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A MONTE CARLO SIMULATION WITH AN "IMPROVED" ACTION FOR THE O(3) NON LINEAR SIGMA MODEL

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ABSTRACT

We present the results of a Monte Carlo simulation for a lattice action improved à la Symanzik. The results seem to give a faster approach of the lattice physics to its continuum limit.

1. - INTRODUCTION

In the last few years lattice theories have been widely investigated by Monte Carlo simulations in order to reconstruct the continuum euclidean field theory. Sources of systematic errors and limitations to numerical simulations are:

a) finite size effects;

b) finite lattice spacing effects (which in asymptotically free theories correspond to a finite bare coupling).

Symanzik(1) suggests to systematically construct a lattice action which minimizes the cutoff dependence (b) to approach more rapidly the continuum limit. This program has been first discussed for the ϕ^4 theory by Symanzik(1), for the two dimensional non-linear sigma model in ref. (2) and (3) and more recently for gauge theories by Weisz(4).

In this paper we present the results of a high statistics Monte Carlo simulation for the two dimensional O(3) non-linear sigma model obtained with the usual action with nearest neighbor interactions and with an "improved" action à la Symanzik and for these two actions we compare the approach of certain relevant physical quantities to the continuum limit. The results for the "improved" action appear to be very encouraging and an analogous effort for lattice gauge theories (including fermions) should be done in the future. It should be noticed however that in the present case the "naive" action already gives results in agreement, within some percent, with what is expected in the continuum limit (see below).

The plan of the paper is the following: in Sect. 2 we recall in more details the basic ideas needed to construct an "improved" lattice action; in Sect. 3 we define the two lattice actions and the relevant physical quantities which have been studied in our Monte Carlo simulation; in Sect. 4 we analyze the strong and weak coupling expansions for several quantities, and in Sect. 5 we compare numerical results for the two actions with the analysis of the strong and weak coupling corresponding series. In the conclusion we will briefly discuss the application of these ideas to gauge theories.

2. - IMPROVING THE LATTICE ACTION.

There are many possible actions on the lattice which formally converge to the same (continuum) limit when the lattice spacing $a \to 0$. In the case of renormalizable interactions for small lattice spacing a given action on the lattice is equivalent to a local theory on the continuum whose lagrangian has the form(1-4):

$$\mathscr{L}_{eff} = \mathscr{L}_o + a^2 \mathscr{L}_1 + a^4 \mathscr{L}_2 + \cdots \qquad (1)$$

$\mathscr{L}_o$ is the ordinary renormalizable continuum lagrangian in D dimensions; $\mathscr{L}_1$ contains operators of dimension D+2, etc. Because of the "irrelevant"(5) operators present in the lagrangian of eq. (1) the physical quantities (masses, string tension, ...) will obey to renormalization group equations which contain scaling violating terms:

$$\left(-a \frac{\partial}{\partial a} + \beta(g) \frac{\partial}{\partial g}\right) \text{Physical quantity} = O\left[a^2(\ln a)\right] \qquad (2)$$

where $\beta(g)$ is the bare coupling constant β-function. For small a the extra term in eq. (2) will cause exponentially small (in g) corrections to ratios of masses (or to any dimensionless quantity with a well defined continuum limit):

$$\frac{m_1}{m_2} = \text{const}\left\{1 + O\left[(am)^2 \ln(am)\right]\right\} \qquad (3)$$

where m is some mass scale.

The suggestion by Symanzik[1] is to choose the lattice action in such a way that the r. h. s. of eqs. (2) and (3) are at least of order $O\left[a^4(\ln a)\right]$. In perturbation theory the effects of the $O(a^2)$ "irrelevant" operators can be eliminated by studying the insertion of non renormalizable operators of dimension D + 2 into appropriate Green functions while their non perturbative effects are usually out of control[f1]. The elimination of these $O(a^2)$ terms implies the introduction on the lattice, beside the usual nearest neighbor interactions, of next to nearest neighbor, diagonal ... interactions with suitable (computable order by order in perturbation theory) coefficients. One has to add six new spin interactions in the usual non linear sigma model lagrangian to correct the $O(a^2)$ terms ; for the four dimensional Yang Mills theory it suffices to add all the plaquettes of length 6a to the usual lagrangian. One should run Monte Carlo simulations for these theories in order to determine experimentally the optimum set of coefficients[4,f2]. This appears to be unpractical: for example in the case of four dimensional gauge theories one should compute at each upgrading of a link 120 plaquettes (to be compared to the 6 plaquettes of the usual action) and span a four dimensional space of coefficients. We decided instead to fix the coefficients of the various terms of the lagrangian by eliminating the $O(a^2)$ terms at the three level; this is a reasonable starting point in view of the fact that, at least for the non linear sigma model, the $O(a^2)$ corrections to the action coming from a one loop computation to the two point function turn out to be rather small[2,f3]. In this way one looses only a factor 2 in CPU time for the non linear sigma model and a factor 4 (and not 20) for gauge theories. We will see that, for the non linear sigma model, even this rather crude modification of the action seems definitely to improve the convergence to the continuum limit.

