ISTITUTO NAZIONALE DI FISICA NUCLEARE
Laboratori Nazionali di Frascati

LNF-83/33

E. Etim : INDUCED GRAVITY IN QUANTUM THEORY IN A CURVED SPACE

Estratto da :
"The High-Energy Limit",
Ed. by A. Zichichi (Plenum, 1983)

Servizio Documentazione
dei Laboratori Nazionali di Frascati
Cas. Postale 13 - Frascati (Roma)

From: THE HIGH-ENERGY LIMIT
Edited by Antonino Zichichi
(Plenum Publishing Corporation, 1983)

INDUCED GRAVITY IN QUANTUM THEORY IN A CURVED SPACE

E. Etim*

Department of Physics

University of Siegen, Siegen, Germany

ABSTRACT: The reason for interest in the unorthodox view of first order ($\sim R(x)$) gravity as a matter-induced quantum effect is really to find an argument not to quantise it. According to this view quantum gravity should be constructed with an action which is, at least, quadratic in the scalar curvature $R(x)$. Such a theory will not contain a dimensional parameter, like Newton's constant, and would probably be renormalisable. This lecture is intended to acquaint the non-expert with the phenomenon of induction of the scalar curvature term in the matter Lagrangian in a curved space in both relativistic and non-relativistic quantum theories.

1. INTRODUCTION

A quantum theory of gravitation seems difficult to formulate without giving up one or more basic concepts characteristic of either quantum theory or general relativity. The two theories appear as if they were created to remain separate; they have structures which are dif-

*Permanent Address:
Laboratori Nazionali, INFN, Frascati, Italy

ficult to reconcile. The problems of uniting quantum theory and general relativity have not been attacked head-on; they have been evaded by reinterpreting or rejecting those concepts of general relativity which are most overtly in conflict with the premises of quantum theory. The metric was the first structure to be sacrificed in this way. The metric tensor $g_{\mu\nu}(x)$ is considered to be just a field, like all the others, which has to be quantised. Its role in determining the distance between two events cannot be maintained in quantum theory. According to quantum theory the distance between two arbitrarily close events is subject to uncontrollable uncertainty and is therefore not unambiguously defined.

The Lagrangian of the quantum spin 2 field $g_{\mu\nu}(x)$ is that of Einstein

$$L(x) = \frac{1}{16\pi G} G(x) \tag{1}$$

where R(x) is the scalar curvature, constructed with $g_{\mu\nu}(x)$, and G is Newton's constant. Since G ($[G] = (\text{length})^2$) is a dimensional parameter the theory described by (1) is not renormalisable. This is accepted as the problem which quantum theory has to solve, and there has been a considerable amount of work aimed at obtaining cancellations, order by order in perturbation theory, of the non-renormalisable divergences. Supergravity[(2)] is claimed to promise to serve in the solution of this problem. While a lot of progress has been made in this direction it is possible that the unification of space-time structure with the quantum theory of all energy-matter forms may not be achieved in this way. It may therefore be necessary to experiment with another kind of evasion. Sakharov[(3)] has suggested one:

reject eq(1) as the fundamental Lagrangian of quantum gravity and reinterpret it as the change induced in the curved space matter Lagrangian by quantum fluctuations. The solution of two key problems is required to make the proposal worth further attention :

(i) $M = \left(\frac{1}{16\pi G}\right)^{1/2}$ is effectively a mass parameter which the matter Lagrangian has to supply. How does one arrange to have such a large mass with which to compute the induced Newton's constant?

(ii) If (1) is not the fundamental Lagrangian of quantum gravity, what is ? The answer is in principle easy: The Lagrangian will have to be proportional to R^2. Such a Lagrangian will not contain a dimensional parameter and hopefully should be renormalisable.

The renormalisability of R^2 gravity was first studied by Stelle[(4)]. It will not be reported here. This lecture is intended to familiarise the non-expert with the induction of the scalar curvature term in the matter Lagrangian in curved space in both non-relativistic and relativistic quantum theories and to comment on the ambiguity of its coefficient. In the relativistic quantum theory the ambiguity of the scalar curvature term is a consequence of regularisation. The induced coefficient is given by a quadratically divergent integral which has to be regularised. Sect.2 is concerned with the non-relativistic limit. I will use there the method of path integrals and compare the kernel in the Feynman path integral for the wave function with the kernel of the corresponding semi-group operator associated with random paths. One comes to the required result also by randomising the metric itself. A derivation of the induced Newton's constant in the relativistic case, discussed by Prof. Adler in his opening lecture[5], is given in sect. 3.

