ISTITUTO NAZIONALE DI FISICA NUCLEARE
Laboratori Nazionali di Frascati

LNF-83/57

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Estratto da:
Siam J. Appl. Math. 43, 565 (1983)

Servizio Documentazione
dei Laboratori Nazionali di Frascati
Cas. Postale 13 - Frascati (Roma)

SIAM J. APPL. MATH.
Vol. 43, No. 3, June 1983

0036-1399/83/4303-0009 $01.25/0

A THEORY OF STOCHASTIC RESONANCE IN CLIMATIC CHANGE*

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Abstract. In this paper we study a one-dimensional, nonlinear stochastic differential equation when small amplitude, long-period forcing is applied. The equation arises in the theory of the climate of the earth. We find that the cooperative effect of the stochastic perturbation and periodic forcing lead to an amplification of the peak of the power spectrum, due to a mechanism that we call stochastic resonance. A heuristic analysis of the resonance condition is presented and our analytical findings are confirmed by numerical calculations.

1. Introduction. In this paper we consider a problem posed by the evolution of the earth's climate in the last 700,000 years.

The recent developments of the methods of acquiring and interpreting climate records indicate that at least seven major climate changes have occurred over this period and, moreover, that these changes have an apparent periodicity. The power spectrum presented in Fig. 1 (Mason (1976)) clearly shows these periodicities. Milankovitch (1930), relating the peaks to the periodical changes in earth orbital parameters, explained climate changes as being induced by external factors. However,

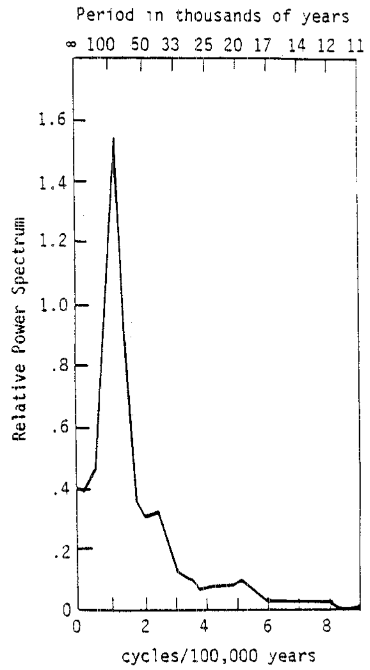

FIG. 1. *Power spectrum of a time series of "observations" over the last* 600,000 *years* (*from Mason* (1976).

* Received by the editors February 1, 1981, and in revised form April 5, 1982.

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focusing attention on the 100,000-years peak, actual calculations of the corresponding orbital parameter, namely, the eccentricity of the earth's orbit, show that for this component the external forcing has a weak amplitude (Berger (1978)). This fact has prompted reconsideration of the causes of climate changes and some authors have advanced the hypothesis that the climate system, controlled by nonlinear coupling of atmosphere-ocean-cryosphere-lithosphere processes, could have a nonlinear response to external forcing so that the 100,000-years period could be an internal period of the system. Some efforts have been made to model the extreme complexity of the climate system, and since the introduction of simple energy-budget models independently by Budyko (1969) and Sellers (1969), a line of research has been followed which develops and exploits these simpler mechanisms in the context of climatic change. Schneider and Dickinson (1974), Ghil (1980), and North et al. (1981) provide reviews.

In this paper we investigate, in the framework of a Budyko–Sellers model, a possible mechanism that, through the nonlinearity, responds to the weak external forcing to reproduce a power spectrum that not only has the desired peak centered around the 100,000-years period but has the correct amplitude (about 10K as reported in Mason (1976) or in other sources listed in Imbrie and Imbrie (1980)). We call this mechanism stochastic resonance. It can be described as an effect resulting from combination of short time-scale phenomena, modeled by stochastic perturbation (Hasselmann (1976)) with the long-term periodicity of the astronomical forcing. In § 2, we discuss some general properties of Budyko–Sellers models which lead to our consideration of stochastic resonance as an explanation of climatic changes. In § 3, we describe the rationale of the model chosen to study the stochastic resonance mechanism. In § 4, we set out an analytical derivation of a relationship among the free parameters of the problem which must be satisfied if stochastic resonance is to occur and, in § 5, these analytical predictions are corroborated by numerical experiments. We note that similar results have been obtained by Nicolis (1982) using a different approach.

2. Some properties of Budyko–Sellers type models. For our purpose, it is convenient to review some properties of Budyko–Sellers type models. These models consider the evolution of a representative earth temperature T caused by slight departure from a perfect balance between the incoming and outgoing radiation, R_{in} and R_{out}, respectively:

$$c\frac{d}{dt}T = R_{in} - R_{out} \tag{2.1}$$

where c is the active thermal inertia of the system.

Generally, R_{in} and R_{out} are parameterized in a suitable manner as functions of T. However, it is possible to formulate these parameterizations in such a way that the RHS satisfies a variational principle (Ghil (1976) and North et al. (1979)). This means that we can define a functional V such that

$$c\frac{d}{dt}T = -\frac{\delta V}{\delta T} \tag{2.2}$$

where $\delta/\delta T$ is the variational derivative. If T is a function of time only (as in the case of globally averaged models), then V is a function and $\delta/\delta T$ reduces to the ordinary derivative d/dT.