3. - LATTICE ACTIONS FOR THE NON LINEAR SIGMA MODEL.

In this section we give the lattice actions used in our Monte Carlo experiment and define the relevant physical quantities we measured. The "naive" and "improved" actions are given by the following equations[f4] :

$$S_N = -\beta \sum_{\vec{n}} \left\{ \vec{s}(\vec{n}) \cdot \left[\vec{s}(\vec{n}+\hat{\mu}_x) + \vec{s}(\vec{n}+\hat{\mu}_y) \right] \right\} \tag{4a}$$

$$S_I = -\beta \sum_{n} \left\{ \frac{4}{3} \vec{s}(\vec{n}) \cdot \left[\vec{s}(\vec{n}+\hat{\mu}_x) + \vec{s}(\vec{n}+\hat{\mu}_y) \right] - \frac{1}{12} \vec{s}(\vec{n}) \cdot \left[\vec{s}(\vec{n}+2\hat{\mu}_x) + \vec{s}(\vec{n}+2\hat{\mu}_y) \right] \right\} \tag{4b}$$

$(a = 1)$

$\vec{s} = (s_1, s_2, s_3)$ with the constraint $\vec{s}(\vec{n}) \cdot \vec{s}(\vec{n}) = 1$. $\hat{\mu}_{x,y}$ are the unit vectors in the x-y directions. The coefficients of the nearest neighbor and next to nearest neighbor interactions in S_I have been choosen to give for $a \to 0$ a bare spin propagator $G_0(k)$:

$$G_o(k) \sim \frac{1}{k^2 + O(a^4 k^6)} \quad . \tag{5}$$

S_N has been widely studied by Monte Carlo simulations[8-13] and there are results for the magnetic susceptibility χ and the inverse mass gap ξ over a large range of β ($0 \lesssim \beta \lesssim 2$). On the continuum these quantities are defined as:

$$\chi = \int d^2\vec{x} < \vec{s}(x)\vec{s}(0)> \ , \qquad \xi^2 = \frac{1}{2} \frac{\int d^2\vec{x}\, x^2 < \vec{s}(x)\cdot\vec{s}(0)>}{\int d^2\vec{x} < \vec{s}(x)\cdot\vec{s}(0)>} \quad . \tag{6}$$

ξ^{-1} is proportional to the mass gap.

A very interesting quantity to study is also the four point dimensionless coupling constant at zero momenta defined as:

$$\lambda = \frac{\int d^2\vec{x}\, d^2\vec{y}\, d^2\vec{z} < \vec{s}(x)\cdot\vec{s}(y)\ \vec{s}(z)\cdot\vec{s}(0)>_c}{\chi^2 \xi^2} = \frac{\chi_4}{\chi^2 \xi^2} \quad . \tag{7}$$

$<\cdots>_c$ stands for connected.

In the low temperature ($\beta \to \infty$) domain, by renormalization group arguments[14,15] one knows the behavior for the quantities defined in eqs. (6) and (7):

$$\lim_{\beta \to \infty} \quad \begin{aligned} \chi(\beta) &\longrightarrow C\,(2\pi\beta)^{-4} \exp(4\pi\beta) \left[1 + O(\tfrac{1}{\beta})\right] , \\ \xi(\beta) &\longrightarrow B\,(2\pi\beta)^{-1} \exp(2\pi\beta) \left[1 + O(\tfrac{1}{\beta})\right] , \\ \lambda(\beta) &\longrightarrow \lambda^* = \text{const.} \end{aligned} \tag{8}$$

The value of λ^* is relevant to establish the convergence of the lattice theory to the continuum limit. In fact, while χ and ξ are rapidly varying functions of β, λ^* is expected to behave as[f5]:

$$\lambda^*_{LATT} = \lambda^* \left\{1 + O\left[a^2 \Lambda_L^2 (\ln a\Lambda_L)\right]\right\} \tag{9}$$

exactly in the same way of ratios of masses (cfr. eq. (3)).