2. ON THE CONNECTION BETWEEN THE SCHRÖDINGER EQUATION IN A CURVED SPACE AND FLUCTUATIONS OF THE METRIC

The fact that the Schrödinger Hamiltonian in a curved space acquires an additional term (with respect to its flat space classical form) proportional to the scalar curvature has been known for a long time[6]. It did come initially as a surprise but is today generating a controversy: the coefficient multiplying the scalar curvature seems to depend on who computes it[7]. The ambiguity derives from the ordering problem of non-commuting operators.

In relativistic quantum theory there is no more to the ambiguity than that it is the consequence of regularisation. It is unavoidable. It becomes of concern however when we try to relate the coefficient of R to Newton's constant. But then it is experiment which is deciding between the various regularised coefficients. Let us therefore limit our considerations to how the induced term arises in the first place. Let $g_{\mu\nu}(x)$ be the Riemann metric of a space-like hypersurface of dimension n and let

$$L(\dot{x}(t),\ x(t)) \quad = \quad \frac{m}{2} g_{\mu\nu}(x)\ \dot{x}^{\mu}\dot{x}^{\nu} - V(x) \qquad (1)$$

be the Lagrangian describing the motion of a non-relativistic particle of mass m under the influence of the potential $v(x)$. $\dot{x}^{\mu}(t) = \frac{dx^{\mu}}{dt}$ where t denotes time.

The classical equation of motion is

$$\frac{d^2x^{\mu}}{dt^2} + \Gamma^{\mu}_{\lambda\nu}\frac{dx^{\lambda}}{dt}\frac{dx^{\nu}}{dt} \quad = \quad -\frac{1}{m} g^{\mu\nu}\nabla_{\nu}V(x) \qquad (2)$$

where the matrix Γ_λ with elements

$$(\Gamma_\lambda)^\mu_\nu = \Gamma^\mu_{\lambda\nu}$$

$$= \frac{1}{2}g^{\mu\sigma}(\partial_\lambda g_{\nu\sigma} + \partial_\sigma g_{\lambda\nu} - \partial_\sigma g_{\lambda\nu}) \quad (3)$$

is the Christoffel symbol (≡ the connection) associated with the Riemann metric $g_{\mu\nu}(x)$ and

$$\nabla_\mu = \partial_\mu - \Gamma_\mu \quad (4)$$

is the covariant derivative.

According to Feynman[8] the wave function $\psi(t,x)$ describing the quantised motion of the particle satisfies the integral equation

$$\psi(t+\Delta t,\ x(t+\Delta t)) = \frac{1}{C}\int d^n x\sqrt{g(x)}\exp\left(\frac{i}{\hbar}\int_t^{t+\Delta t} d\tau L(\dot{x},x)\right) \cdot$$

$$\cdot\ \psi(t,x(t)) \quad (5)$$

C is a normalisation constant and $g(x)$ is the determinant of $g_{\mu\nu}(x)$. It is easy to verify in flat space that eq(5) is equivalent to the Schrödinger equation. It is assumed to continue to do so in a curved space. Our task is to exhibit the corresponding curved space Schrödinger equation. To this end recall that the basic assumption of the path integral representation[8] is that a variation $W^\mu(t)$ about a solution trajectory of eq.(2) in the time interval Δt is bounded and of order $(\Delta t)^{1/2}$. A deformed path is therefore in general continuous but not differentiable since $\lim_{\Delta t\to o} \frac{W^\mu(t)}{\Delta t} \sim (\Delta t)^{-1/2} \to \infty$. The deformed trajectory purports to represent a possible outcome of an experiment designed to follow the quantum motion of the particle. Since the quantum trajectories are non-differentiable one cannot apply the standard methods of the calculus of variations which are used frequently in deriving equations of motion. In this consists the main difficulty of operating with functional integrals. The appropriate calculus rests on the mathematical theory of Brownian motion and more generally of all random (≡ stochastic) motions[9]. Since the potential V(x) is assumed to be a sure function (that is, not subject to random variations) it is sufficient to consider the free geodesic equation

