ISTITUTO NAZIONALE DI FISICA NUCLEARE
Laboratori Nazionali di Frascati

LNF-82/67(R)
10 Settembre 1982
(Revised Version 4 Marzo 1982)

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PART III: ANALYTICAL EXPRESSIONS

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RAY TRACING CALCULATIONS FOR THE PULS FACILITY

PART III: ANALYTICAL EXPRESSIONS

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1. - INTRODUCTION

The ray tracing method is commonly employed for studying and improving the imaging properties of an optical system(1). With the development of X-ray optics and synchrotron radiation facilities, off-axis optical systems formed by mirrors operating at grazing incidence are mandatory for having a high transfer of intensity. However, in off-axis systems the aberrations are large and set a strong limitation to the imaging properties of the mirrors. Often, only one or two mirrors are used for focusing the radiation emitted by the electrons orbiting in the electron accelerator (storage ring) onto the entrance slit of a monochromator. In such a case the use of a ray tracing program, such as that described in the second report of this series(2) (hereafter referred to as R2) is a complicated method for obtaining the most important informations on the image of the source: its shape, the aberrations and the intensity distribution.

Some authors have calculated analytically the aberrations of a toroidal mirror, but they used several approximations(3). In the first report of this series(4) (hereafter referred to as R1), we have derived the mirror equations and an explicit expression of the aberrations in the meridian plane in the case of a source lying on the central ray, by expanding the ray tracing equations in a power series of ϑ, the vertical divergence of the radiation beam. In synchrotron radiation facilities both θ and φ are at most a few milliradians and the transverse dimensions of the source are small with respect to its distance from the mirror. For this reason, we have expanded the equations of the three-dimensional ray tracing reported in R2 for a toroidal mirror, as a power series of the four variables θ, φ, $\frac{\eta}{s}$ and $\frac{\zeta}{s}$ up to quadratic terms (we have seen in R1 that third order terms give a negligible contribution to the aberrations at the image)(*). In this way we get explicit expressions for the cross section of the radiation beam and for the intensity distribution, that are reported and discussed in the next sections.

(*) In this report we shall use the same nomenclature and symbols as in R2, without redefining them.

2. - EXPANSION OF THE BASIC EQUATIONS

In this section we shall recall the fundamental expressions of the ray tracing derived in R2 and we shall give their expansions to second order terms, neglecting all the required lengthy, tedious calculations. Only the preliminary calculations are reported with some detail as an example of the procedure adopted.

The rays leaving the source can be written as:

$$\frac{y}{s} = \frac{\eta}{s} + \frac{m_i}{l_i}\left(\frac{x}{s} + 1\right), \tag{1a}$$

$$\frac{z}{s} = \frac{\zeta}{s} + \frac{n_i}{l_i}\left(\frac{x}{s} + 1\right), \tag{1b}$$

with the direction cosines given by Eq. (3) of R2:

$$l_i = \cos\theta\cos\varphi \approx 1 - \frac{1}{2}(\theta^2 + \varphi^2),$$

$$m_i = \cos\theta\sin\varphi \approx \varphi, \tag{2}$$

$$n_i = \sin\theta \approx \theta.$$

From Eqs. (2) we have that $m_i/l_i = \mathrm{tg}\varphi \approx \varphi$ and that $n_i/l_i = \mathrm{tg}\,\theta\cos\varphi \approx \theta(1 - \frac{1}{2}\varphi^2) \approx \theta$. With the aid of these two relations, Eqs. (1) can be rewritten as:

$$\frac{y}{s} = \varphi\,\frac{x}{s} + \frac{\eta}{s} + \varphi, \tag{1c}$$

$$\frac{z}{s} = \theta\frac{x}{s} + \frac{\zeta}{s} + \theta\;. \tag{1d}$$

The coordinates of the point of intersection between the rays incident on the mirror and the mirror surface are obtained first by solving Eq. (22) of R2, that yields x_c, and then by inserting x_c into Eqs. (1c) and (1d). If we introduce $X = x_c/sl_i$, Eq. (22) of R2 becomes:

$$X^4 + 4PX^3 + 2X^2\left[2P^2 + Q - 2\left(\frac{R-r}{s}\right)^2(l_i^2 + n_i^2)\right] - 4X\left[PQ - 2\left(\frac{R-r}{s}\right)^2 P'\right] + \left[Q^2 - 4Q'\left(\frac{R-r}{s}\right)^2\right] = 0, \tag{3}$$

where P, Q, P' and Q' correspond to p/s, q/s^2, p'/s, and q'/s^2, respectively, and the latter quantities have been defined by Eqs. (23) of R2. P, Q, P' and Q' are expanded in powers of θ, φ, η/s, and ζ/s. For example:

$$P' \approx -\frac{R\sin i_o}{s} + \frac{R\cos i_o}{s}\,\theta - \theta^2\left(1 - \frac{R\sin i_o}{2s}\right) + \varphi^2\,\frac{R\sin i_o}{2s} - \theta\frac{\zeta}{s}\;.$$

Eq. (3) is a fourth degree equation of X. We are looking for a solution near X=0. For most practical cases such a solution will be of the order of 10^{-2} or less, and X can be considered a small quantity alike θ, φ, η/s, and ζ/s. Since we are truncating our expansions to the second order terms, we can neglect the X^4 and X^3 terms in Eq. (3), that becomes an equation of second degree:

$$a_o X^2 - 2(b_o + b_1) X + (c_o + c_1 + c_2) = 0 \; . \tag{4a}$$

The coefficients of Eq. (4a) have been written in a symbolic form, as a sum of terms, each corresponding to the power of θ, φ, η/s, and ζ/s indicated by the subscript. Obviously, from left to right, each coefficient contains powers of higher degree, so that second order terms at most are present through Eq. (4a). For example, the coefficient of X (which is already a first order term) contains a zero order term $b_o = -4rR(R-r)\sin i_o/s^3$, and a first order term $b_1 = (4R/s^2)\left[R\cos i_o \sin i_o(\theta + \frac{\zeta}{s}) + \frac{r(R-r)}{s}\theta\cos i_o\right]$.

The solution of Eq. (4a) near X=0 is easily obtained:

$$X = a_o^{-1}\left[b_o + b_1 - \sqrt{(b_o + b_1)^2 - a_o(c_o + c_1 + c_2)}\right] \approx X_o + X_1 + X_2 \; , \tag{4b}$$

where the right hand side is in symbolic form as the coefficients of Eq. (4a) and it can be expressed explicitely by further expanding the square root of Eq. (4b). Recalling that

$$\frac{x_c}{s} = X\, l_i \approx (X_o + X_1 + X_2)\left[1 - \frac{1}{2}(\theta^2 + \varphi^2)\right] \approx X_o + X_1 + X_2 - \frac{1}{2}X_o(\theta^2 + \varphi^2) \; ,$$

we have the coordinates of the intersection:

$$\begin{aligned}
\frac{x_c}{s} &\approx \operatorname{ctg} i_o\,(\theta + \frac{\zeta}{s}) - \frac{s}{2r\sin i_o}(\varphi + \frac{\eta}{s})^2 - \frac{s}{2R\sin^3 i_o}(\theta + \frac{\zeta}{s})^2 + \operatorname{ctg}^2 i_o\,\theta(\theta + \frac{\zeta}{s}) \; ,\\
\frac{y_c}{s} &\approx (\varphi + \frac{\eta}{s}) + \operatorname{ctg} i_o\,\varphi(\theta + \frac{\zeta}{s}) \; ,\\
\frac{z_c}{s} &\approx (\theta + \frac{\zeta}{s})(1 + \theta \operatorname{ctg} i_o) \; .
\end{aligned} \tag{5}$$