C and B in eqs. (8) are not computable in perturbation theory; they can be extimated by extrapolating the strong coupling expansions for χ and ξ. For the "naive" action eleven and ten terms in powers of β are available for χ[16] and ξ[17] respectively[f6]. The strong coupling expansion is not available (to our knowledge) for the "improved" action. In the weak coupling region one can eventually compare the ratio of the correlation lengths obtained by Monte Carlo simulations for the two actions and the ratio of the corresponding $\Lambda_{N,I}$ which can be computed in the standard way at first order in perturbation theory[11,19,20]:

$$\frac{\xi_N}{\xi_I} = \frac{\Lambda_N}{\Lambda_I} \sim \exp - (\pi + \delta)/2 \quad , \tag{10}$$

where, for O(3):

$$\delta = \frac{4}{3\pi} \int_{-\pi}^{\pi} d^2\vec{p} \left\{ \frac{(\sum_\mu \sin^4 p_\mu / 2)}{\left[\sum_\mu (\frac{16}{3} \sin^2 \frac{p_\mu}{2} - \frac{1}{3} \sin^2 p_\mu)\right]} \left[\frac{1}{4 \sum_\mu \sin^2 \frac{p_\mu}{2}} - 1 \right] \right\} \tag{11}$$

Eqs. (10) and (11) give:

$$\frac{\Lambda_N}{\Lambda_I} \sim 0.45 \, . \tag{12}$$

The first six terms (because ξ enters in its definition) of the strong coupling series of λ have been computed[21] ("naive" action) and Nickel et al.[22] provide us five orders of the weak coupling expansion of λ.

In the next section we report the analysis of the available high temperature and weak coupling series.

4. - SERIES ANALYSIS.

The coefficients of the strong coupling series fo χ_4, χ and ξ are given in Table I. In the following we give the results from the available series for the magnetic susceptibility (i), the correlation length (ii) and λ(iii).

TABLE I - High temperature expansion coefficients for χ_4, χ and ξ defined in eqs. (6) and (7).

χ_4	χ	ξ	n
- 0.6666666667	1.00000000		0
- 3.5555555556	1.33333333	0.333333333	1
- 10.01481482	1.33333333	0.444444444	2
- 20.85925926	1.24444444	0.459259259	3
- 35.74875955	1.05679012	0.434567901	4
- 53.46502058	0.85126396	0.380717225	5
- 72.20750147	0.65956496	0.315547717	6
- 90.03570221	0.49271095		7
	0.35806179		8
	0.25292316		9
	0.17485390		10

(i) We strictly follow the detailed analysis made in refs. (9, 23) to combine the predictions of the high temperature expansion for χ :

$$\chi(\beta) = \sum_{\varrho} \chi_{\varrho} \beta^{\varrho} \tag{13}$$

and its asymptotic ($\beta \to \infty$; cfr. eq. (8)) behavior. In ref. (9) it was found that the approach of χ to the continuum limit is expected to be rather slow in β and the coefficient C in eq. (8) was extimated to be $C \simeq 6 \times 10^{-4}$ which gives good agreement with the experimental points as it is shown in Fig. 1. As noticed in ref. (9), if one assumes that for $\beta \gtrsim 1.4$ χ already follows the predicted renormalization group behavior one finds a value for C greater by a factor $\sim$20 and roughly in agreement with the extimate of ref. (13). The results of refs. (9) and (10) indicate however the presence of large $O(\frac{1}{\beta})$ corrections in eq. (8).