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\lambda\nu}\frac{dx^\lambda}{dt}\frac{dx^\nu}{dt} = 0 \tag{6}$$

in defining random motions, with the property $|W^\mu(t)| \sim (\Delta t)^{1/2}$, also in the presence of a force field. Let us therefore expand $X^\mu(t+\Delta t)$ in powers of Δt and make use of (6)

$$\begin{aligned} W^\mu(t) &= x^\mu(t+\Delta t) - x^\mu(t) = \dot{x}^\mu(t)\Delta t + \\ &\quad + \frac{1}{2!}\ddot{x}^\mu(\Delta t)^2 + \frac{1}{3!}\dddot{x}^\mu(\Delta t)^3 + \ldots\ldots \\ &= \dot{x}^\mu(\Delta t) - \frac{1}{2}\Gamma^\mu_{\lambda\nu}\dot{x}^\mu\dot{x}^\nu(\Delta t)^2 \\ &\quad - \frac{1}{6}(\partial_\sigma\Gamma^\mu_{\lambda\nu} - 2\Gamma^\mu_{\rho\nu}\Gamma^\rho_{\lambda\sigma})\dot{x}^\lambda\dot{x}^\nu\dot{x}^\sigma(\Delta t)^3 \\ &\quad + \ldots\ldots\ldots\ldots \end{aligned} \tag{7}$$

Making use of the assumption that $|W^\mu| \sim (\Delta t)^{1/2}$ in (7) we define the stochastic process (which is independent of Δt)

$$\xi^\mu(t) = \dot{x}^\mu(\Delta t)^{1/2} \tag{8}$$

with the conditional expectations ($\equiv E(\cdot \mid x(t))$)

$$E(\xi^\mu(t)|x(t)) = 0 \tag{9.a}$$

$$E(\xi^\mu(t)\xi^\nu(t)|x(t)) = \frac{\hbar}{m}g^{\mu\nu}(x) \tag{9.b}$$

The constant $\hbar/m$ in eq(9.b) is introduced on dimensional grounds. $\xi^\mu(t)$ defines a Gaussian process whose distribution is taken to be proportional to the kernel in eq(5) considered in Euclidean space-time, that is (Feynman's prescription)[8]

$$P(\xi(t),\Delta t|x(t)) = N\exp\left(-\frac{1}{\hbar}\int_t^{t+\Delta t} d\tau H(x(\tau),x(\tau))\right)$$

$$\simeq N\exp\left(\frac{V(t)\Delta t}{\hbar}\right)\exp\left(-\frac{m}{2\hbar}g_{\mu\nu}(x)\xi^\mu\xi^\nu\,\Delta t\right) \tag{10}$$

N is the constant of proportionality. If the Wick rotated version of eq(10) were straight forwardly used in eq(5) and expanded out for small $\Delta t \to o$ one would get the usual Schrödinger equation as in flat space without a scalar curvature term. The reason why this procedure is incorrect is that $dx^\mu = \xi^\mu(\Delta t)^{1/2}$ is <u>not</u> the stochastic process associated with random variations of paths about classical trajectories in a curved space. According to eq(7) this latter is given by

$$dx^{\mu}(t) = W^{\mu}(t) = \xi^{\mu}(\Delta t)^{1/2} - \frac{1}{2}\Gamma^{\mu}_{\lambda\nu}\xi^{\lambda}\xi^{\nu}\Delta t - \frac{1}{6}(\partial_{\sigma}\Gamma^{\mu}_{\lambda\nu} - 2\Gamma^{\mu}_{\rho\nu}\Gamma^{\rho}_{\lambda\sigma})\,\xi^{\lambda}\xi^{\nu}\xi^{\sigma}(\Delta t)^{3/2} + \ldots\ldots\ldots \tag{11}$$

$W^{\mu}(t) = \xi^{\mu}(\Delta t)^{1/2}$ only in a flat space where the connection coefficients $\Gamma^{\mu}_{\lambda\nu}$ are identically zero. The problem which thus seems to arise is that Feynman's prescription (eq(10)) gives the distribution for $\xi^{\mu}(\Delta t)^{1/2}$ but not for $W^{\mu}(t)$. This latter is not difficult to find from eqs(10) and (11). Let us first consider the conditional expectations of powers of $W^{\mu}(t)$ from eqs(10) and (11).

$$E(W^{\mu}(t)|x(t) = -\frac{\hbar}{2m} g^{\lambda\nu}\Gamma^{\mu}_{\lambda\nu}\Delta t + O(\Delta t)^2) \tag{12.a}$$

$$E(W^{\mu}(t)\ W^{\nu}(t)|x(t) = \frac{\hbar}{m}g^{\mu\nu}\Delta t + O((\Delta t)^2) \tag{12.b}$$

