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PHYSICAL EFFECTS OF BOUNDARY CONDITIONS IN GAUGE THEORIES

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ABSTRACT

It is shown that boundary conditions have large physical effects through the x-independent term in the Fourier expansion of the gauge fields. These unexpected effects have their origin in the transition from a sum over momenta to an integral in the same way as Bose-Einstein condensation in statistical mechanics.

We give an argument whereby the gauge fields should satisfy periodic boundary conditions in QCD and show that they must vanish on the surface of the quantization box in QED.

1. - We show that the choice of boundary conditions (b.c.) in the quantization of gauge theories has observable physical effects. Such effects are due to the x-independent term q_i in the Fourier expansion of the gauge fields

$$A_i(\vec{x},t) = L^{-3/2} \left\{ q_i(t) + \sum_{\vec{n}\neq 0} C_{i\vec{n}}(t) \cos \frac{2\pi}{L} \vec{n}\cdot\vec{x} + S_{i\vec{n}}(t) \sin \frac{2\pi}{L} \vec{n}\cdot\vec{x} \right\} \tag{1}$$

and have their origin in the approximation of a sum over momenta by an integral in the limit $L \to \infty$. In this sense they are analogous to Bose-Einstein condensation in statistical mechanics[1].

The very existence of q_i is determined by b.c.. If one requires the vanishing of the fields on the surface of the quantization box, $q_i=0$, while q_i is different from zero for periodic b.c.. This constant term affects the radiative corrections in QED and the quark-quark interaction in QCD. In this note we will discuss in detail the first point.

As it is well known[2] in QED the cross-sections contain a factor which is a function of the experimental energy resolution ΔE

$$\sigma = \sigma_o\, f(\Delta E). \tag{2}$$

We will see that for particles of very small velocity

$$f(\Delta E) = \begin{cases} 1, \text{ for } q_i = 0 \\ \theta(\Delta E - \bar{\omega}), \text{ for } q_i \neq 0, \end{cases} \tag{3}$$

where θ is the step function and $\bar{\omega}$ is the average radiated energy whose expression is given below. Phenomenology definitely requires $q_i=0$. We will show that such a requirement has another independent reason in QED, while just for this same reason a choice of b.c. for which $q_i \neq 0$ is expected to be appropriate to QCD.

In order to prove Eq. (3) we must evaluate the cross-section for particles of very small velocity. To do this we can either use Galilean QED[3] or perform the limit $c \to \infty$ in the relativistic result. We will follow here the first way which allows us to make contact with QCD for which the relativistic result is not known, and will give a formal proof that the second procedure gives the same result. An explicit evaluation of the limit will be given elsewhere, showing that the unexpected effect of the constant term has its origin in the approximation of a sum over momenta by an integral. Only after the zero-momentum term (the q_i term) has been singled out the approximation is justified.

2. - The Galilean classical Lagrangian density[4] in first order formalism is

$$\begin{aligned} \mathscr{L} = {} & E_i \partial_t A_i - \frac{1}{2} E_i^2 - V(\partial_i E_i + e\varrho) - \Lambda_k \phi_k + \psi^* i \partial_t \psi + \bar{\psi}^* i \partial_t \bar{\psi} \\ & + \frac{1}{2m} (\psi^* \mathscr{D}^2 \psi + \bar{\psi}^* \mathscr{D}^{*2} \bar{\psi}) - mc^2 (\psi^* \psi + \bar{\psi}^* \bar{\psi}). \end{aligned} \tag{4}$$