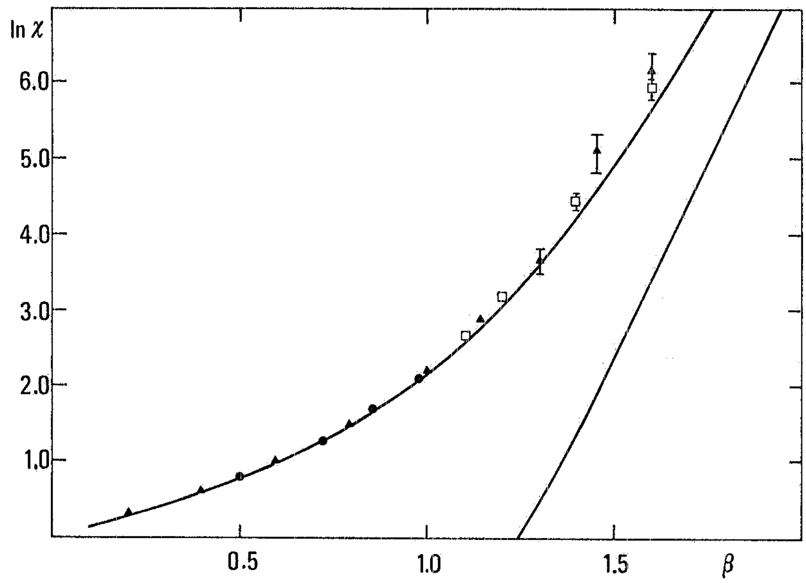

FIG. 1 - The magnetic susceptibility defined in eq. (24) is given as a function of . The Monte Carlo data are taken from ref. (9) (▲), from ref. (10) (□), and from our calculation (●). The solid curve is taken from ref. (9). The line gives the expected asymptotic renormalization group behavior.

(ii) For the correlation length ξ the series is too short to apply the method proposed in ref. (9). We tried to perform a Padè analysis of the six terms at our disposal inorder to find the coefficient B but the series is too short to give any reasonable answer.

(iii) We extimated the behavior of λ as a function of β and λ^* using the high and low temperature series. The results of a Padè analysis (elements (3, 2) or (4, 1)) from the high temperature series give[(f7)]:

$$\lambda^* = -6.18 . \tag{14}$$

Using the series of the triangular lattice[16, 21] one finds :

$$\lambda^*_\Delta = - 6.76 . \qquad (15)$$

We have not pushed the analysis up to give an extimate of the errors in eqs. (14) and (15).

The value of λ^* in eqs. (14) and (15) should be compared with the value one obtains from the analysis of the expansion of the linear sigma model:

$$S = \int d^2\vec{x} \left\{ \frac{1}{2} \left[(\partial_\mu \sigma)^2 + m_o^2 \sigma^2 \right] + \frac{g}{4!} (\sigma^2)^2 \right\} . \qquad (16)$$

We proceed in the standard way[24]:

$$\lambda(g) = g + a_2 g^2 + a_3 g^3 + \cdots . \qquad (17)$$

where g is the four point bare coupling constant. The β-function is defined as:

$$\beta(\lambda) = g \frac{d}{dg} \lambda(g) = g + 2a_2 g^2 + \cdots = \lambda + \tilde{a}_2 \lambda^2 + \cdots \qquad (18)$$

To extract the value of λ^* we use the so called pseudo ϵ-expansion[24]. We define:

$$\tilde{\beta}(\lambda) = \frac{\tilde{a}_2}{\lambda} \beta\left(\frac{\lambda}{\tilde{a}_2}\right) = 1 - \epsilon = 1 + \lambda + \cdots \qquad (19)$$

From eq. (19) we express λ as a series in ϵ:

$$\lambda(\epsilon) = -\epsilon + d_2 \epsilon^2 + \cdots \quad ; \qquad \lambda^* = \lambda(\epsilon = 1) \cdot \frac{40\pi}{33} . \qquad (20)$$

The coefficients of the series in ϵ are reported in Table II.

TABLE II - Weak coupling series for λ. A factor $40\pi/33$ must be included.

λ	d_n
1.000000000	0
0.6502486228	1
0.8168363226E-01	2
0.5501731828E-01	3

To have a more accurate determination of λ^* we added to the computed series[22] the coefficients (starting from the 6th order) obtained by using the asymptotic expressions given in ref. (25)[f8]. One obtains :

$$\lambda^* = - 6.66 \pm 0.06 . \qquad (21)$$

The error in eq. (21) has been extimated by the size of the last known coefficient in ϵ (eq(20)). The values of λ^* in eqs. (14), (15) and (21) are compatible in vue of the fact that it is hard to believe that a rather short high temperature series can give a very good extimate of λ^*.