Equation (2.2) implies that:

(a) its steady solutions are extrema of V, and that

(b) these solutions can only be asymptotically stable or unstable, and they correspond, respectively, to minima and maxima of V.

Moreover the system described by (2.2) relaxes always toward a steady stable solution if the initial data are chosen nearby. These few mathematical facts show the physical principle that the system tends to a state of minimum "energy" (Paltridge (1975)). It also follows that the system described by (2.2) has no possible internal mechanism able to amplify an external forcing and for this reason might be misleading if used to investigate climatic change. However, it is tacitly assumed in the derivation of (2.2) that the faster processes contributing to the determination of T, such as weather fluctuations, are averaged out. To take into account the residual effects of these short frequency phenomena, Hasselmann (1976) proposed to perturb stochastically (2.2) (and, in general, any "climate" equation).

The equation to be considered is

$$c\,dT = -\frac{\delta V}{\delta T}\,dt + \varepsilon^{1/2}\,dW \tag{2.3}$$

where ε is the variance of the stochastic perturbation defined by the Wiener process W. One of us (Sutera (1981) hereafter S), analyzed the effect of this noise on the long-term behavior of a simplified (zero-dimensional) model. It was noted that if an additional stable steady state was assumed in a range of temperature of 10K around the present average temperature, then the observed variance in the climatic data could be obtained. However, the apparent periodicity corresponding to the observed peak in the spectrum did not appear. Benzi et al. (1982) observed that by adding a small periodic forcing (simulating the astronomical forcing) to the same idealized model, the periodicity could be recovered.

3. The model. We propose to study the combined effect of the stochastic perturbations and the small astronomical forcing by means of an energy-budget model of Budyko–Sellers type. To take into account the effect of the periodic variation of the external forcing, we parameterize the incoming radiation as

$$R_{\text{in}} = Q(1 + A\cos\omega t) \tag{3.1}$$

where Q is the solar constant, A the forcing amplitude related to the periodic variation of the earth's orbit eccentricity and $2\pi\omega^{-1} = 92{,}000$ years is the period of this variation. We set $A = 5\times10^{-4}$, corresponding to a total variation of 0.1% of Q over a period.

The outgoing radiation R_{out} is parameterized as the sum of the reflected part, $\alpha(T)Q$, and the radiated part

$$R_{\text{out}} = \alpha(T)R_{\text{in}} + E(T). \tag{3.2}$$

We note that $\alpha(T)$ represents the albedo of the system and that $E(T)$ depends on its emissivity. In the first instance, we consider the thermal inertia of the system as independent of the temperature so that

$$c\frac{d}{dt}T = Q(1 + A\cos\omega t)(1 - \alpha(T)) - E(T). \tag{3.3}$$

We wish to examine qualitatively certain properties of (3.3). For this purpose we will construct a RHS not directly from the physics of the earth's climate, but by reference to the general mathematical form of the Budyko–Sellers type models which have been constructed by other investigators. We postulate that (3.3) has three steady states, of which two are stable and separated by 10K, as suggested by climatic observations (e.g., Mason (1976)). Moreover, since we can define a function V such that the RHS of (3.3) can be obtained from its derivative (see the discussion of § 2), it follows that the third steady state is an unstable one. To simplify the computations we also assume that the two steady states are symmetrically placed with respect to the unstable one. Under these assumptions, let us discuss the properties of (3.3) if $A = 0$. For this case, the equation is

$$c\frac{d}{dt}T = Q(1-\alpha(T)) - E(T). \tag{3.4}$$

Its general form can be written as

$$c\frac{d}{dt}T = F(T). \tag{3.5}$$

Invoking the variational nature of (3.5), we consider the function $V(T)$ such that

$$V(T) = -\int F(T)\,dT, \qquad -\frac{d}{dT}V = F(T) \tag{3.6}$$

corresponding to (2.2). We shall hereafter call V the pseudopotential. As previously discussed, the minima of V are stable steady states of (3.5) and we shall call these solutions "observable climates". Let T_3 be the present climatic representative temperature (about 288.6K, see Ghil (1976) and Fraedrich (1978), (1979)). By our assumptions, the other stable steady state of (3.5) is $T_1 = 278.6$K and the unstable one is $T_2 = 283.3$.K. We note that these assumptions lead to the results considered in S. We remark also that a more complex model with realistic parameterizations and suitable choice of the albedo function has led Bhattacharya and Ghil (1982) to discover, within the 10K interval here considered, a structure of steady states similar to the one here proposed.