$W^{\mu}(t)$ is therefore also Gaussian distributed but with a non-zero mean. To find the corresponding distribution we argue that it is sufficient to solve eq(11) for $\xi^{\mu}(t)$ in terms of $W^{\mu}(t)$, that is

$$\xi^{\mu}(\Delta t)^{1/2} = W^{\mu} + \frac{1}{2}\Gamma^{\mu}_{\lambda\nu} W^{\lambda} W^{\nu} + \\ + \frac{1}{6}(\partial_{\sigma}\Gamma^{\mu}_{\lambda\nu} + \Gamma^{\mu}_{\rho\nu}\Gamma^{\rho}_{\lambda\sigma}) W^{\lambda} W^{\nu} W^{\sigma} + \\ + \ldots\ldots\ldots\ldots \qquad (13)$$

and then substitute into eq(10) to have

$$P(\xi[W], \Delta t|x(t)) \equiv \hat{P}(W,\Delta t|x(t)) \\ = P(W,\Delta t|x(t))\Big[1 - \frac{m}{2\hbar\Delta t} g_{\mu\nu}\Gamma^{\mu}_{\lambda\sigma} W^{\lambda}W^{\nu}W^{\sigma} \\ - \frac{m}{8\hbar\Delta t} g_{\mu\nu}\Gamma^{\mu}_{\lambda\tau}\Gamma^{\nu}_{\rho\sigma} W^{\lambda} W^{\tau} W^{\rho} W^{\sigma} \\ + \frac{m}{6\hbar\Delta t} g_{\mu\nu}(\partial_{\sigma}\Gamma^{\mu}_{\lambda\tau} - \Gamma^{\mu}_{\rho\sigma}\Gamma^{\rho}_{\lambda\tau}) W^{\lambda}W^{\tau}W^{\nu}W^{\sigma} \\ + \frac{m^2}{8\hbar^2(\Delta t)^2} g_{\mu\nu}\, g_{\lambda\tau}\Gamma^{\mu}_{\rho\sigma}\Gamma^{\lambda}_{\alpha\beta} W^{\nu}W^{\tau}W^{\rho}W^{\sigma}W^{\alpha}W^{\beta} \\ + \ldots\ldots\ldots\ldots\ldots\ldots\Big] \qquad (14)$$

The Wick rotated distribution $\hat{P}(W,i\Delta t|x(t))$ is now to be used as the kernel in the Feynman path integral in eq(5). The additional terms in eq(14) induce a change in the normalisation of $\hat{P}(W,\Delta t|x(t))$ with respect to that of $P(W,\Delta t|x(t))$. This change, in turn, accounts for the different form of the Schrödinger equation obtained by using the Wick rotated form of the one or the other in eq(5). The change is found immediately by evaluating the integral

$$\hat{I}(\Delta t) = \int d^n W \; \hat{P}(W,\Delta t|x(t)) \cdot \Big[\sqrt{g} + \partial\mu \sqrt{g}\, W^\mu + \frac{1}{2} \partial_\nu \partial_\nu \sqrt{g}\, W^\mu W^\nu + \ldots\ldots\ldots\ldots \Big] \tag{15}$$

and making use of the identities

$$\int d^n W \exp\left(-\frac{m}{2\hbar\Delta t} g_{\mu\nu} W^\mu W^\nu\right) = \frac{(\pi\hbar\Delta t/m)^{n/2}}{\sqrt{g}} \tag{16.a}$$

$$\int d^n W \exp\left(-\frac{m}{2\hbar\Delta t} g_{\mu\nu} W^\mu W^\nu\right) \cdot W^{\lambda 1}, W^{\lambda 2}, \ldots W^{\lambda 2k} = \left[\sum_{(i)} g^{\lambda i_1, \lambda i_2}, g^{\lambda i_3, \lambda i_4}, \ldots g^{\lambda i_{2k-1}, \lambda i_{2k}}\right] \cdot$$

$$\cdot \left(\frac{\hbar\Delta t}{m}\right)^k \left(\frac{\pi\hbar\Delta t}{m}\right)^{n/2} / \sqrt{g} \tag{16.b}$$