This is confirmed by the analysis we made for the two dimensional (soluble) Ising model. By analyzing the (ten order) available series for the triangular and quadratic lattice one finds:

$$\lambda^*_\Delta \simeq -2.03 \;; \qquad \lambda^*_\Box \simeq -1.73^{(26)} \tag{22}$$

while the weak coupling series would give[f9]:

$$\lambda^* = \lambda(\varepsilon=1) = -1.85 \pm 0.01 \tag{23}$$

(This definition of λ^* is accordingly to ref. (24)).

The difference between λ^*_Δ and $\lambda^*_\Box$ gives us an idea of the large errors, even with a rather long series, in extrapolating high temperature series to large value of β. The situation for the non-linear sigma model is even worse because the critical temperature is zero ($\beta \to \infty$) and we have only six orders in β. We will then assume that the best extimate of λ^* is that given in eq. (21).

5. - NUMERICAL RESULTS AND THEIR COMPARISON WITH THEORETICAL EXPECTATIONS.

On a finite lattice eqs. (6) and (7) become:

$$\chi = \frac{1}{L^2} \sum_{\vec{n}_1,\vec{n}_2} <\vec{s}(\vec{n}_1)\cdot\vec{s}(\vec{n}_2)> = \frac{1}{L^2} <\vec{M}^2> \;;$$

$$(\xi a)^2 = \frac{1}{\left[2 - 2\cos\left(\frac{2\pi}{L}\right)\right]} \left[\frac{<\vec{M}^2>}{C_1} - 1\right] \tag{24}$$

with

$$C_1 = \sum_{\vec{n}_2} \sum_{(n_1)_x} e^{i\frac{2\pi}{L}x} \sum_{(n_1)_y} <\vec{s}(\vec{n}_1)\cdot\vec{s}(\vec{n}_2)> \,,$$

and

$$\lambda = \left(\frac{L}{\xi a}\right)^2 \left[\frac{<(\vec{M}^2)^2> - \frac{5}{3}<\vec{M}^2>^2}{<\vec{M}^2>^2}\right] . \tag{25}$$

$\vec{n}_{1,2}$ are the position vectors on the lattice; L is the (dimensionless) lattice size. ξ can also be evaluated through the exponential decay of the two point function: all the results with this definition are undistinguishable from those obtained with the definition of eq. (24). In Figs. 1-4 we report the Monte Carlo results for the magnetic susceptibility and the cor

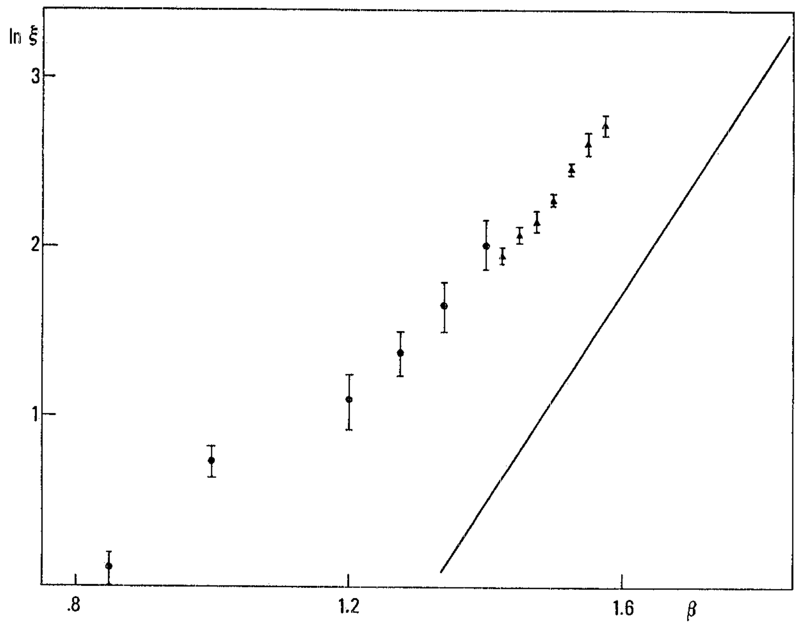

FIG. 2 - The Monte Carlo data for the correlation length (ξa) defined in eq. (24) as a function of β from ref. (8) (⧳), and ref. (12) (⧳). The line gives the expected renormalization group behavior.

