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QUANTIZATION OF GALILEAN GAUGE THEORIES

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ABSTRACT

Galilean gauge theories are quantized according to Dirac's theory of canonical quantization of constrained systems. Only the constant term in the Fourier expansion of the gauge fields is compatible with the constraints, and it is different from zero for periodic boundary conditions (b.c.), while it is zero if the fields are required to vanish on the surface of the quantization box. Such term has physical effects which therefore depend on b.c..

If the constant term of the electric potential does not vanish only states of vanishing charge can exist. This constraint holds both in the abelian and nonabelian case and it is true also in the relativistic theory.

The constant term of the magnetic potential gives rise in the abelian case to radiative corrections which are the $c \to \infty$ limit of the relativistic ones, but does not alter the matter fields interaction which remains purely Coulombic. The matter fields interaction is instead affected by the constant magnetic potential in the nonabelian case.

1. - INTRODUCTION

Looking for the features of nonabelian gauge theories which could possibly provide the confinement of color, it would be desirable to introduce the most drastic approximations which could respect such features. We have considered(1) the approximation obtained by letting the velocity of light $c \to \infty$. This limit is not unique in general, but it is strongly constrained by requiring that Poincarè invariance should contract into Galilei invariance while other symmetries (gauge invariance, chiral invariance, charge symmetry) should be conserved. If the limit is done in this way, the low energy behaviour of relativistic theories (for which such behaviour is konwn) is reproduced in the limiting theories. This has been shown for the Goldstone and Higgs models, the Schwinger mechanism for spontaneous mass generation, the Wess-Zumino and the Fayet models of supersymmetry[2] and for QED. Concerning the latter, which is most relevant to QCD, we have shown[1] that in Galilean QED there are radiative corrections which factorize as in the relativistic case.

The above results give some confidence that the $c \to \infty$ limit should conserve the infrared features of nonabelian gauge theories as well. The study of such features requires these theories in the quantum form. This has already been formally obtained by performing the limit of the relativistic theory in the path integral formulation[3]. The resulting Galilean theory contains constraints which, in the nonabelian case, we have been able to solve only in a gauge in which the canonical quantization of the relativistic theory is not known. In addition the choice of the gauge proves to be crucial, because as we will see the Faddev-Popov determinant for the Coulomb and Landau gauge vanishes identically in the Galilean limit. We prefer therefore as a first step to quantize directly classical Galilean gauge theories, using Dirac's formalism[4] of canonical quantization of constrained systems. In the abelian case we recover the result obtained by performing the limit in the relativistic theory.

A general feature of Galilean quantum gauge theories is that only the space-independent term of the Fourier expansion of the gauge fields survives in the limit, consistently with the fact a field propagating with infinite velocity must be constant over space. Such a constant term is compatible with periodic boundary conditions (p.b.c.), but must be zero if the fields are required to vanish on the surface of the quantization volume. We will refer to the latter b.c. by the elliptic expression: vanishing boundary conditions (v.b.c.).

We will consider in this paper only p.b.c. and v.b.c.. Constant gauge fields are shown to have physical effects which therefore depend on b.c.. We will show that comparison with experiment requires v.b.c. in QED. This conclusion is made possible by the exact solution of the infrared sector in QED. We do not try to solve the infrared sector of Galilean QCD in the present paper, and therefore we cannot draw a definite conclusion concerning b.c., but we argue that in this case they should be periodic. Although the infrared sector is not solved, it follows from our results that in Galilean QCD there is not colored soft radiation.

The paper is organized in the following way. In Sect. 2. we prove the identical vanishing of the Faddeev-Popov determinant in Galilean gauge theories and in Sect. 3. we perform the canonical quantization. In Sect. 4. and 5. we solve the constraints for the abelian and nonabelian case respectively. In Sect. 6. we solve the infrared sector of Galilean QED. In Sect. 7 we discuss the effects of b.c. and in Sect. 8. we present our conclusions.

2. - IDENTICAL VANISHING OF THE FADDEEV-POPOV DETERMINANT IN THE $c \to \infty$ LIMIT

A necessary condition for the quantization of gauge theories in the Coulomb gauge

$$\partial_k A_k = 0 , \tag{2.1}$$

is that the determinant of the Faddeev-Popov operator $\mathscr{D}_k \partial_k$ be different from zero. This condition is necessary both in the canonical quantization à la Dirac and in the formalism of longitudinal and transverse fields[5].

We will show in this section that in the Galilean limit only field configurations survive for which such determinant is identically zero. As a consequence we cannot perform the $c \to \infty$ limit on the quantum theory in the Coulomb or the Landau gauges which coincide in the limit.