By definition the steady solutions of (3.5) satisfy the algebraic equation

$$F(T) = 0. \tag{3.7}$$

Using (3.4), (3.7) implies

$$Q(1-\alpha(T)) = E(T). \tag{3.8}$$

We introduce the function $\gamma(T)$, defined as

$$\begin{aligned} &\gamma(T) = [Q(1-\alpha(T))/E(T)] - 1, \\ &\gamma(T) = 0 \quad \text{if } T = T_i, \quad i = 1, 2, 3. \end{aligned} \tag{3.9}$$

From our assumptions, it also follows that within the range of temperature here considered, $\gamma(T)$ can be approximated by

$$\gamma(T) = \beta\left(1-\frac{T}{T_1}\right)\left(1-\frac{T}{T_2}\right)\left(1-\frac{T}{T_3}\right). \tag{3.10}$$

Here β is a dimensionless constant which has to be determined from further considerations (see below).

Combining (3.10) and (3.8), we obtain

$$Q(1-\alpha(T)) = E(T)(1+\gamma(T)) = E(T)\left[1+\beta\left(1-\frac{T}{T_1}\right)\left(1-\frac{T}{T_2}\right)\left(1-\frac{T}{T_3}\right)\right]. \tag{3.11}$$

From (3.11) and (3.4), we can write

$$\frac{d}{dt}T=\frac{E(T)}{c}(1+\gamma(T))(1+A\cos\omega t)-\frac{E(T)}{c}$$

$$=\frac{E(T)}{c}[\gamma(T)A\cos\omega t+A\cos\omega t+\gamma(T)] \tag{3.12}$$

$$=\frac{E(T)}{c}\left[\beta\left(1-\frac{T}{T_1}\right)\left(1-\frac{T}{T_2}\right)\left(1-\frac{T}{T_3}\right)(A\cos\omega t+1)+A\cos\omega t\right].$$

The numerical value of β can be derived from the following considerations. With $A=0$, (3.12) around $T=T_3$ is approximately

$$\frac{d}{dt}T=\frac{1}{\tau}(T_3-T) \tag{3.13}$$

where

$$\frac{1}{\tau}=-\frac{E(T_3)}{c}\left[\beta\frac{1}{T_3}\left(1-\frac{T_3}{T_1}\right)\left(1-\frac{T_3}{T_2}\right)\right]. \tag{3.14}$$

In (3.13) τ is the e-folding time to reach equilibrium (see also S and Fraedrich (1978), (1979)). The corresponding time constant has been estimated in other models with more complex physically justified parameterizations, and using these values we may compute by means of (3.14). In the following, we use $\tau=2.5\times10^8$ sec (8 years), following Ghil (1976).

4. The stochastic resonance mechanism. We will study the effect of the small periodic forcing $A\neq 0$ combined with an input of perturbations continuous in time simulated by a "white noise". Formally our equation is

$$dT=\frac{F(T,t)}{c}dt+\varepsilon^{1/2}dW \tag{4.1}$$

where ε is the variance of the noise and dW are the differential increments of a Wiener process.

We start by analyzing the effect of the small periodic forcing in the deterministic case $\varepsilon=0$, using a perturbation analysis. Without losing generality we assume

$$E(T)=\langle E\rangle\equiv\frac{1}{T_3-T_1}\int_{T_1}^{T_3}E(T)\,dT. \tag{4.2}$$

Since $\Delta T\equiv(T_3-T_1)/2\ll T_2$, we choose $\Delta T/T_2$ as a perturbative parameter in all the calculations. We wish to evaluate the variation of the steady-states as a function of the periodic forcing.

Considering

$$1-\frac{T_3}{T_1}=1-\frac{T_2+\Delta T}{T_2-\Delta T}\simeq-2\frac{\Delta T}{T_2},$$

$$1-\frac{T_3}{T_2}=1-\frac{T_2+\Delta T}{T_2}=-\frac{\Delta T}{T_2},$$

it follows that

$$\frac{1}{\tau}\simeq-2\frac{\langle E\rangle}{c}\beta\frac{1}{T_2}\left(\frac{\Delta T}{T_2}\right)^2. \tag{4.3}$$

From the condition $F(T)=0$, using (4.3) we obtain

$$-\frac{1}{T_3}\left(\frac{T_2}{\Delta T}\right)^2\frac{c}{\langle E\rangle}\frac{1}{2\tau}\left(1-\frac{T}{T_1}\right)\left(1-\frac{T}{T_2}\right)\left(1-\frac{T}{T_3}\right)(A\cos\omega t+1)+A\cos\omega t=0. \tag{4.4}$$

Let $T_3+\delta T_3\cos\omega t$ be the solution of (4.4). Neglecting small terms of order $(\delta T_3/T_2)^2$, $(\delta T_3\Delta T/T_2)^2$, $(A\delta T_3/T_2)^2$ we obtain from (4.4)

$$-T_3\frac{T_2^2}{(\Delta T)^2}\frac{c}{\langle E\rangle}\frac{1}{2\tau}\left(-2\frac{\Delta T}{T_2}\right)\left(-\frac{\Delta T}{T_2}\right)\left(-\frac{\delta T_3}{T_3}\right)+A\cos\omega t=0, \tag{4.5}$$

$$-\delta T_3=\frac{\langle E\rangle}{c}\tau A. \tag{4.6}$$

Substituting the numerical values of $\langle E\rangle=8.77\times10^{-3}$, and $c=4000$ it follows that $\delta T_3=0.3$K, too small a change compared to the 10K marked in the record, but in accordance with the results of recent investigations of the effect of orbital changes in Budyko–Sellers type models, (North and Coakley (1979)).