The sum over the indices (i) extends over all the distinct pairs that can be formed from 2k objects. The sum contains $(2k)!/_2 k.k!$ terms. We find for $\hat{I}(\Delta t)$

$$\hat{I}(\Delta t) \simeq \left(\frac{\pi\hbar\Delta t}{m}\right)^{n/2} N \exp\left[-\frac{1}{\hbar}\int_t^{t+\Delta t} d\tau\,(V + \frac{\hbar^2}{6m} R)\right] \tag{17}$$

where $R(x) = g^{\mu\nu}(x)\, R_{\mu\nu}(x)$ with

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}$$

$$R^{\rho}_{\mu\sigma\nu} = \partial_\nu\, \Gamma^{\rho}_{\mu\sigma} - \partial_\sigma\, \Gamma^{\rho}_{\mu\nu} + \Gamma^{\rho}_{\tau\sigma}\, \Gamma^{\tau}_{\mu\nu} - \Gamma^{\rho}_{\tau\nu}\, \Gamma^{\tau}_{\mu\sigma} \tag{18}$$

Comparing eq(17) with the normalisation of (10) i.e.

$$I(\Delta t) = \int d^n W \sqrt{g}\, P(W,\Delta t|\, x(t))$$

$$\simeq \frac{\pi\hbar\Delta t}{m}^{n/2} N \exp\left(-\frac{1}{\hbar}\int_t^{t+\Delta t} d\tau\, V(x(\tau))\right) \tag{19}$$

we see that the effective potential in the curved space is

$$V'(x) = V(x) + \frac{\hbar^2}{6m} R(x) \tag{20}$$

We emphasise that the additional term in eq(20) is induced by a certain averaging procedure over the matter distribution. To bring this out clearly we did not try to evaluate the path integral directly through discretisation prescriptions(7). Quantum fluctuations in the relativistic case induce the Einstein Lagrangian through a similar averaging procedure. That the coefficient of R(x) is arbitrary is clear from our derivation, since we could very well have used the correlation function

$$E(\xi^{\mu}(t)\ \xi^{\nu}(t)\,|\,x(t)) = \frac{c\hbar}{m}\, g^{\mu\nu}(x) \tag{21}$$

with C an arbitrary constant, instead of eq(9.b). Eq (20) would then have been replaced by

$$V'(x) = V(x) + \frac{c\hbar^2}{6m} R(x) \tag{22}$$

Random variations about a given trajectory form a semi-group (existence of an identity element ≡ no variation; the closure of the composition law follows from the fact that any two variations always combine to give a third) but not a group. In the space of random functions of the coordinates these operations are represented by the kernel in eq(10). In terms of the kernel the composition law is expressed by the fact that the kernels are self-reproducing i.e. if we write $P(\xi,\Delta t|x(t)$ as the transition probability $W(t + \Delta t,\ x(t + \Delta t)|\ t,\ x(t))$ then one has

$$W(t_2,x(t_2)\,|\,t_1,x(t_1)) = \int d^n x(t)\ \sqrt{g(x(t))}\ \cdot$$
$$\cdot\ W(t_2,x(t_2)\,|\,t,x(t))\ W(t,x(t)\,|\,t_1,x(t_1)) \tag{23}$$

The constant N can be chosen such that eq(23) is satisfied. The kernel of the semi-group operator of Brownian motions is proportional to the Wick rotated kernel of the Feynman path integral. This is where quantum and classical random field theories overlap.

The induced potential in eq(20) or (22) may be related to second order random fluctuations of the metric tensor $g_{\mu\nu}(x)$. To this end let us rewrite the curved space Schrödiner equation

$$i\hbar \frac{\partial\psi(t,x)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^{\mu} \nabla_{\mu} \psi(t,x) + (V + \kappa\frac{\hbar^2}{2m} R)\psi(t,x) \tag{24}$$

with $\kappa = \frac{C}{3}$, in the form

$$i\hbar \frac{\partial\psi}{\partial t} = [\frac{1}{2m} D_{\mu} D_{\nu} + \kappa\hbar^2 R_{\mu\nu})g^{\mu\nu} + V]\psi(t,x) \tag{25}$$

D_{μ} is the extended covariant derivative

$$D_{\mu} = -i(\hbar\nabla_{\mu} + mu_{\mu}) \tag{26}$$

with

$$u_{\mu}(t,x) = \frac{\hbar}{m} \nabla_{\mu}|\psi (t,x)| \tag{27}$$

Quantum fluctuations of the metric in the presence of matter ($\psi(t,x) \neq 0$ and $\xi^{\mu} \neq 0$) may be defined by

$$dg^{\mu\nu}(t,x) = \frac{m}{\hbar(1+n)}(u^{\mu} \xi^{\nu} + u^{\nu} \xi^{\mu})(\Delta t)^{1/2} + \frac{\kappa}{2n} R(x) \xi^{\mu} \xi^{\nu} \Delta t + O((\Delta t)^{3/2}) \tag{28}$$

The coefficient of the second order term in $\xi^{\mu}(t)$ is determined by dimensional arguments.