In order to establish the notation let us write the Galilean expressions of the stress tensors

$$
\begin{aligned}
F^a_{ij} &= \partial_i A^a_j - \partial_j A^a_i + g f^{abc} A^b_i A^c_j \\
F^a_{oi} &= \partial_t A^a_i + \partial_i V^a + g f^{abc} A^b_i V^c,
\end{aligned}
\tag{2.2}
$$

and the covariant derivatives

$$
\begin{aligned}
\mathscr{D}_t &= \partial_t + i g t^a V^a \\
\mathscr{D}_k &= \partial_k - i g t^a A^a_k .
\end{aligned}
\tag{2.3}
$$

In the above equations A^a_i and V^a are the gauge fields, f^{abc} are the structure constants and t^a the generators of the color group in the appropriate representation. In the regular real representation

$$(t^a)^{bc} = i f^{abc}. \tag{2.4}$$

The gauge-field Lagrangian density is[1]

$$\mathscr{L}_G = \frac{1}{2} F^a_{oi} F^a_{oi} - \Lambda^a_i \phi^a_i, \tag{2.5}$$

where

$$\phi^a_i = \frac{1}{2} \varepsilon_{ijk} F^a_{jk}. \tag{2.6}$$

The Λ^a_i are Lagrange multipliers, whose variation generates the constraints

$$\phi^a_i = 0 . \tag{2.7}$$

These constraints can be rewritten

$$\mathscr{D}_j A_k = \partial_k A_j. \tag{2.8}$$

By taking the $\mathscr{D}_k$- derivative of the above equation and using the commutativity of the covariant derivatives which follows from the constraint (2.7) we get

$$\mathscr{D}_k \partial_k A_j = \mathscr{D}_j \mathscr{D}_k A_k = \mathscr{D}_j \partial_k A_k = 0, \tag{2.9}$$

showing that the Faddeev-Popov operator $\mathscr{D}_k \partial_k$ has at least one eigenstate with vanishing eigenvalue, so that its determinant must vanish.

3. - CANONICAL QUANTIZATION OF GALILEAN GAUGE THEORIES

Gauge theories contain constraints. The theory of canonical quantization of constrained systems has been developed by Dirac(4). This theory is well known and will not be reviewed here. In order to establish the notation, however, we summarize the main points.

One requires that the original constraints ϕ_σ, called primary constaints, be valid at all the times, by imposing the vanishing of their commutator with the Hamiltonian. This condition generates new constraints χ_σ, called secondary constraints, for which again commutativity with the Hamiltonian is required and so on. If the Lagrangian is consistent this process goes to an end. The necessary gauge fixing contraints are then added to the set of the χ_σ's. From the whole set of ϕ_σ's and χ_σ's a maximal subset is chosen for which the determinant of the Poisson brackets does not vanish. The constraints of this set are called first class. If we denote the first class constraints by Ξ_σ, by definition

$$\det \Delta_{\sigma\tau} \neq 0; \qquad \Delta_{\sigma\tau} = \left[\Xi_\sigma \, , \, \Xi_\tau \right] . \tag{3.1}$$

First class constraints allow the elimination of pairs of conjugate variables. The remaining variables no longer satisfy, in general, canonical Poisson brackets. Their commutation relations can be deduced from the commutation relations obeyed by the old variables once the constraints are taken into account. Such commutation relations are called Dirac's brackets and read

$$[A, B]^* = [A,B] - [A, \Xi_\sigma] \; \Delta^{-1}_{\sigma\tau} \; [\Xi_\sigma , B] . \tag{3.2}$$

An essential point to be recalled is that the constraints should not be used until all the commutation relations have been worked out.

The matter field Lagrangian density is(2)

$$\mathscr{L}_M = \psi^* i \, \mathscr{D}_t \psi + \bar\psi^* i \, \mathscr{D}^*_t \bar\psi + \psi^* \, \frac{\mathscr{D}^2}{2m} \psi + \bar\psi^* \, \frac{\mathscr{D}^{*2}}{2m} - m \, c^2 (\psi^* \psi + \bar\psi^* \bar\psi), \tag{3.3}$$

where $\psi, \bar\psi$ are the matter, antimatter fields respectively. The above expression holds both for bosonic (commuting) and fermion (anticommuting) fields.

For the gauge field Lagrangian it is convenient to use the first order formulation

$$\mathscr{L}_G = E_i^a\ \partial_t A_i^a - \frac{1}{2} E_i^a E_i^a - V^a\ \mathscr{D}_i^{ab} E_i^b - \Lambda_i^a\ \phi_i^a\ . \tag{3.4}$$

From the total Lagrangian we find that the canonical variables are A_i^a, ψ and $\overline{\psi}$ with canonical momenta

$$\begin{aligned} \pi &= \frac{\partial \mathscr{L}}{\partial \dot{\psi}} = i\,\psi^* \\ \overline{\pi} &= \frac{\partial \mathscr{L}}{\partial \dot{\overline{\psi}}} = i\,\overline{\psi}^* \\ E_i^a &= \frac{\partial \mathscr{L}}{\partial \dot{A}_i^a} \end{aligned} \tag{3.5}$$