The knowledge of (x_c/s), (y_c/s), and (z_c/s) allows the determination of the direction cosines of the normal to the mirror, given by Eqs. (25) of R2, following the same procedure as above, and then of the tangent angle of incidence

$$\begin{aligned}
\sin i \approx\; & \sin i_o + (\frac{s}{R\sin i_o} - 1)\cos i_o\,\theta + \frac{s}{R}\operatorname{ctg} i_o \frac{\zeta}{s} + \frac{1}{\sin i_o} \cdot \\
& \cdot\left[\frac{s}{R\sin i_o}\theta(\theta + \frac{\zeta}{s}) - \frac{s^2}{2R^2\sin^2 i_o}(\theta + \frac{\zeta}{s})^2 - \frac{s^2}{2Rr}(\cos^2 i_o + \frac{R}{r}\sin^2 i_o) \right.\cdot\\
& \left. \cdot(\varphi + \frac{\eta}{s})^2 + \frac{s\sin i_o}{r}\varphi(\varphi + \frac{\eta}{s}) - \frac{1}{2}\sin^2 i_o(\theta^2 + \varphi^2)\right] \; .
\end{aligned} \tag{6}$$

In Eq. (6) the zero order term is the tangent angle of incidence of the central ray and the other terms represent the corrections due to the divergence of the beam, the shifted position of the source and the curvature of the reflecting surface.

The direction cosines of the reflected rays in the rotated coordinate system Σ', obtained from Eqs. (9) and (12) of R2, are given by:

$$l'_r \approx 1 - \frac{1}{2}(\theta^2 + \varphi^2) - \frac{2\,s^2}{R^2 \sin^2 i_o}\,(\theta + \frac{\zeta}{s})^2 + \frac{2\,s}{R \sin i_o}\,\theta\,(\theta + \frac{\zeta}{s}) - \frac{2\,s^2 \sin^2 i_o}{r^2}\,(\varphi + \frac{\eta}{s})^2 + \tag{7a}$$

$$+ \frac{2\,s}{r} \sin i_o\, \varphi\,(\varphi + \frac{\eta}{s})\ ,$$

$$m'_r \approx \varphi - \frac{2\,s}{r} \sin i_o\,(\varphi + \frac{\eta}{s}) - \frac{2\,s^2}{rR} \operatorname{ctg} i_o\,(\varphi + \frac{\eta}{s})\,(\theta + \frac{\zeta}{s}), \tag{7b}$$

$$n'_r \approx -\theta + \frac{2\,s}{R \sin i_o}\,(\theta + \frac{\zeta}{s}) - \frac{s^2}{R^2}\,\frac{\cos i_o}{\sin^3 i_o}\,(\theta + \frac{\zeta}{s})^2 - \frac{s^2}{rR} \operatorname{ctg} i_o$$

$$\cdot\,(1 + \frac{2R}{r} \sin^2 i_o)\,(\varphi + \frac{\eta}{s})^2 + \frac{2\,s}{R \sin i_o} \operatorname{ctg} i_o\,\theta\,(\theta + \frac{\zeta}{s}) + \frac{2\,s}{r} \cos i_o\,\varphi\,(\varphi + \frac{\eta}{s})\,. \tag{7c}$$

Finally, the expressions of the coordinates of the intersection of the reflected rays with planes x'=constant, that give the imaging properties of the mirror, are given by:

$$y' \approx y_1 + y_2$$
$$z' \approx z_1 + z_2 \tag{8}$$

where, as for Eq.(4a), subscripts 1 and 2 mean first and second order terms of θ, φ, η/s, and ζ/s, respectively. Note that also in the present case, as in Eq. (12) of R1, we do not find zero order terms: when all the variables tend to zero also y' and z' must be identically zero (central ray and central source).