From eq(28) the operator D_{μ} can be defined as the limit

$$\frac{i}{m} D_{\mu} g^{\mu\nu}(x) = \lim_{\Delta t \downarrow o} \frac{1}{\Delta t} E(\xi_{\mu}(\Delta t)^{1/2} dg^{\mu\nu} | x(t))$$

$$= u^{\nu}(t,x) \tag{29}$$

Similarly from the second term in eq(28) we have in the limit $\Delta t \rightarrow 0_{+}$

$$Dg^{\mu\nu}(x) = \lim_{\Delta t \downarrow o} \frac{1}{\Delta t} E(dg^{\mu\nu} | x(t)$$

$$= \frac{\kappa\hbar}{2mn} R(x) \; g^{\mu\nu}(x) \tag{30}$$

whence

$$\frac{\kappa\hbar^2}{2m} R(x) = - \lim_{\Delta t \downarrow o} \frac{1}{\Delta t} E(\hbar d \ell n \; g(x) | \; x(t)) \tag{31}$$

This formula is general and exists in almost identical form for relativistic quantum fields. The characteristics of the Schrödinger equation in a curved space and those of the fluctuations of the metric are therefore intimately related. Note that there is a priori no reason for the coefficients of R in eqs(25) and (28) to be related. However, since both are arbitrary and are connected with the same effect caused by fluctuations of the matter field ($\sim \xi^{\mu}(\Delta t)^{1/2}$) the principle of energy balance leads one to identify them.

3. INDUCED GRAVITY IN RELATIVISTIC QUANTUM THEORY

The original argument of Sakharov[3] for a matter-induced gravitational action is approximate. It may be formulated as follows: consider for simplicity the Lagrangian density

$$L[R;\varphi] = \frac{1}{2}\sqrt{-g}\left[g^{\mu\nu}(x)\ \partial_\mu\ \varphi(x)\ \partial_\nu\varphi(x) - m^2\ \varphi^2(x)\right] \quad (32)$$

for a scalar field $\varphi(x)$ in n dimensional space-time with metric $g_{\mu\nu}(x)$. g is, as before, the determinant of $g_{\mu\nu}(x)$. L depends functionally on the curvature $R(x)$ of the spacetime manifold. Let us expand it in powers of $R(x)$

$$L[R;\varphi] = L[R=o;\varphi] + \left(\frac{\delta L[R;\varphi]}{\delta R(x)}\right)_{R=o} \cdot R(x) + \frac{1}{2}\left(\frac{\delta^2 L[R;\varphi]}{\delta R^2(x)}\right)_{R=o} R^2(x) + \dots . \quad (33)$$

and try to identify the expansion coefficients. Clearly

$$L[R=0;\varphi] = \frac{1}{2}[\eta^{\mu\nu}\ \partial_\mu\ \varphi(x)\ \partial_\nu\ \varphi(x) - m^2\varphi^2(x)] \quad (34)$$

where $\eta^{\mu\nu}$ is the Lorentz metric. To find the coefficient of $R(x)$ we make use of the functional generalisation of eq(28), that is we substitute

$$g^{\mu\nu}[\varphi;x] = \eta^{\mu\nu} + \delta g^{\mu\nu}[\varphi;x] \tag{35}$$

for $g^{\mu\nu}(x)$ in eq(32), where

$$\delta g^{\mu\nu}[\varphi;x] = (U^{\mu} W^{\nu} + U^{\nu} W^{\mu}) + R(x) W^{\mu} W^{\nu} + O(W^3) \tag{36}$$

$U^{\mu} = U^{\mu}[\varphi]$ and $W^{\mu} = W^{\mu}[\varphi]$ are functionals of $\varphi(x)$, with W^{μ} the fluctuation in $\varphi(x)$. The coefficient of $R(x)$ in eq(33) is then found to be

$$\left(\frac{\delta L[R;\varphi]}{\delta R(x)}\right)_{r=o} = \frac{1}{2} W^{\mu}[\varphi]\, W^{\nu}[\varphi]\, \partial_{\mu} \varphi(x)\, \partial_{\nu} \varphi(x) \tag{37}$$