Let us now consider the effect of the noise in our model in the case $A=0$. The mean exit time $\langle\theta_{T_2T_3}\rangle$ from the domain of attraction of T_3 is given approximately by

$$\langle\theta_{T_2T_3}\rangle\cong\frac{\pi}{[|V''(T_2)V''(T_3)|]^{1/2}}\exp\frac{2\Delta V(T_2,T_3)}{\varepsilon} \tag{4.7}$$

where

$$\Delta V(T_2,T_3)=-\int_{T_2}^{T_3}F(T)\,dT \tag{4.8}$$

and $V''(T_i)$, $i=1,2$ is the second derivative at T_i of V with respect to T (we recall that the formula (4.7) has been derived for the double-well potential by Kramers; see Wax (1954, p. 65)). From our model it follows that

$$\langle\theta_{T_2T_3}\rangle=\pi 2^{1/2}\tau\exp\left[\frac{(\Delta T)^2}{4\tau\varepsilon}\right] \tag{4.9}$$

(see the Appendix for details of the calculation of ΔV). Substituting numerical values, we find that if $\varepsilon=0.38\,\mathrm{K}^2\,\mathrm{yr}^{-1}$, the system leaves the domain of attraction of T_3 in 10^5 years on the average. We note that this value of the noise level is close to the temperature fluctuations around the present climate average (about $0.1\,\mathrm{K}^2\,\mathrm{yr}^{-1}$) (Fraedrich (1978), (1979); Budyko (1969) and Frankignoul and Hasselmann (1977)). However, as shown by Williams (1982), the probability density for the first exit from the domain of attraction between the random time θ and $\theta+d\theta$ is given by

$$\frac{1}{\langle\theta_{T_2T_3}\rangle}\exp\left(-\frac{\theta}{\langle\theta_{T_2T_3}\rangle}\right).$$

Therefore, the correlation function $r(t)=\langle T(t)T(0)\rangle/\langle T^2\rangle-1$ behaves as:

$$r(t)\simeq\exp\left(\frac{-t}{\langle\theta_{T_2T_3}\rangle}\right). \tag{4.10}$$

Since the power spectrum is the Fourier transform of $r(t)$, it follows that for $A=0$ no peaks will appear.

We now consider the combined effect of periodic and stochastic forcing. Since the correlation function depends on the exit time and since the exit time depends exponentially on ΔV, we compute the effect of the term $A\cos\omega t$ in ΔV. Because of the symmetry we shall compute only

$$\Delta V(t, T_2, T_3) = -\int_{T_2}^{T_3} \frac{F(t,T)}{c}\,dT. \tag{4.11}$$

Using (3.12) we obtain, after some algebraic manipulations,

$$\begin{aligned}\Delta V(t, T_2, T_3) &\simeq \frac{(\Delta T)^2}{8\tau} + \int_{T_2}^{T_3} \frac{\langle E\rangle}{c}(1+\gamma(T))A\cos\omega t\,dT \\ &\simeq \left(\frac{\Delta T}{8\tau}\right)^2 + \frac{\langle E\rangle}{c}A\,\Delta T\cos\omega t \\ &\simeq \left(\frac{\Delta T}{8\tau}\right)^2\left[1+\left(1-\frac{\delta\langle E\rangle\tau A}{c\,\Delta T}\right)\cos\omega t\right].\end{aligned} \tag{4.12}$$

In (4.12) we have neglected terms of order $(\Delta T/T_2)^2$. The dimensionless constant $M = 8(\langle E\rangle/c)A\tau/\Delta T$ is a measure of the forcing effect on ΔV. Our numerical values give

$$M = 8\frac{\langle E\rangle A\tau}{c\,\Delta T} = .4385. \tag{4.13}$$

V is a periodic function of time with a period of $2\pi\omega^{-1}$, with a maximum value $\Delta V_{\text{down}} = 1.44(\Delta T)^2/8\tau$ and a minimum value $\Delta V_{\text{up}} = 0.56(\Delta T)^2/\delta\tau$. Let $\langle\theta_{T_2T_3}^{\text{up}}\rangle$ and $\langle\theta_{T_2T_3}^{\text{down}}\rangle$ be the average exit times corresponding, respectively, to ΔV_{up} and ΔV_{down}. From (4.3) we obtain

$$\frac{\langle\theta_{T_2T_3}^{\text{up}}\rangle}{2^{1/2}\pi\tau} = \exp\left\{\frac{2\Delta V_{\text{up}}}{\varepsilon}\right\} = \exp\frac{2\Delta V_{\text{down}}}{\varepsilon}\cdot\frac{\Delta V_{\text{up}}}{\Delta V_{\text{down}}} \cong \left[\frac{\langle\theta_{T_2T_3}^{\text{down}}\rangle}{2^{1/2}\pi\tau}\right]. \tag{4.14}$$