The nonvanishing Poisson brackets are

$$\begin{aligned} \left\{\psi_\alpha^r(x), \pi_\beta^s(y)\right\} &= \delta_{\alpha\beta}\,\delta^{rs}\,\delta^3(x-y) \\ \left\{\overline{\psi}_\alpha^r(x), \overline{\pi}_\beta^s(y)\right\} &= \delta_{\alpha\beta}\,\delta^{rs}\,\delta^3(x-y) \\ \left[\ A_i^a(x), E_j^b(y)\right] &= \delta^{ab}\,\delta_{ij}\,\delta^3(x-y) \end{aligned} \tag{3.6}$$

where r,s are color indices and α, β spinor indices. The fields Λ^a and V^a are Lagrange multipliers for the primary constraints (2.7) and

$$\phi^a(x) = \mathscr{D}_k^{ab}(x)\ E_k^b\ (x) + g\ \varrho^a\ (x) = 0, \tag{3.7}$$

with

$$\varrho^a\ (x) = \psi^*\ t^a\,\psi - \overline{\psi}^*\ t^a\,\overline{\psi}\ . \tag{3.8}$$

The constraint (3.7)is the same as in the relativistic theory.

Following Dirac's theory we assume the Hamiltonian density

$$\mathscr{H} = E_i^a\ \partial_t\ A_i^a + \psi^*\ i\ \partial_t\psi\ + \overline{\psi}^*\ i\ \partial_t\overline{\psi}\ - (\mathscr{L}_M + \mathscr{L}_G), \tag{3.9}$$

and require the vanishing of the time derivative of the primary constraints

$$\begin{aligned} \left[\phi^a, H\right] &= 0, \\ \left[\phi_i^a, H\right] &= 0. \end{aligned} \tag{3.10}$$

The first of Eqs. (3.10) is automatically satisfied as in the relativistic case, while the second one gives

$$\left[\phi_i^a(y), H\right] = g\, f^{abc}\, V^b(y)\, \phi_i^c(y) + \chi_i^a(y) = 0\,, \tag{3.11}$$

generating the secondary constraints

$$\chi_i^a(y) = \varepsilon_{kij}\, \mathscr{D}_i^{ab}(y)\, E_j^b(y) = 0\,. \tag{3.12}$$

These are the only secondary constraints, because the vanishing of $[\chi_i^a(y), H]$ yields only conditions on Λ_i^a.

We quantize in a cubic box of edge L and volume $\Omega = L^3$ with p.b.c.. Fields and constraints can therefore be expanded in Fourier series

$$A_i^a(x) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{n}} A_{i\vec{n}}^a\, e^{i\frac{2\pi}{L} n\cdot x}\,, \qquad A_{i,\vec{n}}^a = A_{i,-\vec{n}}^a$$

$$E_i^a(x) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{n}} E_{i\vec{n}}^a\, e^{-i\frac{2\pi}{L} n\cdot x}\,, \qquad E_{i,\vec{n}}^a = E_{i,-\vec{n}}^a\,, \tag{3.13}$$

$$\phi^a(x) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{n}} \phi_{\vec{n}}^a\, e^{i\frac{2\pi}{L} n\cdot x}\,, \qquad \phi_{\vec{n}}^a = \phi_{-\vec{n}}^a \tag{3.14}$$

and so on.

The nonvanishing Poisson brackets among the Fourier coefficients are

$$\left[A_{i\vec{m}}^a, E_{j\vec{n}}^b\right] = \delta_{ij}\, \delta^{ab}\, \delta_{\vec{m},\vec{n}}\,. \tag{3.16}$$

Of particular significance is the constraint $\Phi_o^a = 0$. This is only present if the constant term $V_o^a \neq 0$. The choice of the values of V_o is a choice of b.c. which will be discussed later.

4. - SOLUTION OF THE CONSTRAINTS IN THE ABELIAN CASE

4.1. - The Coulomb Gauge

Let us write down the full set of constraints, including the gauge fixing one

$$\phi_{i\vec{n}} = \frac{1}{2}\varepsilon_{ijk}\, i\, \frac{2\pi}{L} n_j A_{k\vec{n}} = 0$$

$$\chi_{i\vec{n}} = -\, \varepsilon_{ijk}\, i\, \frac{2\pi}{L} n_j E_{k\vec{n}} = 0 \qquad (4.1)$$

$$\phi_{\vec{n}} = -i\, \frac{2\pi}{L} n_i\, E_{i\vec{n}} + e\,\varrho_{\vec{n}} = 0$$

$$\chi_{\vec{n}} = i\, \frac{2\pi}{L} n_i A_{i\vec{n}} = 0.$$

We do not evaluate the determinant of the constraint matrix, but show that the constraints are first class by the explicit solution. Assuming p.b.c.