The explicit expressions of y_1, y_2, z_1 and z_2 are:

$$y_1 = \varphi\, s \left[1 + (1 - \frac{2s}{r} \sin i_o)\, \frac{x'}{s}\right] + \eta\,(1 - \frac{2s}{r} \sin i_o\, \frac{x'}{s}), \tag{9a}$$

$$z_1 = -\theta\, s \left[1 + (1 - \frac{2s}{R \sin i_o})\,\frac{x'}{s}\right] - \zeta\,(1 - \frac{2s}{R \sin i_o}\,\frac{x'}{s}), \tag{9b}$$

$$y_2 = \frac{2s^2}{r} \operatorname{ctg} i_o\,(\sin i_o - \frac{s}{R}\,\frac{x'}{s})\,(\theta + \frac{\zeta}{s})\,(\varphi + \frac{\eta}{s}), \tag{9c}$$

$$z_2 = -x' \operatorname{ctg} i_o \left[\frac{s^2}{R^2 \sin^2 i_o}\,(\theta + \frac{\zeta}{s})^2 + \frac{s^2}{rR}\,(1 + \frac{2R}{r} \sin^2 i_o)\,(\varphi + \frac{\eta}{s})^2 - \right.$$

$$\left. - \frac{2s}{R \sin i_o}\,\theta\,(\theta + \frac{\zeta}{s}) - \frac{2s}{r} \sin i_o\,\varphi(\varphi + \frac{\eta}{s})\right] - s^2 \operatorname{ctg} i_o \left[\frac{1}{R \sin i_o}\ \cdot \right. \tag{9d}$$

$$\left. \cdot\,(\theta + \frac{\zeta}{s})^2 - \frac{x \sin i_o}{r}\,(\varphi + \frac{\eta}{s})^2\right].$$

Together with Eqs. (7), they represent the explicit expressions of the ray tracing.

3. - DISCUSSION

If the source is on the central ray,i.e. $\eta = \zeta=0$, in Eqs. (7c), (9b) and (9d), corresponding to the z coordinate and associated variation, θ appears in both the linear and the quadratic terms, while φ appears only as φ^2. This corresponds to the symmetry of the system with respect to the meridian plane, so that, changing the sign of φ does not alter the results. If we let also $\varphi \rightarrow 0$, we get the meridian plane case and we recover the equations of R1. In fact, for example, Eqs. (5) become:

$$x_c = s \operatorname{ctg} i_o \, \theta \left[1 + \theta \left(\operatorname{ctg} i_o - \frac{s}{2 R \sin^2 i_o \cos i_o} \right) \right],$$

$$y_c = 0 \quad ,$$

$$z_c = \theta s (1 + \theta \operatorname{ctg} i_o),$$

that are the same as Eqs. (4a) and (5a) of R1.

As we have already discussed in Sect. 3 of R1, the linear terms of Eqs. (8), i.e. Eqs. (9a) and (9b), give the focusing properties of the toroidal mirror in the sagittal and in the meridian planes, respectively. From Eq. (9b) we find that in the meridian plane the source is imaged at a distance $x'=q_{//}$ from the mirror, such that the coefficient of θ is zero, yielding the relation:

$$\frac{1}{q_{//}} + \frac{1}{s} = \frac{2}{R \sin i_o} \, , \tag{10}$$

which is the same as Eq. (13) of R1. The dimensions of the image of a finite source are given by the second term of Eq. (9b):

$$z_1 = \frac{q_{//}}{s} \zeta = M_{//} \zeta, \tag{11}$$

where we have introduced $M_{//}= \frac{q_{//}}{s}$, the magnification in the meridian plane of the mirror.

The sagittal image is found at $x'=q_{\perp}$, such that the coefficient of φ in Eq. (9a) is zero, yielding:

$$\frac{1}{q_{\perp}} + \frac{1}{s} = \frac{2 \sin i_o}{r} \, . \tag{12}$$

By introcing the sagittal magnification $M_{\perp} = \frac{q_{\perp}}{s}$, the dimension of the image of a finite source is given by:

$$y_1 = - M_{\perp} \eta \, . \tag{13}$$

As we can see, the linear terms of the expressions derived in Sect. 3 contain all the informations relative to the imaging properties of the toroidal mirror. Eqs. (10) and (12) are well-known formulae, derived previously from more general principles, as shown, for example, in Ref. (5). At normal incidence (i_o=90°) we obtain again the familiar expressions of Gauss optics.