Now take the vacuum expection value of (37) and average over the distribution of the $W^{\mu}[\varphi]$ to get

$$<o|E\left(\left(\frac{\delta L[R;\varphi]}{\delta R(x)}\right)_{R=o} |\varphi(x)\right)|o> =$$

$$= \frac{1}{2} \int d^n k E(W^{\mu}[\varphi]W^{\nu}[\varphi]\ \varphi(x))\ \theta(k_o)\delta(k^2-m^2)k_{\mu}\, k_{\nu} \tag{38}$$

We use the following normalisation of states

$$\langle o|[a(k),\ a^{+}(k')]|o\rangle \ = \ 2E\delta^{(n-1)}(\vec{k}-\vec{k}') \tag{39}$$

Note that $W^{\mu}[\varphi]$ has the dimension of length. The assumption made by Sakharov is that

$$E(W^{\mu}[\varphi]W^{\nu}[\varphi]|\varphi(x))\ k_{\mu}\ k_{\nu} \sim \text{const.} \tag{40}$$

so that the coeficient of R(x) in eq(33) is (n-2)-fold divergent. Making use of (38) in (33) we therefore have

$$E(L[R;\varphi]|\varphi(x)) \ = \ E(L[R=o;\varphi]|\ \varphi(x)) + \\ + \frac{1}{16\pi\ G_{ind}} R(x) + O(R^2) \tag{41}$$

where the induced Newton's constant is defined by a regularisation of (38), i.e.

$$\frac{1}{16\pi\ G_{ind}} R(x) \sim \text{Reg} \int \frac{d^{n-1}\vec{k}}{\sqrt{\vec{k}^2+m^2}} \tag{42}$$

As a formula for computing G_{ind} eq(38) is too general to be useful since we have neither specified the functionals $W^{\mu}[\varphi]$ nor their distribution. This may be remedied in the following way. Use the formula

$$\frac{\delta g(x)}{\delta g^{\mu\nu}(x)} = - g(x) g_{\mu\nu}(x) \tag{43}$$

to recognise that the energy-momentum tensor

$$T_{\mu\nu}(x) = \partial_\mu \varphi \frac{\delta L[R;\varphi]}{\delta(\partial_\nu \varphi)} - g_{\mu\nu} L[R;\varphi] \tag{44}$$

for the field $\varphi(x)$ can be put in the form

$$T_{\mu\nu}(x) = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}(x)} (\sqrt{-g} L[R;\varphi]) \tag{45}$$

Applying the operator $\frac{g^{\mu\nu}}{\sqrt{-g}} \frac{\delta}{\delta g \mu\nu} \sqrt{-g}$ to both sides of eq(41) and making use of (45) we therefore have

$$E(T^\mu_\mu(x) | \varphi(x))_{curved} = E(T^\mu_\mu(x) | \varphi(x))_{flat} - \frac{nR(x)}{32\pi G_{ind}}$$

or equivalently

(46)

$$\frac{n}{32\pi G_{ind}} R(x) = -[E(\delta T^\mu_\mu(x) | \varphi(x))]_{g_{\mu\nu} = \eta_{\mu\nu}} \tag{47}$$

Eq(47) should be compared with (31). We now specify the conditional expectation $E(\cdot|\varphi)$ using Feynman's prescription (c.f.eq(10); the distribution of φ is given by the kernel of the functional integral, that is

$$P[R;\varphi] = N \exp\left(\frac{i}{\hbar} d^n x\sqrt{-g}L[R;\varphi]\right) \tag{48}$$

The right hand side of (47) is now computed by averaging $T^{\mu}_{\mu}(x)$ over

$$\delta P[R;\varphi] = N \exp\left(\frac{i}{\hbar}\int d^n x L[R=o;\varphi]\right) \cdot$$

$$\cdot \left[\left(\frac{i}{\hbar}\right)\int d^n x \delta(\sqrt{-g}L[R;\varphi])\right) \tag{49}$$

(Consequently

$$\frac{n}{32\pi\ G_{ind}} R(x) = -\frac{i}{\hbar}\int d^n y[\sqrt{-g}(y)\delta g^{\mu\nu}(y) \cdot$$

$$\cdot E(T_{\mu\nu}(y)T^{\lambda}_{\lambda}(x)\,|\,\varphi(x))]_{g_{\mu\nu}=\eta_{\mu\nu}} \tag{50}$$

Eq(50) is true for all metrics. We shall specialise it to the case of a conformally flat, constant curvature space where

$$g^{\mu\nu}(y) = \eta^{\mu\nu} - \eta^{\mu\nu}\frac{Ry^2}{n!} + \ldots\ldots\ldots \tag{51}$$