Moreover, from the probability density of θ we can also calculate the variance of $\langle\theta_{T_2T_3}^{\text{up}}\rangle$ and $\langle\theta_{T_2T_3}^{\text{down}}\rangle$. It follows that

$$\operatorname{Var}\frac{\langle\theta_{T_2,T_3}^{\text{up}}\rangle}{\tau} \simeq [\operatorname{Var}\langle\theta_{T_2,T_3}^{\text{down}}\rangle]^{1/3}. \tag{4.15}$$

Therefore, not only is $\langle\theta_{T_2T_3}^{\text{up}}\rangle$ much smaller than $\langle\theta_{T_2T_3}^{\text{down}}\rangle$, but also its variance is small. So if the noise level guarantees the condition $\langle\theta_{T_2T_3}^{\text{up}}\rangle \ll \pi/\omega$, then with a high probability the solution of (4.1) leaves the domain of attraction of T_3 with the pseudopotential difference in position "up".

We can conclude that if the previous condition on $\langle\theta_{T_2T_3}^{\text{up}}\rangle$ is satisfied, our solution will show almost periodic jumps between the two stable steady states (see also the numerical results presented in the next section). Therefore, the power spectrum will show a peak around $2\pi\omega^{-1}$ with an amplitude of about 10K.

Of course there will be limits to the range of the noise for which the previous result holds. Besides the lower bound given by the inequality $\langle\theta_{T_2T_3}^{\text{up}}\rangle \ll \pi/\omega$, the noise level must have an upper bound. This follows because, if the noise has too large a variance, the solution will jump at a time much before that at which the pseudopotential difference reaches ΔV_{up}. A good upper bound for the noise variance can be obtained if $\langle\theta_{T_2T_3}^{\text{down}}\rangle \simeq 2\pi/\omega$ is satisfied. Using both conditions, we obtain the range of the noise variance $[\varepsilon_1, \varepsilon_2]$ as follows:

$$2^{1/2}\pi\tau \exp[2\Delta V_{\text{up}}/\varepsilon_1] = \frac{\pi}{\omega},$$

(4.16)

$$2^{1/2}\pi\tau \exp[2\Delta V_{\text{down}}/\varepsilon_2] = \frac{2\pi}{\omega},$$

which gives $\varepsilon_1 = 0.06\ \text{K}^2\ \text{yr}^{-1}$ and $\varepsilon_2 = 0.14\ \text{K}^2\ \text{yr}^{-1}$.

5. Numerical results. First, we verify the bounds of the noise level, viz. $\varepsilon \in [.06, .15]$ developed in the previous section by numerical integration of (4.1) with $F(T)$, the deterministic part, given by the right-hand side of (3.12):

$$(5.1)\quad dT = \frac{\langle E(T)\rangle}{c}\left[\beta\left(1-\frac{T}{T_2}\right)\left(1-\frac{T}{T_3}\right)\left(1-\frac{T}{T_3}\right)(A\cos\omega t+1)+A\cos\omega t\right]dt + \varepsilon^{1/2}\,dW.$$

For the numerical procedure we choose a second order implicit Euler–Cauchy method for the deterministic part and a Gaussian random number generator for the stochastic component. The time step is one year and the other parameters are as defined in § 3. Therefore, the total variance of the perturbing process will be ε. We believe that a good convergence has been obtained since over 500 realizations, for an initial condition chosen at the top of the barrier we get a nearly 0.5 ratio of trajectories reaching each side of the barrier. Of course, more detailed analysis would be needed to show perfect convergence, but out test seems a reasonably good indicator. Moreover, increasing the order of the numerical scheme does not produce significant changes in the nature of the solution. The values of ε chosen in our numerical calculation are summarized in Table 1. In Case 1, we note that in the absence of the periodic forcing there is no periodic exit, although it is clear that with this amplitude the noise drives the system to an exit (see Fig. 2). In Case 2, the absence of resonance is evident from Fig. 3, in agreement with our analytical estimation. Cases 3 and 4 show the resonance effect, as is seen from Figs. 4a, 4b, 5a, 5b, which comprise plots of $T(t)$ versus t and of the associated Fourier spectrum. Again the noise level values are in the predicted range.

TABLE 1
Values of the forcing parameters used in the computation.

Case	ε	A	Stochastic resonance
1	.10	0	no
2	.05	5×10^{-4}	no
3	.08	5×10^{-4}	yes
4	.10	5×10^{-4}	yes
5	.15	5×10^{-4}	no

Finally, in Case 5, we lose resonance because, although the power spectrum of the time series in Fig. 6 still shows resonance, the plot of $T(t)$ versus time t shows a large number of exits at apparently random separations in time.