$$A_{i\vec{n}} = q_i\, \delta_{\vec{n},0}$$

$$E_{i\vec{n}} = p_i\, \delta_{\vec{n},0} - e\,(\Delta^{-1}\,\partial_i\,\varrho)_{\vec{n}}. \qquad (4.2)$$

Since there is no constraint for the constant terms, they satisfy their original Poisson brackets

$$\left[q_i, p_j\right] = \delta_{ij}. \qquad (4.3)$$

The Hamiltonian is

$$H = H_o + H_c, \qquad (4.4)$$

where H_c is Coulomb interaction and

$$H_o = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 - \omega q_i I_i - \frac{1}{2m}\int_\Omega d^3x\, (\psi^*\Delta\,\psi + \bar{\psi}^*\Delta\,\bar{\psi}) + mc^2\int_\Omega d^3x\, (\psi^*\psi + \bar{\psi}^*\bar{\psi}). \qquad (4.5)$$

$$I_i = \sqrt{\frac{m}{N}}\int_\Omega d^3x\, \frac{1}{m\,i}\,(\psi^*\,\partial_i\psi - \bar{\psi}^*\,\partial_i\,\bar{\psi}) \qquad (4.6)$$

$$\omega^2 = \frac{g^2N}{m\Omega} \qquad (4.7)$$

$$N = \int_\Omega d^3x\, (\psi^*\psi + \bar{\psi}^*\bar{\psi}). \qquad (4.8)$$

For v.b.c., q_i, p_i=0. We have put the terms involving q_i, p_i in H_o because their effect is to change the definition of in- and out- states for matter fields, rather than to give rise to a real interaction. We will see this in detail in Sect. 6 where we will solve the infrared sector of the Hamiltonian (4.4).

4.2. - The Gauge $A_3=0$

The gauge fixing $A_3(x)=0$ is not consistent in the case of p.b.c.. The reason is that the Fourier components of this constraint are $A_{3\,\vec{n}} = 0$, and the solution of the constraints would not leave us with pairs of conjugate variables because $q_3=0$ and $p_3 \neq 0$. In the Dirac's formalism this incosistency is related to the vanishing of the determinant of the Poisson brackets matrix Δ, which follows from the fact that $A_{30}=0$ commutes with all the other constraints. A gauge fixing which makes the set of constraints first class is

$$A_{3\,\vec{n}} = 0, \qquad \vec{n} \neq 0. \tag{4.9}$$

The solution of the constraints with p.b.c. is now

$$\begin{aligned} &A_{3\,\vec{n}} = q_3\, \delta_{\vec{n},0} \\ &A_{k\,\vec{n}} = q_k\, \delta_{\vec{n},0} + n_k\, A_{\vec{n}}\, \frac{L}{2\pi i} \quad, k = 1,2 \\ &E_{i\,\vec{n}} = p_i\, \delta_{\vec{n},0}\, . \end{aligned} \tag{4.10}$$

From the above Eqs. we infer that

$$\begin{aligned} &A_i\,(x) = \frac{1}{\sqrt{\Omega}}\, q_i + \partial_i\, A(x) \qquad , i = 1,2 \\ &A_3(x) = \frac{1}{\sqrt{\Omega}}\, q_3\, , \end{aligned} \tag{4.11}$$

with $A(x)$ satisfying p.b.c.. The term $\partial_i\, A(x)$, however, can be reabsorbed by a transformation on the matter fields

$$\begin{aligned} &\psi\,(x) \rightarrow e^{i\,A(x)}\, \psi(x) \\ &\overline{\psi}\,(x) \rightarrow e^{-i\,A(x)}\, \overline{\psi}(x). \end{aligned} \tag{4.12}$$

Note that the terms q_i cannot be absorbed in the same way because the transformed matter fields would no longer satisfy p.b.c..

In conclusion the gauge (4.9) gives the some result of the Coulomb gauge.

5. - SOLUTION OF THE CONSTRAINTS FOR SU(2)

For SU(2) we choose the gauge fixing

$$A_3^a(x,t) = \delta^{a3}\left[\frac{1}{\sqrt{\Omega}}\, q_3(t) + \partial_3 A(x,t)\right] , \tag{5.1}$$

whose consistency is shown below by the explicit solution of the constraints. This solution is obtained by steps. We first solve the constraints (2.6) and (5.1) which determine the form of the gauge fields, and then the constraints (3.7) and (3.12) which determine the form of the conjugate fields E_i. The general solution of Eq. (2.6) is implicitely given by

$$G\ \partial_k\, G^{-1} = - g\ \varepsilon_a\ A_k^a , \tag{5.2}$$

where G is a unitary operator and

$$(\varepsilon_a)_{bc} = \varepsilon_{abc} = (\varepsilon^a)^{bc} . \tag{5.3}$$

By taking k=3 and using Eq. (5.1) we have

$$\partial_3\, G^{-1} = - g\, G^{-1}\ \varepsilon_3 \left[\frac{1}{\sqrt{\Omega}}\, q_3 + \partial_3\, A\right]$$

whose general integral is

$$G = \exp\ \left[g\ \varepsilon_3\, (\frac{1}{\sqrt{\Omega}}\ \ q_3\, x_3 + A)\right]\ \ T\,(x_1, x_2) ,$$

$T(x_1, x_2)$ being a unitary operator independent of x_3.

