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Y. Srivastava: RADIATION AND NOISE

Lectures delivered at Escuela Latino Americana  
de Fisica, Cali, Colombia, June 1982

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LECTURE 1 - INTRODUCTION

The central theme of these lectures may be summed succinctly as follows :

Radiation is Noise

and in particular,

Low-frequency Noise is given by Soft Radiation.

Thus, we shall study noise through a description of the radiation spectrum corresponding to the specific process under consideration. But, what do I mean by radiation? By radiation is meant the emission (and/absorption) of appropriate massless quanta. It is the photon in quantum electrodynamics (QED), gluons in quantum chromo-dynamics (QCD) and the graviton in quantum gravodynamics (QGD). (Perhaps also, the phonon for quantized acoustical excitations).

Hopefully by the end of these lectures you will see through explicit computations in a wide variety of phenomena ranging from Josephson junctions and Electronic shot noise to high energy electron scattering, super high energy hadronic processes, gravitational energy loss, that it is the same mechanism at play. Consequently, the results are rather similar: one finds a "power" spectrum  $\sim (1/\omega)^{1-\beta}$  for low frequencies  $\omega$  and one is left ubiquitously with Poisson distributions. The latter helps us understand at a microscopic level why generically stochastic descriptions are applicable (and successful) for noise.

Before discussing the applications, let us take time out to discuss the various underlying theories: QED, QCD and QGD (Einstein theory). Usually, they are studied separately, but it is possible to give a general formalism - in terms of connections - which is applicable to them

all. Now, we wouldn't need much of it for later discussions, but this is a school and you are supposed to be learning new things and it is so beautiful . . . .

Making Connections :

There is a deep relationship between an underlying continuous symmetry and existence of massless quanta (gauge particles). The well-known argument goes as follows :

Consider a field  $\psi(x)$  which transforms as a "tensor" under a gauge group G. That is let

$$\psi'(x) \equiv U(x) \psi(x) \tag{1.1}$$

$$= e^{i \lambda_a(x) T_a} \psi(x) \tag{1.2}$$

where  $T_a$ 's are generators of G.  $\lambda_a(x)$  are called the parameter of the gauge transformation.

Now, if we wish to construct an invariant action for the fields  $\psi$  through a Lagrange density  $\mathcal{L}(x)$ , which is a functional of polynomials and derivatives of  $\psi$ , we must require that  $\mathcal{L}(x)$  be a scalar under G. But,  $\partial_\mu \psi \equiv \frac{\partial}{\partial x^\mu} \psi(x)$ ,  $x^\mu = (x^0, \underline{x})$ , does not transform as a tensor even though  $\psi$  does. Hence the need for a covariant derivative (and of a connection). Introduce  $\Gamma_\mu(x)$ , a connection such that the covariant derivative of  $\psi$ ,

$$D_\mu \psi \equiv (\partial_\mu + g \Gamma_\mu) \psi \tag{1.3}$$

transforms under G as does  $\psi$  ( $g$  is the charge of  $\psi$ ) that is, we require

$$(D_\mu \psi)' = U(D_\mu \psi). \tag{1.4}$$

Using (1.1), (1.3) and (1.4) we discover then how the connection itself must transform. It is easily seen that

$$\Gamma'_\mu = U \Gamma_\mu U^{-1} + \frac{1}{g} (\partial_\mu U) U^{-1} = U \Gamma_\mu U^{-1} - \frac{1}{g} U (\partial_\mu U^{-1}). \tag{1.5}$$

It is important to note that the connection  $\Gamma_\mu$  is itself not a tensor, due to the "displacement" or inhomogeneous term in eq. (1.5).

Examples :

1. QED: Here the gauge group G is abelian, so :

$$U(x) = e^{ie\lambda(x)} \quad \text{and customarily} \quad \Gamma_\mu = iA_\mu(x),$$

where  $A_\mu$  is the EM vector potential, which transforms as

$$A'_\mu(x) = A_\mu(x) - \partial_\mu \lambda(x) \tag{1.6}$$

where  $\lambda(x)$  is the gauge function.

2. QCD: Here the gauge group G is SU(3) color and  $\psi(x)$  are the colored quarks (transforming as the fundamental representation  $\underline{3}$ ) and we may write

$$U(x) = e^{ig \sum_{a=1}^8 \lambda_a(x) T_a} . \quad (1.7)$$

The 8 generators of SU(3),  $T_a$  obey the Lie algebra

$$[T_a, T_b] = if_{abc} T_c \quad (1.8)$$

where  $f_{abc}$  are the structure constants of SU(3). The connection  $\Gamma_\mu(x) = iA_{\mu a}(x) T_a$ , then introduces 8 color gauge potentials  $A_{\mu a}(x)$ , the so called gluon fields.