Substituting for $\delta g^{\mu\nu}(y)$ from (51) into (50) gives finally

$$\frac{n}{32\pi\ G_{ind}} = \frac{i}{n!\hbar}\int d^n y\ y^2 E(T^{\mu}_{\mu}(y)T^{\nu}_{\nu}(o)\,|\,\varphi)$$

$$\sim \frac{i}{n!\hbar}\int d^n y\ y^2 E[T(T^{\mu}_{\mu}(y)T^{\nu}_{\nu}(o))\,|\,\varphi] \tag{52}$$

T stands for time ordering. Eq(50) admits other representations but we shall not consider them here. We have shown in this lecture that the induction of the Einstein action for gravity by quantum fluctuations of matter fields in a curved space closely parallels the analogous non-relativistic quantum effect (compare eqs(31) and (47)). Other powers of the scalar curvature R(x) are of course also induced; they involve higher order correlations of the fluctuations of the matter field and may hopefully be neglected. The main reason for interest in induced first order (~ R) gravity is to find an argument for not quantizing it. According to this argument quantum gravity should be constructed with an action which is at least quadratic in R(x). For details on the prospects of a theory of this kind the reader is referred to the lecture of Adler in these proceedings[5].

REFERENCES

1. For a review see the article of C.J.Isham in Quantum Gravity, Proceedings of the Oxford Symposium (1975). Eds.C.J.Isham,R.Penrose and D.W.Sciama.
2. See the lectures in:Recents Developments in Gravitational Theory (Plenum Press 1979).Eds.S.Deser and B.Zumino.
3. A.D.Sakharov,Sov.Phys.Doklady 12(1968)1040.
4. K.Stelle; Phys.Rev.D16(1977)953.
5. S.Adler: These proceedings.
6. B.S.De Witt; Rev.Mod.Phys.29(1957)29.
7. See the lectures in:Functional Integrals and Their Applications.(Plenum Press,1980)Ed.J.P.Antoine
8. R.D.Feynman;Rev.Mod.Phys.20(1948)367.
9. J.Yeh;Stochastic Processes and The Weiner Integral; Marcel Dekker,Inc.New York (1973).

D I S C U S S I O N

CHAIRMAN: Prof. E. Etim

Scientific Secretary: E. Guadagnini

DISCUSSION

- *DRAGON:*

I am confused concerning the dimension of the Riemannian space. Is R still the curvature of a surface at constant time or has the dimension of the manifold increased, by some mechanism, to include time?

- *ETIM:*

R is still the Riemannian curvature of space only.

- *DRAGON:*

The affine connection entering R is independent of the metric at this stage or is it explained by the metric?

- *ETIM:*

It is given by the metric.

In order to get the result shown by Adler, at this point, I have to assume that your affine connection is defined by the metric, otherwise you don't get this result.

- *GUADAGNINI:*

Could you explain the meaning of the correlation equation

$$<dw_\mu dw_\nu> \propto h \ g_{\mu\nu}(x) \ dt \ ?$$

- *ETIM:*

This is a rule of path-integral and stochastic quantization that will lead to the Schroedinger equation.

If we use canonical quantization the kinetic energy term is given by the Laplace-Beltrami operator:

$$\Delta = \nabla_\mu \nabla^\mu = \frac{1}{\sqrt{g}} \partial_\mu (\partial_\nu g^{\mu\nu})$$

There is no induced R term. This procedure is wrong for scalar fields.

- *SORACE:*

In order to obtain the Schroedinger equation by canonical quantization have we to take into account the ordering problem?

- *ETIM:*

Yes, in quantum theory; but we have a classical theory in the path-integral formulation. This ordering problem will enter only in the determination of the coefficient R. The way to solve this is to fix yourself in a geodesic coordinate system. Then in that coordinate system the Christoffel symbols vanish at that point; the second derivatives will just give you R and that coefficient is determined.

- *ADLER:*

I don't quite agree. The canonical quantization procedure is well defined for spin ½ and spin 1 fields. It is only for the scalar field where one finds that there is an ambiguity as to the correct wave equation in curved space-time, and the naive kinetic energy term that we write down, which is $\nabla\mu \nabla^\mu$ has the property that it should be conformal invariant and in fact it is, and it happens that there is a ⅙ R term. There are two possible scalar wave equations and you can quantize either of them. For ½ and spin 1 there is essentially no ambiguity; for this reason it was necessary to exclude scalar fields in my lectures to get a determined induced R term.