Let us now write down Eq. (5.2) for k=1,2

$$G\,\partial_k\, G^{-1} = \exp\ \left[g\ \varepsilon_3\, (\ \frac{1}{\sqrt{\Omega}}\ q_3\, x_3 + A)\right]\ (T\ \partial_k\ T^{-1}). \\ \exp\ \left[-g\, (\frac{1}{\sqrt{\Omega}}\ q_3\, x_3 + A)\right]\ - g\ \varepsilon_3\ \partial_k\, A , \qquad k=1,2 .$$

Putting

$$T\ \partial_k\ T^{-1} = g\ \varepsilon_a\ \tilde{A}_k^a\,(x_1, x_2) , \qquad k=1,2,$$

we can rewrite it as

$$G\ \partial_k\ G^{-1} = \left\{\exp\left[\ g\ \varepsilon_3\, (\frac{1}{\sqrt{\Omega}}\ q_3\, x_3 + A)\ \right]\right\}_d^b\ \ \tilde{A}_k^d\,(x_1, x_2)\ \varepsilon^b - g\ \varepsilon_3\, \partial_k\, A.$$

We now use b.c.. The last term satisfies p.b.c. if A does, so that a necessary condition for the first term to satisfy the same conditions is

$$\tilde{A}_k^d(x_1, x_2) = \delta^{d3} \tilde{A}_k(x_1, x_2) ,$$

which allows to write

$$T = \exp \left\{ g\, \varepsilon_3 \left[\frac{1}{\sqrt{\Omega}} (q_1 x_1 + q_2 x_2) + B \right] \right\}$$

with B satisying p.b.c. but otherwise arbitrary. By a redifinition of A we have finally

$$G = \exp \left[g\, \varepsilon_3 \left(\frac{1}{\sqrt{\Omega}} \vec{q}\; \vec{x} + A \right) \right] . \tag{5.4}$$

from which we derive

$$A_i^a = \delta^{a3} \left[\frac{1}{\sqrt{\Omega}} q_i + \partial_i A \right] . \tag{5.5}$$

We must now solve the constraints (3.7) and (3.12). From these constraints we derive the equation

$$\mathscr{D}^2 E_i = - g\, \mathscr{D}_i \varrho , \tag{5.6}$$

whose general integral is

$$E_i = P_i - g\, \mathscr{D}^{-2} \mathscr{D}_i \varrho , \tag{5.7}$$

P_i satisfying the homogenous equation

$$\mathscr{D}^2 P_i = 0 . \tag{5.8}$$

This equation can be rewriten

$$\Delta \exp \left(- g\, \varepsilon_3 \frac{q \cdot x}{\sqrt{\Omega}} \right) R_i = 0 , \tag{5.9}$$

with

$$R_i = \exp (- g\, \varepsilon_3 A)\, P_i . \tag{5.10}$$

R_i can be expandend in Fourier series

$$R_i^a = \frac{1}{\sqrt{\Omega}} \sum_{\vec{n}} c_{i\vec{n}}^a \cos \frac{2\pi}{L} n \cdot x + s_{i\vec{n}}^a \sin \frac{2\pi}{L} n \cdot x . \tag{5.11}$$

Eq. (5.9) gives the following equations for the Fourier coefficient

$$\left|\left[\left(g\frac{1}{\sqrt{\Omega}}\varepsilon_3 q\right)^2-\left(\frac{2\pi}{L}n\right)^2\right]^2-\left(\frac{2\pi}{L}g\frac{1}{\sqrt{\Omega}}\varepsilon_3\vec{q}\cdot\vec{n}\right)^2\right| c_{i\vec{n}}=0$$
$$\left|\left[\left(g\frac{1}{\sqrt{\Omega}}\varepsilon_3 q\right)^2-\left(\frac{2\pi}{L}n\right)^2\right]^2-\left(\frac{2\pi}{L}g\frac{1}{\sqrt{\Omega}}\varepsilon_3\vec{q}\cdot\vec{n}\right)^2\right| s_{i\vec{n}}=0 \qquad (5.12)$$

The solution to the above equations is

$$c^a_{i\vec{n}}=\delta^{a3}\,p_i\,\delta_{\vec{n},0}$$
$$s^a_{i\vec{n}}=\delta^{a3}\,s_i\,\delta_{\vec{n},0}\,, \qquad (5.13)$$

so that finally

$$p^a_i=\frac{1}{\sqrt{\Omega}}\,\delta^{a3}\,p_i\,. \qquad (5.14)$$

Reasoning as in the abelian case we can absorb the function A of Eq. (5.4) into the matter fields, so that we are left with the variables q_i, p_i

$$A^a_i=\frac{1}{\sqrt{\Omega}}\,\delta^{a3}\,q_i\,(t)$$
$$E^a_i=\frac{1}{\sqrt{\Omega}}\,\delta^{a3}\,p_i\,(t)\,. \qquad (5.15)$$

Unlike the abelian case, however, the constraints affect the constant fields A^a_{io} and E^a_{io}, determining their dependence on the color index given in Eq. (5.15).