In the above formula ψ and $\bar{\psi}$ are the matter, **antimatter** fields resp., V is the electric potential and

$$\varrho = \psi^* \psi - \bar{\psi}^* \bar{\psi}, \tag{5}$$

$$\mathscr{D}_i = \partial_i - ie A_i, \tag{6}$$

$$\phi_k = \frac{1}{2} \varepsilon_{kij} \partial_i A_j. \tag{7}$$

V and Λ_k are Lagrange multipliers for the primary constraints

$$\partial_i E_i + e\varrho = 0 \tag{8}$$

$$\phi_k = 0\ . \tag{9}$$

Following Dirac's theory of canonical quantization of constrained systems[5] we find that the only secondary contraint is

$$\varepsilon_{kij}\, \partial_i\, E_j = 0 \ . \tag{10}$$

Choosing the Coulomb gauge

$$\partial_i\, A_i = 0 \ , \tag{11}$$

we have the set of first class constraints (8) to (11). If we now choose periodic b.c. the solution to the constraint equations is

$$\begin{aligned} A_i &= \frac{1}{L^{3/2}}\, q_i \\ E_i &= \frac{1}{L^{3/2}}\, p_i - e\, \Delta^{-1}\, \partial_i\, \varrho , \end{aligned} \tag{12}$$

where q_i, p_i are x-independent and satisfy canonical Poisson brackets. If we instead require the vanishing of the fields on the surface of the quantization box the solution to the constraint equations is

$$\begin{aligned} A_i &= 0 \\ E_i &= -e\, \Delta^{-1}\, \partial_i\, \varrho . \end{aligned} \tag{13}$$

In this latter case the Hamiltonian is the Schrödinger Hamiltonian with Coulomb interaction only, there are no radiative corrections and $f(\Delta E)=1$. In the first case the Hamiltonian is

$$H = H_o + H_I , \tag{14}$$

where H_I is the Coulomb interaction and

$$H_o = \frac{1}{2}\, p^2 + \frac{1}{2}\omega^2\, q^2 - \omega\, q_k\, I_k - \frac{1}{2m}\int d^3x\, (\psi^* \Delta\, \psi + \overline{\psi}^* \Delta\, \overline{\psi}), \tag{15}$$

$$I_k = \sqrt{\frac{m}{N}} \int d^3x\, \frac{1}{2m_i}\, (\psi^*\, \partial_k \psi - \overline{\psi}^* \partial_k\, \overline{\psi}), \tag{16}$$

$$N = \int d^3x\, (\psi^* \psi + \overline{\psi}^* \overline{\psi}), \tag{17}$$

$$\omega^2 = \frac{e^2 N}{mL^3} \ . \tag{18}$$

The above Hamiltonian has been shown[6] to be the $c \to \infty$ limit of the relativistic Hamiltonian, and this provides the formal proof that the results we will obtain for the radiative corrections are the $c \to \infty$ limit of the relativistic ones.

Introducing creation and destruction operators we can rewrite H_o in the form

$$H_o = \frac{3}{2}(\omega + I_k^2) + \omega (a_k - \frac{i}{\sqrt{2\omega}} I_k)^+ (a_k - \frac{i}{\sqrt{2\omega}} I_k) - \frac{1}{2m} \int d^3x (\psi^* \Delta \psi + \overline{\psi}^* \Delta \overline{\psi}) . \tag{19}$$

Let us denote by $|i\rangle$, $|f\rangle$,... the normalized eigenstates of the kinetic energy operator. Such states are also eigenstates of the operator I_k with eigenvalues $I_k(i)$, $I_k(f)$,... The normalized eigenstates of H_o are

$$|f, n_1, n_2, n_3 \rangle = \prod_{k=1}^{3} \frac{1}{\sqrt{n_k!}} \left[a_k^+ + \frac{i}{\sqrt{2\omega}} I_k(f) \right]^{n_k} . \\ e^{\frac{i}{\sqrt{2\omega}} (a_k^+ + a_k) I_k(f)} |f\rangle . \tag{20}$$

To evaluate cross-sections we need the S-matrix elements

$$\langle i, 0, 0, 0, | S | f, n_1, n_2, n_3 \rangle = \langle 0 | e^{-\frac{i}{\sqrt{2\omega}} (a_k^+ + a_k) I_k(i)} \\ \prod_{h=1}^{3} \left[a_h^+ + \frac{i}{\sqrt{2\omega}} I_h(f) \right]^{n_h} e^{\frac{i}{\sqrt{2\omega}} (a_l^+ + a_l) I_l(f)} |0\rangle . \\ \langle i | S [(n_1 + n_2 + n_3)\omega] | f \rangle . \tag{21}$$