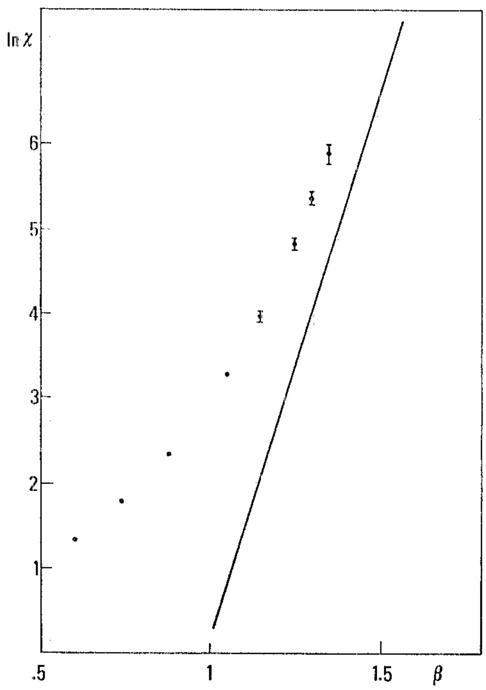

FIG. 3 - Same as in Fig. 1 but for the "improved" action.

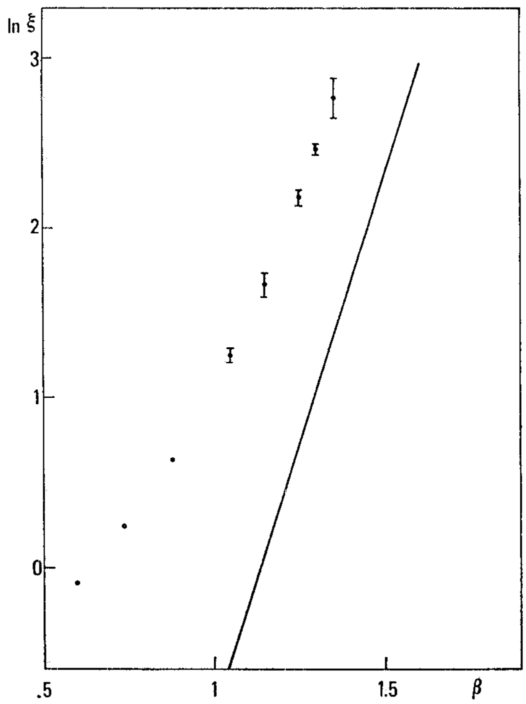

FIG. 4 - Same as in Fig. 2 but for the "improved" action.

relation length taken from refs. (9, 10, 12) and measured in our simulation for the "naive" and "improved" actions. At large β we expect for these quantities the predicted renormalization group behavior : discrepancies from the exponential growth indicate that the asymptotic regime is not reached and give a large error in the determination of the coefficients C and B in eqs. (8) (ref. (9)). We found convenient to fit the Monte Carlo data at large β, after the onset of the low temperature regime, with the following parametrization:

$$\chi = C(2\pi\beta)^{-4} \exp(4\pi \epsilon_1 \beta) , \qquad \xi = B(2\pi\beta)^{-1} \exp(2\pi\epsilon_2\beta) . \tag{26}$$

The deviation of $\epsilon_{1,2}$ from 1 is a measure of the rate of convergence to the asymptotic regime. One finds:

$$\begin{aligned} &\epsilon_1 = 0.80 \pm 5\% \quad (\text{ref } (10)) ; \quad &&\epsilon_2 = 0.92 \pm 10\% \quad (\text{ref. } (12)) \quad &&\text{"naive"} \\ &\epsilon_1 = 1.07 \pm 10\% ; &&\epsilon_2 = 1.02 \pm 10\% &&\text{"improved"} \end{aligned} \tag{27}$$

Fitting our data with $\epsilon_1 = \epsilon_2 = 1$ we obtain

$$C_I = 0.76 \pm 0.007 , \qquad B_I = 0.027 \pm 0.002 .$$

The corresponding value of B with the naive action (B_N) should be, from eq. (12), $B_N = 0.012 \pm 0.001$. This value is in definite disagreement with the one extimated directly with the naive action, i.e. $B = 0.0085 \pm 0.0003$ (ref. (12)) (where the error quoted is only statistical): the origin of this effects should be found in the slow set in of the scaling with the naive action, which can produce systematical errors.