We must therefore find out the commutation relations between q_i, p_i starting from Dirac's brackets

$$\left[A^3_{io},E^3_{jo}\right]^*=\left[q_i,p_j\right]=\delta_{ij}-\left[A^3_{io},\Xi^a_{\sigma\vec{m}}\right](\Delta^{-1})^{ab}_{\sigma\vec{m},\tau\vec{n}}\left[\Xi^b_{\tau\vec{n}},E^3_{j0}\right]\,, \qquad (5.16)$$

and analogous expressions for $[A^3_{io},A^3_{jo}]^*$ and $[E^3_{io},E^3_{jo}]^*$. Here the index σ specifies the constraint as well as the spatial index when present. The evaluation of Δ^{-1} is a much tedious work, the more so because our constraint equations are not independent as shown by a simple counting. We have althoughether 24 equations for the 18 variables A^a_{io}, E^a_{io}, and the solution leaves us with the 6 variables q_i, p_i. Now the constraints Ξ_σ of Eq. (5.16) must be independent, so that we must single them out of the full set of constraints. We do not need to do such a work, however. After evaluating commutators we can indeed use the constraint solutions. It is then easy to check that

$$\left[A^a_{io},\ \Xi^b_{\sigma\vec{m}}\right] \propto \delta_{\vec{m},o}\,\varepsilon^{ab3} \tag{5.17}$$

so that A^3_{io} commutes with all the constaint. It follows that

$$\left[q_i,\ p_j\right] = \delta_{ij}$$
$$\left[q_i, q_j\right] = 0\,. \tag{5.18}$$

The commumator of E^3_{io} with the constraints also vanishes with the exception of

$$\left[E^a_{jo}, \chi^b_{\vec{m}}\right] = \frac{1}{\sqrt{\Omega}}\ \delta_{i3}\ \delta^{ab}\ \delta_{\vec{m}\,o} \tag{5.19}$$

We can of course choose as independent constraints the χ^b plus others, so that

$$\left[E^3_{io}, E^3_{jo}\right]^* = \left[p_i, p_j\right] = \frac{1}{\Omega}\ \delta_{i3}\ \delta_{j3}\,(\Delta^{-1})^{33}_{\sigma o,\,\sigma o} = 0 \tag{5.20}$$

due to the antisymmetry of Δ.

As in the abelian case if we require v.b.c. we have $q_i = p_i = 0$.

We can finally write the Hamiltonian

$$H = H_o = H_I \tag{5.21}$$

where

$$H_I = -\frac{1}{2}g^2\int_\Omega d^3x\, d^3y\ \varrho^a(x)\left[\mathscr{D}^{-2}(x-y)\right]^{ab}\ \varrho^b(y)\,, \tag{5.22}$$

$$H_o = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 - \omega q_k I^3_k - \frac{1}{2m}\int_\Omega d^3x\,(\psi^*\Delta\psi + \overline{\psi}^*\Delta\,\overline{\psi}) + mc^2\int_\Omega d^3x(\psi^*\psi + \overline{\psi}^*\overline{\psi}) \tag{5.23}$$

In the last equation

$$I^a_k = \sqrt{\frac{m}{N}}\int_\Omega d^3x\ \frac{1}{m\,i}\ (\psi^*\sigma^a\,\partial_k\,\psi - \psi^*\sigma^a\partial_k\overline{\psi}) \tag{5.24}$$

$$\omega^2 = \frac{g^2 N}{m\Omega} \tag{5.25}$$

$$N = \int_\Omega d^3x\,(\psi^*\psi + \overline{\psi}^*\overline{\psi})\,. \tag{5.26}$$

For v.b.c. we must put $q_i = p_i = 0$ in the above formulae so that $H_o = -\frac{1}{2m}\int_\Omega d^3x(\psi^*\Delta\psi + \overline{\psi}^*\Delta\overline{\psi}) + mc^2\int_\Omega dx^3(\psi^*\psi + \overline{\psi}^*\overline{\psi})$ and H_I becomes a purely Coulombic interaction.