For given values of R, r and s, in general $q_{//}$ and $q_{\perp}$ do not coincide. Thus, in the limit of a point source, the image in the position of $q_{//}$ ($q_{\perp}$) is a straight segment perpendicular (parallel) to the meridian plane. The length of the segment is given by $y_1=1-(q_{//}/q_{\perp})$ ($z_1=1-(q_{\perp}/q_{//})$). At normal incidence, for spherical mirrors, it is well known that $q_{//}=q_{\perp}$. The same condition can be achieved also at non normal incidence by the use of toroidal mirrors by choosing properly R and r. For fixed positions of the source and the image with respect to the mirror and for a certain angle of incidence, R and r are obtained by solving Eqs. (10) and (12), with $q_{/\!/}=q_{\perp}=q$ and the two radii satisfying the relation $r/R=\sin^2 i_o$. The two magnifications are coincident as well.

The second order terms give the aberrations of the toroidal mirror. Below we shall consider only the case $q_{/\!/}=q_{\perp}=q$. At the image $x'=q$ Eqs. (9c) and (9d) become

$$y_2 = \frac{s\ \mathrm{ctg}\ i_o}{2}\ \frac{1-M^2}{M}\ (\theta+\frac{\zeta}{s})(\varphi+\frac{\eta}{s})\ , \tag{14a}$$

$$z_2 = \frac{s\ \mathrm{ctg}\ i_o}{4}\ \frac{M+1}{M}\ \Big[\,3\,\theta^2(M-1)+\varphi^2(M-1)-\frac{\zeta^2}{s^2}(M+3) -\frac{\eta^2}{s^2}(3M+1)+\frac{2\theta\zeta}{s}(M-3)-\frac{2\varphi\eta}{s}(M+1)\Big]\ . \tag{14b}$$

Eqs. (14) contain the contributions to the aberrations from both the divergence of the beams and the finite dimensions of the source. In R1 we have studied the aberrations from the divergence of the beam only, since the finite dimensions of the source give a finite image. For this reason, below we shall consider the case $\eta=\zeta=0$, and Eqs. (14) simplify to

$$y_2 = -2\,A\,\theta\,\varphi\ , \tag{15a}$$

$$z_2 = A\,(3\,\theta^2+\varphi^2), \tag{15b}$$

with

$$A = \frac{(M^2-1)}{4M}\,s\ \mathrm{ctg}\ i_o\ . \tag{16}$$

According to Eq. (15a) the aberration extends symmetrically with respect to the meridian plane. Instead, from Eq. (15b), the aberration lies on the same side of the sagittal plane containing the central ray, with $z'>0$ when $M>1$, and $z'<0$ when $M<1$. We note also that Eqs. (15) are almost symmetric in the two variables θ and φ. In the special case of either θ or $\varphi=0$, $y_2=0$ and the image is only a short segment in the meridian plane. The reason for this is trivial for $\varphi=0$. In the case of $\theta=0$ it is a consequence of the grazing incidence. Finally, for $M=1$ the coefficient A, expressed by Eq. (16), is zero and the aberrations cancel out. By choosing either θ or φ as a parameter, Eqs. (15) can be joined together to give an analytical relationship between z_2 and y_2, that holds for either θ or φ or both different from zero:

$$z_2 = A\,(3\,\theta^2+\frac{y_2^2}{4\,A^2\,\theta^2})\ . \tag{17}$$

Eq. (17) describes a family of parabolas of increasing aperture and with vertex moving further away from the central ray for increasing θ. The envelope of this family of parabolas is given by two half-lines of equation

$$z = +\sqrt{3}\,y \qquad y>0\ ,$$
$$z = -\sqrt{3}\,y \qquad y<0\ . \tag{18}$$

All the aberration extends above these two half-lines for $A>0$. For $A<0$, Eqs. (15) change for the sign of z only.