At fixed ξa the renormalization group behavior for ξ and χ is reached much faster with the new action and give more confidence in the determination of the constants C and B. If this is true also for gauge theories a much better determination of the quark potential[27] should be possible with an "improved" action. We note that, at large β, the ratio ξ_N/ξ_I is in agreement within a 30% of accuracy with the prediction from eq. (12) which corresponds to a shift in β $\beta_N - \beta_I \sim 0.127$. A precise evaluation of λ^* starting from lattice Monte Carlo results is very difficult because of very large statistical fluctuations and finite size effects[f10]. In Fig. 5 we report, at fixed β the values of λ as a function of $L/\xi a$. We expect $\lambda[L/\xi a]$ to go as:

$$\lambda\left[L/\xi a\right] = \lambda\left[1 + O(\exp - L/\xi a)\right] . \tag{28}$$

We choosed $L/\xi a \sim 8$ so that finite size effects in fixing this ratio are smaller ($\lesssim 5\%$) with respect to the splitting of the couplings λ's from the two different actions and to their

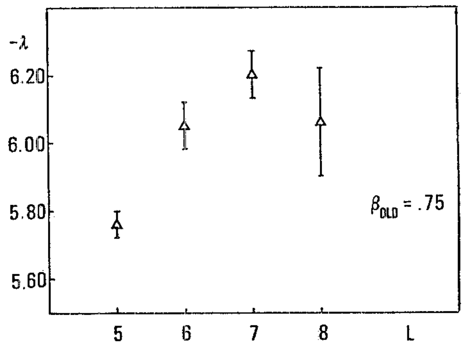

FIG. 5 - $\lambda(L/\xi a)$ for the "naive" action at β = 0.75 for several values of $L/\xi a$.

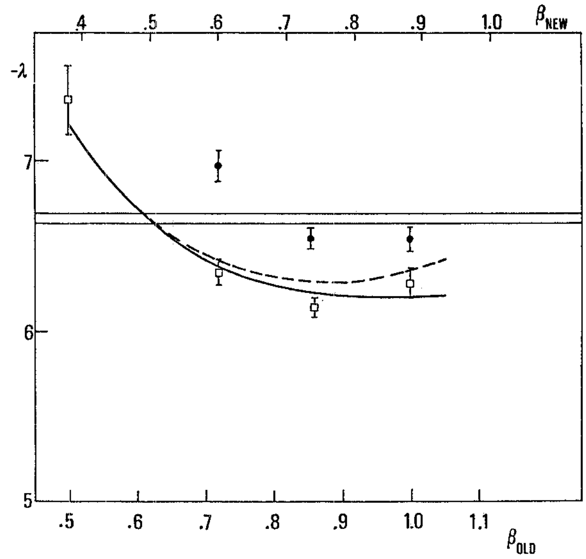

FIG. 6 - $\lambda(\beta)$ as a function of β with $L/\xi a \simeq 8$ for the "naive" and "improved" action. The expected asymptotic value $\lambda^* = -6.66$ is represented by a straightline. The scale of β for the improved action has been shifted by 0.12 accordingly to eq. (12). The high temperature series ("naive" action) is given by the solid line, its Padè approximant (3, 2) by a dashed line.

statistical errors (with $\sim$1-4 milions of sweeps at fixed β). In Fig. 6 we report $\lambda(L/\xi a \sim 8)$ for the two actions as a function of β together with λ^* extimated from the low temperature series (eq. (21)) and the high temperature behavior for $\lambda(\beta)$ ("naive" action). We averaged λ, at different β's, over 5×10^5 - 4×10^6 (at larger β) events (1 sweep x 4 hits). The statistical error was extimated by dividing the events in groups of 5000 and averaging over these groups. We checked that successive groups were uncorrelated. Assuming that the value of λ^* given in eq. (21) is a realistic extimate of the true continuum value the "improved" action gives a better approximation of it already at moderate values of β(f11). A similar analysis at higher values of β, because the difference between the λ's with the two actions goes to zero exponentially in β would render prohibitive the computer time needed to reduce systematic and statistical errors. The overall picture which emerges from our data is that our "improved" action works definitely better than the "naive" one. The CPU time used on CDC 7600 was approximately 65 hs.