The Green's function $\mathscr{D}^{-2}(x-y)$ has the following expression

$$\mathscr{D}^{-2}(x-y) = \frac{1}{\Omega} \sum_{\vec{n}\neq 0} \left[i \frac{2\pi}{L} \vec{n} - \varepsilon_3 \frac{g}{\sqrt{\Omega}} \vec{q} (\vec{x}-\vec{y}) \right]^{-2} e^{i \frac{2\pi}{L} \vec{n}\cdot(\vec{x}-\vec{y})}$$

$$= \frac{1}{(2\pi)^3} \int d^3p \frac{1}{-p^2} e^{i\vec{p}(\vec{x}-\vec{y}) - \varepsilon_3 \frac{g}{\sqrt{\Omega}} \vec{q}\cdot(\vec{x}-\vec{y})} + O\left(\frac{1}{L}\right) = \tag{5.27}$$

$$= -\frac{1}{4\pi} \frac{1}{|\vec{x}-\vec{y}|} e^{-\varepsilon_3 \frac{g}{\sqrt{\Omega}} \vec{q}\cdot(\vec{x}-\vec{y})} + O(L^{-5/2})$$

It remains to show whether the terms of order $L^{-5/2}$ can be neglected. If they can H_I can be written in the form of the potential

$$H_I = \frac{1}{8\pi} g^2 \sum_{i<j} \frac{1}{|\vec{x}_i - \vec{x}_j|} \left\{ \vec{\sigma}(i)\, \vec{\sigma}(j) + \left[1 - \cos \frac{g}{\sqrt{\Omega}} \vec{q}\, (\vec{x}_i - \vec{x}_j) \right] \right. \tag{5.28}$$

$$\left. \cdot \left[\vec{\sigma}(i)\cdot\vec{\sigma}(j) - \sigma_3(i)\,\sigma_3(j) \right] - \sin \frac{g}{\sqrt{\Omega}} \vec{q}\cdot(\vec{x}_i - \vec{x}_j)\, \varepsilon_{ab3}\, \sigma_a(i)\, \sigma_b(j) \right\}.$$

which, for $\vec{q} = 0$, becomes purely Coulombic. This potential is not invariant under color rotations, but this does not course any difficulty because (as we will see) it must be used only with color singlet states.

6. - THE INFRARED SECTOR IN GALILEAN QED

Introducing creation and destruction operators:

$$a_k^+ = (2\omega)^{-1/2}(p_k + i\omega q_k)$$
$$a_k = (2\omega)^{-1/2}(p_k - i\omega q_k), \tag{6.1}$$

we can rewrite H_o of Eq. (4.5) as

$$H_o = \frac{3}{2}\omega + \omega a_k^+ a_k + (a_k^+ - a_k)\, i \sqrt{\frac{\omega}{2}}\, I_k - \frac{1}{2m} \int_\Omega d^3x (\psi^* \Delta \psi + \overline{\psi}^* \Delta \overline{\psi}) \tag{6.2}$$
$$+ mc^2 \int_\Omega dx^3 (\psi^* \psi + \overline{\psi}^* \overline{\psi})$$

Let us denote by $|i\rangle$, $|f\rangle$,..the normalized eigenstates of the kinetic energy operator. Such states are also eigenstates of the operator I_k with eigenvalues $I_k(i)$, $I_k(f)$, ... H_o can then be put in diagonal form by introducing the shifted operatos

$$b_k(f) = a_k - \frac{i}{\sqrt{2\omega}} I_k(f), \tag{6.3}$$

$$H_o(f) = \frac{3}{2}(\omega^2 + I_k^2(f)) + \omega b_k^+(f)\, b_k(f) - \frac{1}{2m}\int_\Omega d^3x\, (\psi^* \Delta \psi + \bar{\psi}^* \Delta \bar{\psi}) + mc^2 \int_\Omega dx^3 (\psi^* \psi + \bar{\psi}^* \bar{\psi}) \tag{6.4}$$

The normalized eigenstates of H_o are

$$| f, n_1, n_2\, n_3 \rangle = \prod_{k=1}^{3} \frac{1}{\sqrt{n_k!}} \left[a_k^+ + \frac{i}{\sqrt{2\omega}} I_k(f) \right]^{n_k} \cdot e^{\frac{i}{\sqrt{2\omega}}(a_k^+ + a_k) I_k(f)} \; | f \rangle . \tag{6.5}$$

To evaluate cross-sections we need the matrix elements

$$\langle i, 0, 0, 0, | S | f, n_1, n_2, n_3 \rangle = \langle 0 | e^{-\frac{i}{\sqrt{2\omega}}(a_k^+ + a_k) I_k(i)} \prod_{h=1}^{3} \left[a_h^+ + \frac{i}{\sqrt{2\omega}} I_h(f) \right]^{n_h} e^{\frac{i}{\sqrt{2\omega}}(a_\ell^+ + a_\ell) I_\ell(f)} | 0 \rangle . \tag{6.6}$$

$$\langle i | S \left[(n_1 + n_2 + n_3)\omega \right] | f \rangle$$

The factorization in the above Equation is due to the fact that H_c of Eq. (4.4) contains only matter fields. The S-matrix is however a function of the radiated energy $(n_1 + n_2 + n_3)\omega$. By choosing the axes in the appropriate way we can put

$$\Delta I_h = I_h(f) - I_h(i) = \Delta I\, \delta_{h3} , \tag{6.7}$$

so that

$$|\langle i, 0, 0, 0, | S | f, n_1, n_2, n_3 \rangle|^2 = \delta_{on_1}\, \delta_{on_2} |\langle i | S(n_3\omega) | f \rangle|^2 \frac{1}{n_3!} \left[\frac{(\Delta I)^2}{2\omega} \right]^{n_3} e^{-\frac{(\Delta I)^2}{2\omega}} , \tag{6.8}$$

which is proportional to the probability for a classical source ΔI to radiate n_3 photons[1].