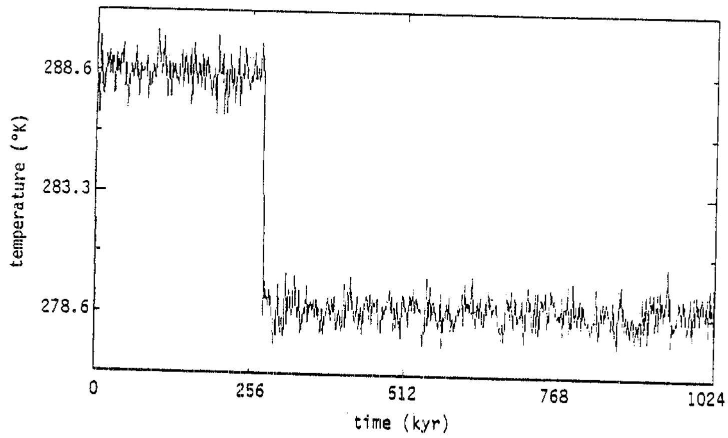

FIG. 2. *Numerical integration of* (5.1) *for Case* 1 *of Table* 1.

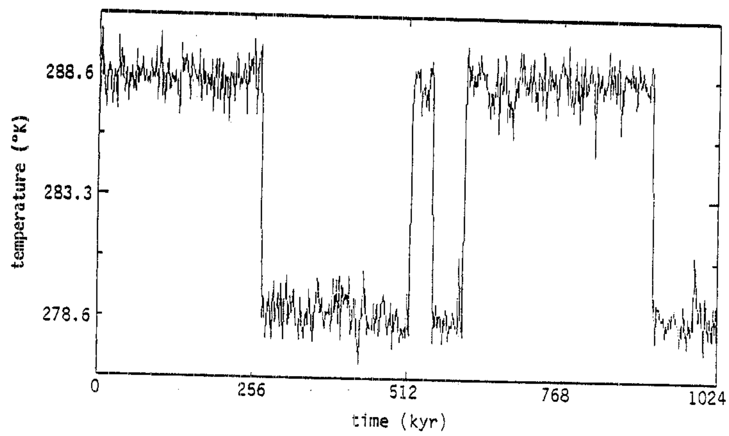

FIG. 3. *Numerical integration for Case* 2 *of Table* 1. *Note that the noise level is too small to produce periodic jumping.*

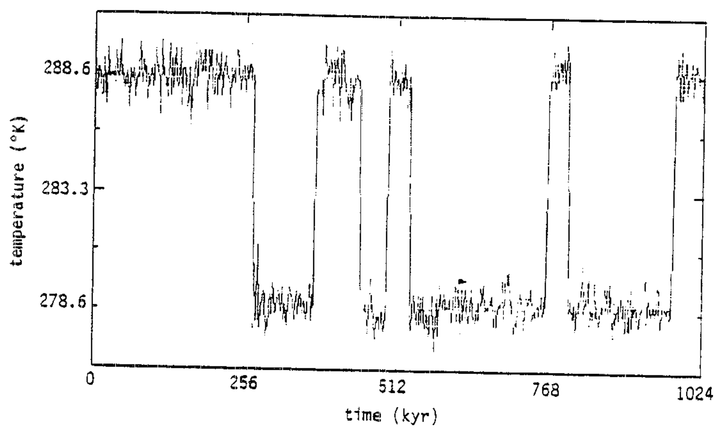

FIG. 4a. *Numerical integration of Case* 3 *of Table* 1. *Although there is a long residence time after the first jump, the stochastic resonance mechanism is clearly evident for a longer time.*