6. - OUTLOOK.

The extension to gauge theories of our crude method to improve the lattice action is straightforward. The "naive" action is simply the Wilson (Manton, Villain, ...) action involving only the elementary plaquettes P shown in Fig. 7a:

$$S_N = - \frac{1}{2g^2} \sum_P \mathrm{tr}\left[1 - U(p)\right] \quad , \tag{29}$$

while the "improved" action is a suitable combination of elementary plaquettes and six link rectangular plaquettes P' (Fig. 7b)[4]:

$$S_I = - \frac{1}{2g^2} \left\{ \frac{5}{3} \sum_P \mathrm{tr}\left[1 - U(p)\right] - \frac{1}{12} \sum_{P'} \mathrm{tr}\left[1 - U(p')\right] \right\} \tag{30}$$

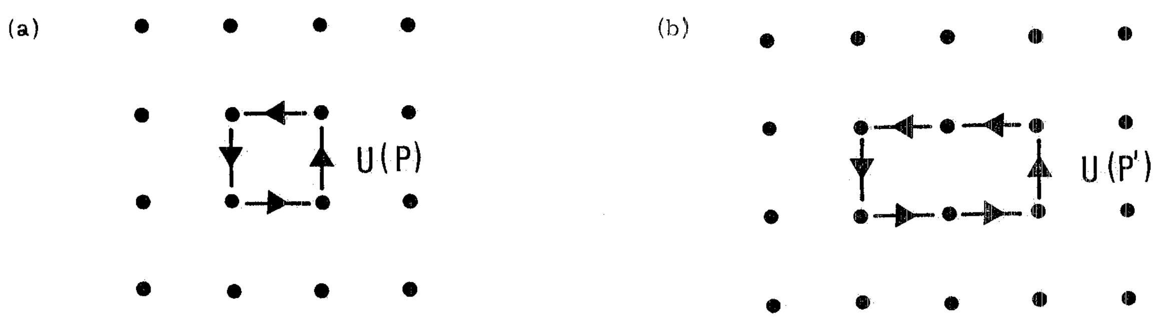

FIG. 7 - The plaquettes P (a) and P' (b) defined in the text are shown.

This implies to put to zero in the lagrangian the (still undetermined at lowest order in g) coefficients of the non planar six links plaquettes. Each link belongs to 6 square and 18 rectangular plaquettes so we loose only a factor 4 in computer time. In vue of the good results obtained for the two dimensional non linear sigma model it would be worthwhile to try to measure with the action of eq. (30) the ratio of the glueball O^{++} mass over Λ_L (obtained by measuring the string tension) as a function of $1/g^2$.

FOOTNOTES

(f1) - They could be studied within some approximation e. g. the 1/N expansion.

(f2) - In the framework of real space renormalization group a similar approach has been already tried for gauge theories by Wilson[6].

(f3) - No one loop computation has been yet done for gauge theories but the work is in progress[7].

(f4) - Because of a misprint the next to neighbor interaction as well as the coefficients $A_{1,2,3}$ in eq. (4) of ref. (2) had a wrong sign.

(f5) - In the infinite volume limit.

(f6) - Because the results of ref. (17) were not available to us, we computed the first six orders of the strong coupling series for ξ^2 (defined trough the second moment of the two point function) on a square lattice using the diagrams given in ref. (18). We thank F. Nicodemi and R. Pettorino for a useful check of our computation with their results for ξ up to the fifth order.

(f7) - This value has been obtained at $\beta \sim 1$ where the Padè are almost constant in β. Note that the (3, 2) approximant gives a constant $\lambda(\beta)$ for $\beta \to \infty$.

(f8) - Note that the approach of the coefficients to their asymptotic expressions is rather slow as we checked also in the three dimensional Ising model where also the coefficient of the sixth order is computed.

(f9) - This value is in agreement with the value reported in ref. (24).

(f10) - Note the large prefactor $(L/\xi a)^2$ in eq. (25).

(f11) - This is true also in the $N \to \infty$ limit. Lo Musto, F. Nicodemi and R. Pettorino, private communication.

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