The cross section for a process measured with infinite energy resolution in the initial state $| i \rangle$ and energy resolution ΔE in the final state $| f \rangle$ is[6]

$$\sigma \sim \lim_{\Omega \to \infty} \sum_{n_3=0}^{\infty} \left| < i \left| S(n_3\omega) \right| f > \right|^2 \frac{1}{n_3!} \left[\frac{(\Delta I)^2}{2\omega} \right]^{n_3}$$

$$e^{-\frac{(\Delta I)^2}{2\omega}} \theta(\Delta E - n_3 \omega) = \left| < i \left| S \left[\frac{1}{2} (\Delta I)^2 \right] \right| f> \right|^2 \tag{6.9}$$

$$. \theta \left[\Delta E - \frac{1}{2} (\Delta I)^2 \right] .$$

The average radiated energy is

$$\bar{\omega} = \lim_{\Omega \to \infty} \sum_{n_3=0}^{\infty} \frac{1}{n_3!} \left[\frac{(\Delta I)^2}{2\omega} \right]^{n_3} n_3 \omega \, e^{-\frac{(\Delta I)^2}{2\omega}} = \frac{1}{2} (\Delta I)^2. \tag{6.10}$$

According to the above result the cross section vanishes unless the energy uncertainty is greater than the average radiated energy. This is true with p.b.c.. For v.b.c. there are no radiative corrections.

The radiative correction of Eq. (6.9) can also be obtained as the $c \to \infty$ limit from the relativistic theory. The formal proof is that the $c \to \infty$ limit of the relativistic Hamiltonian provides[3] the Hamiltonian we have obtained by quantization of the Galilean Lagrangian. An explicit evaluation of the limit, given elsewhere, shows that this unexpected effect comes from a zero-momentum contribution which must be singled out before a sum over momenta can be approximated by an integral. In this sense such an effect is analogous to Bose-Einstein condensation in statistical mechaniscs.

7. - BOUNDARY CONDITIONS

Let us first discuss b.c. for the electric potential. This discussion holds also for the relativistic case, because b.c. affect the constraint (3.7) which is the same in the Galilean and in the relativistic theory. The constant term of the constraint

$$\phi^a_0 = g \left[\varepsilon^{abc} (A^c_k E^b_k)_0 + \varrho^a_o \right] = 0 \tag{7.1}$$

comes from the variation of V^a_o, and it is therefore present only if $V^a_0 \neq 0$, i.e. for p.b.c.. This term is proportional to the color charge, which must vanish for p.b.c., while it is arbitrary for v.b.c.. In the Galilean limit there is no color charge associated with gluons because the first term of Eq. (7.1) vanishes, so that the matter color charge must vanish. Obviously the same conclusion holds in the abelian case. According to phenomenology, we must then choose p.b.c. for QCD and v.b.c. for QED. This last choice is in fact the usual one, if we remember that the Lagrange multiplier V has the actual meaning of electric potential, which is required to vanish at infinity.

Coming now to the magnetic potential, we first observe that its b.c. must be the same as for the electric potential if do not want to break Galilean (and Poincarè) invariance. V and A_k are infact related[7] by the Galilei transformation

$$V' = V + v_k A_k \qquad (7.2)$$
$$A'_k = A_k,$$

v_k being the parameters of the transformation. We must therefore have $q_i=0$ in QED and $q_i \neq 0$ in QCD. The first requirement is in agreement with phenomenology. We have in fact seen that $q_i \neq 0$ gives rise to radiative corrections incompatible with experiment.

As to the second requirement, it can only be discussed once the infrared sector of Galilean QCD has been solved.

8. - CONCLUSIONS

The main gool of the present investigation was to get insigth into the problem of confinement. Our results in this connection are

i) the restriction of physical states to be color singlet does not appear to be an intrinsic feature of QCD, but depends on our choice of b.c..

ii) in Galilean QCD there is no confining potential, but in order to be in agreement with phenomenology it is not necessary that only quark bound states should exist. It is sufficient that color singlet quark bound states do not disintegrate into color singlet states of free quarks. This condition coul be realized in Galilean QCD if the state of the gluon cloud relative to bound states were orthogonal to the state of the gluon cloud relative to unbound states.

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