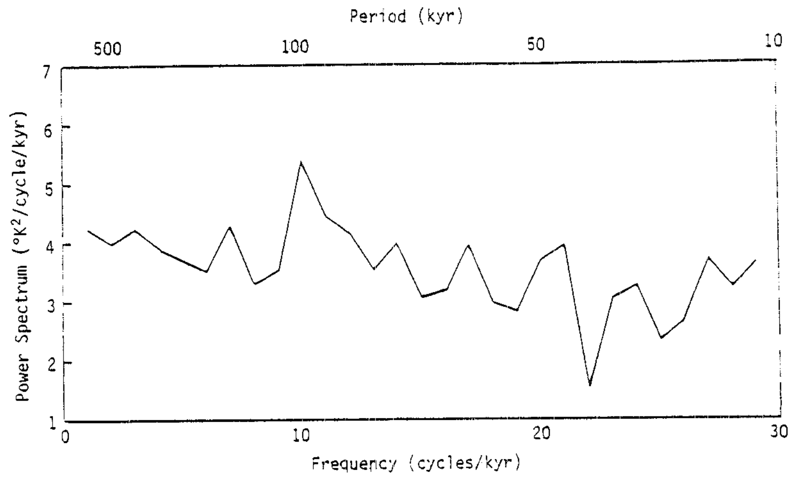

FIG. 4b. *Power spectra of the record shown in Fig. 4a.*

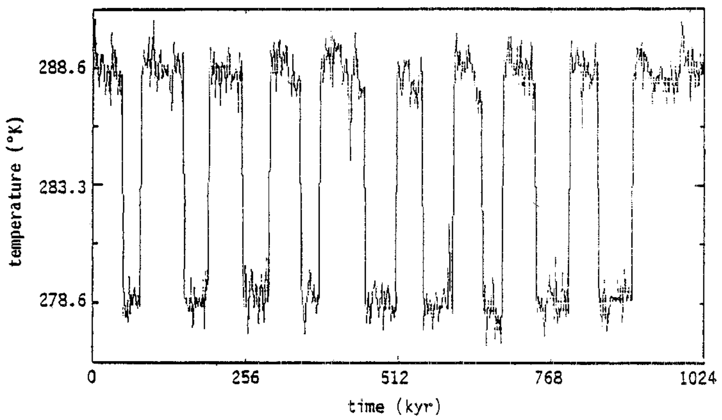

FIG. 5a. *Numerical integration of Case 4 of Table 1.*

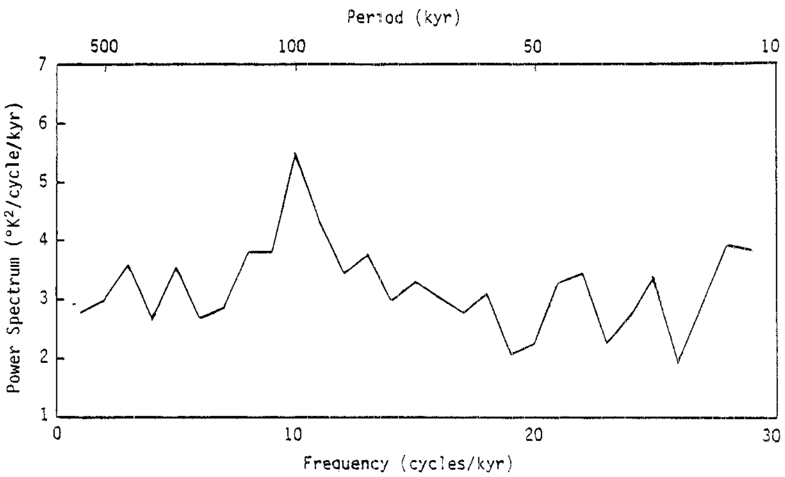

FIG. 5b. *Power spectra of the record shown in Fig. 5a.*

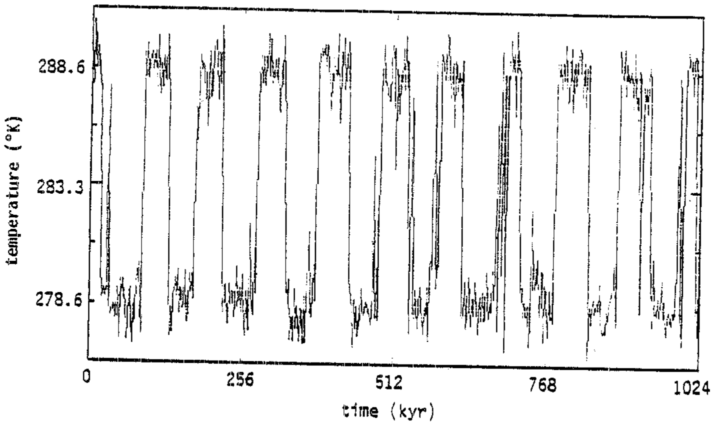

FIG. 6. *Numerical integration of Case 5 of Table 1. There is still evidence of some periodic behavior but we can note several flips between the two stable steady states that occur at random times.*

The second set of numerical verifications that we present are intended to heuristically confirm the existence of the stochastic resonance mechanism for a double well periodically perturbed. We renormalize (5.1) in such a way it will reduce to

$$d\xi = \xi(1-\xi^2) + A\cos\omega t + \tilde{\varepsilon}^{1/2}\, dW \tag{5.2}$$

where $A = .11$, $\omega = 2\pi/6{,}000$ time units. We have integrated previous equations for a large range of $\tilde{\varepsilon}$, namely $\tilde{\varepsilon} \in [.01, .3]$. In Fig. 7, we show the peak of the power spectrum, P, versus $\tilde{\varepsilon}$. We observe a sharp increase in P in a small interval of noise, while P decreases to a much slower rate where the noise increases. However, in Fig. 8, we show the behavior of the ratio between the variance of the exit time and its mean value, indicated as R. It appears that the range of the noise where $R \to 0$ is small. Figure 9 shows P and R plotted together. This allows us to conclude that we have heuristically demonstrated that (5.2) shows stochastic resonance as previously defined.

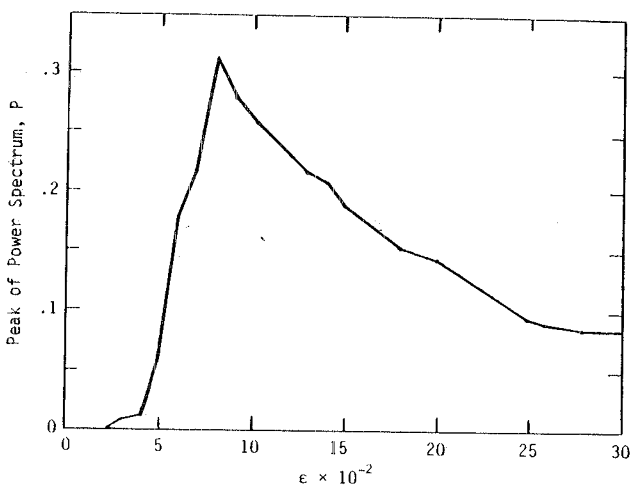

FIG. 7. *Peaks of power spectra of numerical solutions versus $\tilde{\varepsilon}$.*