The factorization in the above equation is due to the fact that H_I contains only matter fields. The S matrix is however a function of the radiated energy $(n_1 + n_2 + n_3)\omega$. By choosing the axes in the appropriate way we can put

$$\Delta I_h = I_h(f) - I_h(i) = \Delta I \, \delta_{h3} , \tag{22}$$

so that

$$|\langle i, 0, 0, 0 | S | f, n_1, n_2, n_3 \rangle|^2 = \delta_{on_1} \delta_{on_2} |\langle i | S(n_3 \omega) | f \rangle|^2 . \\ \frac{1}{n_3!} \left[\frac{(\Delta I)^2}{2\omega} \right]^{n_3} e^{-\frac{(\Delta I)^2}{2\omega}} , \tag{23}$$

which coincides with the probability for a classical source ΔI to radiate n_3 photons[3].

The cross-section for a process measured with infinite energy resolution in the initial state and energy resolution ΔE in the final state is[7]

$$\sigma \sim \lim_{L\to\infty} \sum_{n_3=0}^{\infty} |<i|\, S(n_3\omega)\,|\, f>|^2 \frac{1}{n_3!} \left[\frac{(\Delta I)^2}{2\omega}\right]^{n_3} \cdot \tag{24}$$

$$e^{-\frac{(\Delta I)^2}{2\omega}} \theta(\Delta E - n_3\omega) = |<i\,|S\,[\tfrac{1}{2}(\Delta I)^2]\,|\, f>|^2 \;\theta(\Delta E - \tfrac{1}{2}(\Delta I)^2)$$

An anologous calculation shows that the average radiated energy is

$$\bar{\omega} = \frac{1}{2}(\Delta I)^2 \tag{25}$$

completing the proof of Eq. (3).

3. - Let us now come to the other reason mentioned at the beginning to have $q_i=0$ in QED. This is related to the value of the constant term V_o in the Fourier expansion of the electric potential. If we choose b.c. for V such that $V_o \neq 0$, it follows from Eq. (8) that the constant term ϱ_o in the Fourier expansion of the charge density ϱ must vanish. For $V_o=0$, on the other hand, the constant term is absent in Eq. (8) and there is no condition on ϱ_o. Since ϱ_o is proportional to the total electric charge which can take any value, we are forced to choose b.c. such that the electric potential should vanish at large distances, namely $V_o=0$. Now the b.c. for V and A_i are not independent because V and A_i are related by Galilei transformations[8]

$$\begin{aligned} V &\to V + v_k A_k \\ A_k &\to A_k , \end{aligned} \tag{26}$$

where v_k are the parameters of the transformation. We must then have either $V_o=q_i=0$ or $V_o, q_i \neq 0$. It is then satisfactory that independent experimental facts, like the existence of charged states and the absence of the radiative correction of Eq. (3) require compatible b.c. $V_o=q_i=0$.

The situation is reversed in QCD. In this case the electric potential need not be zero at infinity. Indeed in the Galilean limit the choice $V_o \neq 0$ implies the vanishing of the color charge[4] in agreement with experiment. We expect therefore that comparison with experiment requires $q_i \neq 0$.

The effect of q_i on the quark-quark interaction is presently under investigation. It is already known, however, that it does not give rise to colored radiation[4].

A last remark concerning solid state physics. Here one has to do will chargless states (crystals), so that $\varrho_o=0$, and V_o, q_i can be different from zero. This does not lead to any inconsistency, because the number N of Eq. (17) is in this case proportional to L^3, so that the photon energy is finite and there are no radiative corrections in the Galilean limit.

This has been shown to be the many-body content of the Higgs effect[9].

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