Assuming a uniform distribution of intensity among the rays leaving the source, the intensity per unit area of the cross section of the beam is given by:

$$\frac{dI}{d\sigma} \propto \frac{d\theta\, d\varphi}{dy\, dz} = \left| J \right|^{-1}. \tag{19}$$

The Jacobian J follows immediately from Eqs. (15):

$$\left| J \right| = 4 \left| A^2(3\,\theta^2 - \varphi^2) \right| = 4 \left| A \right| \sqrt{z^2 - 3\,y^2}.$$

Thus

$$\frac{dI}{d\sigma} \propto \left| \frac{1}{4A} \right| (z^2 - 3\,y^2)^{-1/2}. \tag{20}$$

The condition of reality of the square root of Eq. (20) gives again the envelope function of the family of parabolas expressed by Eq. (17). For $A > 0$, all the radiation lies above the half-lines given by Eq. (18). The situation is symmetric with respect to the y axis for $A < 0$. The integration of Eq. (20) over a certain area, that may represent an aperture inserted along the beam path, such as the entrance slit of a monochromator, gives the fraction of intensity transmitted through that aperture, a quantity very important to know in designing and/or comparing optical systems.

Let us consider now the direction cosines of the reflected rays. Inserting Eqs. (10) and (12) into the linear terms of Eqs. (7), we obtain:

$$l'_r = 1 - \frac{1}{2}\left(\frac{\theta^2 + \varphi^2}{M^2}\right) \qquad \eta = \zeta = 0;$$

$$m'_r \approx -\frac{\varphi}{M} - \frac{M+1}{M}\,\frac{\eta}{s}, \tag{21}$$

$$n'_r = \frac{\theta}{M} + \frac{M+1}{M}\,\frac{\zeta}{s}.$$

As in R1, we find again the well-known result that the angles of convergence of the reflected rays are the same as those of the incident rays, divided by the magnification of the mirror. An extra term arises if the source does not lie on the central ray ($\eta, \zeta \neq 0$). Neglecting such a contribution, the quadratic terms of Eqs. (7) give a deviation from perfect imaging and thus are responsible for the aberrations. In fact, in the case of perfect imaging, n'_r should equal $\sin\theta' \sim \theta' - \theta'^3$, and the quadratic term should be missing from its expansion. The negative sign of m'_r means that the rays, that leave the source with positive values of φ, after being focused, leave the image with a negative φ'. This is well known for Gauss optics. Obviously, this is not the case for θ at grazing incidence. In this case, the rays leaving the source with positive θ, leave also the image with positive θ'.

4. - CYLINDRICAL MIRRORS

The focusing properties of a single toroidal mirror can be approximated well with two cylindrical mirrors curved in the meridian and in the sagittal planes respectively. In this way it is possible to obtain a good image of the source with different magnifications in the two planes, with the advantage of matching more closely the f-number of a monochromator when the maximum divergence from the source is not the same for θ and φ. Below we give the main equations for a cylindrical mirror in both configurations.

4.1. - Curvature in the meridian plane

With this configuration, the mirror focuses in the meridian plane and behaves as a flat mirror in the sagittal plane. The main equations can be derived from those of Sect. 2 for the toroidal mirror by letting $r \rightarrow \infty$. x_c, z_c

and related "meridian" quantities should not depend on φ, at least in the first order of approximation. In fact:

$$\frac{x_c}{s} = \mathrm{ctg}\, i_o \,(\theta + \frac{\zeta}{s}) - \frac{s}{2\,R\,\sin^3 i_o}\,(\theta + \frac{\zeta}{s})^2 + \mathrm{ctg}^2 i_o\, \theta\,(\theta + \frac{\zeta}{s}) \;. \tag{22}$$

The direction cosines of the reflected rays and the reflected beam cross section are given by:

