \theta \left[\Delta E - \tfrac{1}{2} (\Delta I)^2 \right] .$$

The average radiated energy is

$$\omega = \lim_{L \to \infty} \sum_{n_3=0}^{\infty} \frac{1}{n_3!} \left[\frac{(\Delta I)^2}{2\omega} \right]^{n_3} n_3 \omega \, e^{-\frac{(\Delta I)^2}{2\omega}} = \tfrac{1}{2} (\Delta I)^2 . \tag{6.10}$$

According to the above result the cross section vanishes unless the energy uncertainty is greater than the average radiated energy. This is true with p.b.c.. For v.b.c. there are no radiative corrections.

The radiative correction of Eq. (6.7) can also be obtained as the $c \to \infty$ limit from the relativistic theory. The formal proof is that the $c \to \infty$ limit of the relativistic Hamiltonian provides[3] the Hamiltonian we have obtained by quantization of the Galilean Lagrangian. An explicit evaluation of the limit[9], shows that this unexpected effect comes from a zero-momentum contribution which must be singled out before a sum over momenta can be approximated by an integral. In this sense such effect is analogous to Bose-Einstein condensation in statistical mechaniscs.

7. - BOUNDARY CONDITIONS

We have left the constraint ϕ_o^a, the zero-momentum component of the constraint (3.7) without a corresponding gauge fixing. According to Dirac's theory it must be satisfied by physical states.

The reason why we have singled out ϕ_o^a is twofold. First, unlike the other constraints, its existence depends on b.c.. It comes in fact from the variation of the zero-momentum component of the electric potential V_o^a. For v.b.c. $V_o^a=0$ and the constraint $\phi_o^a=0$ is absent.

Second, ϕ_o^a has a direct physical meaning

$$\phi_o^a = g\, L^{3/2}\ (Q_G^a + Q_M^a), \tag{7.1}$$

Q_G^a and Q_M^a being the gluon and matter charge respectively.

Eq. (7.1) holds also in the relativistic case because the constraint (3.7) is the same. In the c→ ∞ limit

$$Q_G^a = \varepsilon^{abc}\, v^b \quad c, \quad Q_G^2 = l^2\,. \tag{7.2}$$

A result analogous to Eq. (7.1) holds in the abelian case stating that the electric charge of physical states must vanish for p.b.c..

Boundary conditions must be the same for the electric and the magnetic potentials, because they are related by Lorents or Galilean transformations[10]

$$V' = V = v_k\, A_k$$

$$A'_k = A_k\ , \tag{7.3}$$

v_k being the parameters of the transformation. We must therefore have $q_i \neq 0$ for p.b.c. and $q_i = 0$ for v.b.c..

Independent experimental facts, like the existence of electrically charged states and the absence of the radiative corrections of Eq. (6.9) require v.b.c. for QED. On the other hand the absence of colored states is in favor of p.b.c. for QCD.

8. - CONCLUSIONS

The gool of the present investigation was to get insigth into the problem of confinement. Our results in this connection are

i) the restriction of physical states to be color singlets does not appear to be necessarily an intrinsic feature of QCD, but could depend on our choice of b.c..

ii) in Galilean QCD there is no confining potential. In order to be in agreement with phenomenology, however, it is not necessary that only quark bound states should exist. It is sufficient that color singlet quark bound states do not disintegrate. This condition could be realized in Galilean QCD if the state of the gluons cloud relative to bound states were orthogonal to the state of the gluon could relative to unbound states[5].

In this connection it is interesting to note that the gluon state, unlike the photon state, does not depend only on the quark current but also on the quark intrinsic state as shown by the couplings in the Hamiltonian.

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