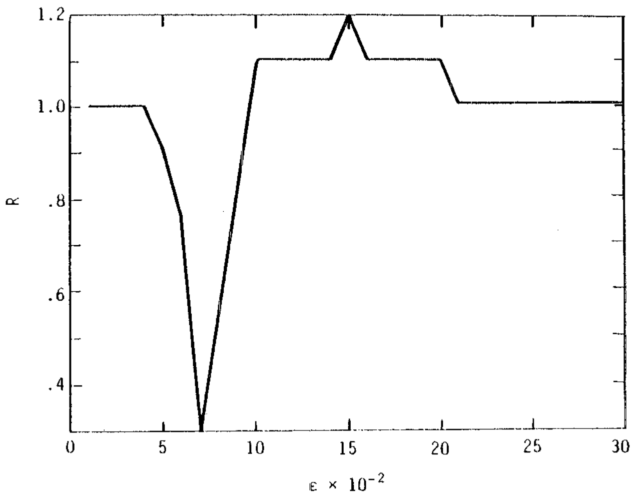

FIG. 8. *Ratio R versus noise level.*

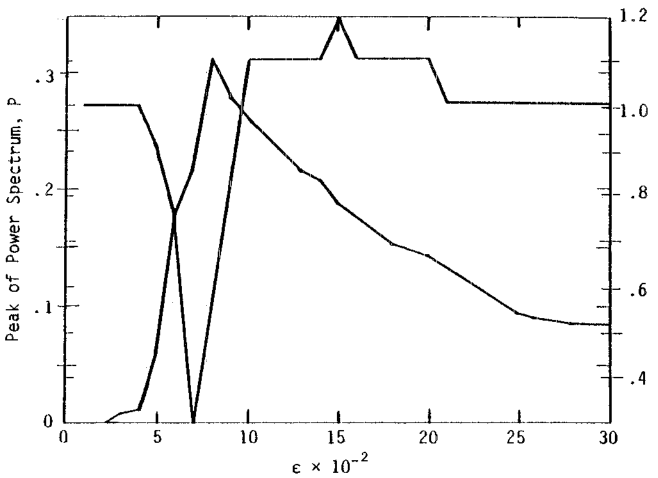

FIG. 9. *Figs.* 7 *and* 8 *together. Left scale refers to the power spectrum peaks, while right scale refers to R ratios.*

6. Conclusions. In this paper we have analyzed a possible mechanism for glaciation cycles using a simple zero-dimensional (spatially averaged) energy-budget model including short-term phenomena modelled by stochastic perturbation and long-term variation of the energy input prescribed by astronomical theory. The two effects produce the same periodic behavior as is observed in the climatic record. We note as our main conclusion the importance of the nonlinearities (implied in the assumption of two stable steady states) in producing the observed behavior. Therefore, we stress the importance of correct parameterization of the several nonlinear processes in more complex models of the climate system. A similar but linear model would exhibit peaks in the power spectrum at the prescribed frequencies but the temperature changes at those frequencies would be limited to a few tenths of a degree, contrary to the climatic observations. We also note that the mechanism has a wide applicability and that it appears as a property of a nonautonomous stochastic differential equation with two deterministically stable equilibria.

Appendix. In this appendix we compute the power series perturbation for the pseudopotential difference. By definition:

$$\Delta V=-\int_{T_2}^{T_3} F(T)\,dT=-\frac{\langle E\rangle}{c}\int_{T_2}^{T_3}\beta\left(1-\frac{T}{T_1}\right)\left(1-\frac{T}{T_2}\right)\left(1-\frac{T}{T_3}\right)dT. \tag{A1}$$

Defining the dimensionless parameter ξ as follows:

$$T=T_2+\xi\Delta T$$

We can write (A.1) in Taylor series of $\Delta T/T_2$:

$$\begin{aligned}\Delta V&=\beta\frac{\langle E\rangle}{c}\Delta T\left(\frac{\Delta T}{T_2}\right)^3\int_0^1 \xi(\xi^2-1)\,d\xi\\&=\frac{\langle E\rangle}{c}\frac{c}{\langle E\rangle}\frac{1}{2\tau}T_2\left(\frac{T_2}{\Delta T}\right)^2\Delta T\left(\frac{\Delta T}{T_2}\right)^3\frac{1}{4}\\&\simeq\frac{(\Delta T)^2}{8\tau}.\end{aligned}\tag{A2}$$

Inserting (A2) into (4.7), we finally get (4.9).

Acknowledgments. It is a pleasure to thank J. P. Eckman and L. Thomas for discussing some of their results with us prior to publication.

We also thank Dr. G. D. Robinson for a careful reading of the manuscript, Professor M. Ghil for suggestions and constant encouragement, and Drs. G. North, J. Pandolfo and Mr. D. Dee for discussions.

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