$$\begin{aligned}
l'_r &= 1 - \frac{1}{2}\,\frac{\theta^2}{M^2} - \frac{1}{2}\varphi^2 - \frac{1}{2}(\frac{M+1}{M}\,\frac{\zeta}{s})^2 - \frac{M+1}{M}\,\frac{\zeta}{s}\,\frac{\theta}{M} \,, \\
m'_r &= \varphi \,, \\
n'_r &= \frac{\theta}{M} + \frac{M+1}{M}\,\frac{\zeta}{s} - (\frac{M+1}{4M})\,\mathrm{ctg}\, i_o\,(\theta + \frac{\zeta}{s})\,(- \frac{3\,M+1}{M}\,\theta + \frac{M+1}{M}\,\frac{\zeta}{s}) \,, \\
y_1 &= \varphi\,(s + x') + \eta \,, \\
y_2 &= 0 \,, \\
z_1 &= -\theta\,(s - \frac{x'}{M}) - \zeta(1 - \frac{M+1}{M}\,\frac{x'}{s}) \,, \\
z_2 &= -x'\,\mathrm{ctg}\, i_o\,\frac{M+1}{4M^2}(\theta + \frac{\zeta}{s})\left[(M+1)\frac{\zeta}{s} - (3\,M-1)\,\theta\right] - \frac{s}{2}\,\mathrm{ctg}\, i_o\,(\theta + \frac{\zeta}{s})^2\,\frac{M+1}{M}
\end{aligned} \tag{23}$$

The expressions of m'_r and $y'=y_1+y_2$ in Eqs. (23) correspond to rays not deflected by the mirror and thus aberration free (note $y_2=0$). They are the same as m'_r and y' in Eqs. (18) and (19) of R2, derived for a flat mirror. Thus, as expected, this position of the cylindrical mirror yields only the meridian focus, with the aberration given by the same expression as Eq. (16) of R1, as one can verify easily putting $\zeta=0$ and $x'=q$. The magnification in this case is defined only for the meridian plane. Thus, in deriving Eqs. (23), we had to start from Eqs. (7) and (9) of Sect. 2, rather than from the equations of Sect. 3.

4.2. - Curvature in the sagittal plane

This mounting of a cylindrical mirror focuses in the sagittal plane and should behave as a flat mirror in the meridian plane. The imaging equations are obtained from those of Sect. 2 of the toroidal mirror by letting $R \to \infty$:

$$\begin{aligned}
l'_r &= 1 - \frac{1}{2}\theta^2 - \frac{1}{2}\,\frac{\varphi^2}{M^2} - \frac{\varphi}{M}\,\frac{M+1}{M}\,\frac{\eta}{s} - \frac{1}{2}\,(\frac{M+1}{M})^2\,(\frac{\eta}{s})^2 \,, \\
m'_r &= -\frac{\varphi}{M} - \frac{M+1}{M}\,\frac{\eta}{s} \,, \\
n'_r &= -\theta + \frac{M+1}{M}\,\varphi\,(\varphi + \frac{\eta}{s}) \,, \\
y_1 &= \varphi\,(s - \frac{x'}{M}) + \eta\,(1 - \frac{M+1}{M}\,\frac{x'}{s}) \,, \\
y_2 &= \frac{M+1}{M}\,s\,\mathrm{ctg}\, i_o\,(\theta + \frac{\zeta}{s})\,(\varphi + \frac{\eta}{s}) \,, \\
z_1 &= -\theta\,(s + x') - \zeta \\
z_2 &= x'\,\frac{M+1}{M}\,\mathrm{ctg}\, i_o\,\varphi\,(\varphi + \frac{\eta}{s}) + \frac{1}{2}\,s\,\mathrm{ctg}\, i_o\,\frac{M+1}{M}\,(\varphi + \frac{\eta}{s})^2
\end{aligned} \tag{24}$$

From Eqs. (24), the first order term of n'_r and z_1 correspond to Eqs. (18) and (19) of R2 for a flat mirror as expected. Unlike the previous mounting with the curvature in the meridian plane, second order terms are still present here ($z_2 \neq 0$), since the plane of incidence is tilted with respect to the meridian plane. However, these terms are much smaller than the first order terms and we can neglect their contributions. The imaging properties are given by y_1, with a sagittal image position and a sagittal magnification.

ACKNOWLEDGEMENTS

We acknowledge Prof. F. Bassani, Director of PULS, and Prof. R. Scrimaglio, Director of the LNF, for their interest in this work. We thank also Dr. A. Savoia for several stimulating discussions. PULS is a research program run jointly by the Italian National Research Council and the Italian National Institution for Nuclear Physics.

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