#### Gravitation:

The theory of Gravitation can be viewed as a gauge theory of coordinate transformations, i. e. a theory invariant under any reparametrization of the coordinates,  $x^\mu \rightarrow x'^\mu = x'^\mu(x)$ . The gauge group is now GL(4, R), the general linear group of real transformations in 4-dim. The generators  $G_\mu^\nu$  satisfy the Lie algebra

$$[G_\mu^\nu, G_\lambda^\sigma] = \delta_\lambda^\nu G_\mu^\sigma - \delta_\mu^\sigma G_\lambda^\nu . \quad (1.9)$$

For the connection  $\Gamma_\mu$ , we obtain

$$\Gamma_\mu'(x') = \left( \frac{\partial x^\nu}{\partial x'^\mu} \right) \left[ U \Gamma_\nu U^{-1} + U \partial_\nu U^{-1} \right] . \quad (1.10)$$

This differs from eq. (1.5) in the extra differential which "compensates" for the change in the coordinate frame where derivatives are taken. (A crucial difference between "internal" symmetries in which coordinates themselves do not change and space-time symmetries where they do). The more familiar Christoffel symbol  $\Gamma_{\lambda\mu}^\nu$  is related to the gravitational connection  $\Gamma_\mu$  as

$$\Gamma_\mu = G_\nu^\lambda \Gamma_{\lambda\mu}^\nu + \tau_\mu \quad (1.11)$$

where  $\tau_\mu$  is any arbitrary tensor under GL(4, R) which can be added. In the Einstein theory, without torsion,  $\tau_\mu = 0$ .

Writing,

$$U = e^{G_\mu^\nu \omega_\nu^\mu(x)} \quad (1.12)$$

where  $\omega_\nu^\mu$  is related to infinitesimal coordinate transformations,

$$x'^\mu \simeq x^\mu + \xi^\mu(x) \quad \text{through} \quad \omega_\nu^\mu(x) \simeq \partial_\nu \xi^\mu(x) . \quad (1.13)$$

We have

$$(1.14)$$

Now, we have the tools necessary for obtaining a gauge invariant  $\mathcal{L}(x)$  for  $\psi(x)$ : any functional  $\mathcal{L}(\psi, D\psi)$  will do, e. g. in QED, QCD

$$\mathcal{L}_{\text{Fermion}} = i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi \quad (1.15)$$

where  $\psi$  may be a lepton (electron, muon, ...) field when  $\not{D} = (\partial_\mu - ieA_\mu)\gamma^\mu$ , or  $\psi$  may be a quark field and for QCD,  $\not{D} = (\partial_\mu - igA_{\mu a}T_a)\gamma^\mu$ , where  $a$  is the color index and  $g$  is the strong coupling constant.

How do we introduce the kinetic energy for the gauge fields, i. e. what is  $\mathcal{L}_{\text{gauge}}$ ? Once again we construct a gauge invariant quantity,  $F_{\mu\nu}(x)$ , the field strength tensor, made out of non-gauge invariant connection  $\Gamma_\mu$ . We form

$$F_{\mu\nu}(x) = \partial_\mu \Gamma_\nu(x) - \partial_\nu \Gamma_\mu(x) + g[\Gamma_\mu(x), \Gamma_\nu(x)]. \quad (1.16)$$

Using (1.5) it is easy to show that

$$F'_{\mu\nu}(x) = U F_{\mu\nu}(x) U^{-1} \quad (1.17)$$

i. e.  $F_{\mu\nu}$  is a tensor. Thus, any scalar formed out of  $F_{\mu\nu}$  would do for a gauge invariant Lagrangian. If we want only second order derivative in  $\mathcal{L}_{\text{gauge}}$ , there is a unique choice

$$\mathcal{L}_{\text{gauge}}(x) = -\frac{1}{4} \text{tr}(F_{\mu\nu}(x)F^{\mu\nu}(x)). \quad (1.18)$$

For QED,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (1.19)$$

and

$$\mathcal{L}_{\text{QED}} = (\vec{E}^2 - \vec{B}^2) \quad (1.20)$$

in terms of electric field  $\vec{E}$  and magnetic field  $\vec{B}$ . For QCD

$$F_{\mu\nu} = F_{\mu\nu a}(x) T_a, \quad (1.21)$$

we have

$$F_{\mu\nu a}(x) = \partial_\mu A_{\nu a}(x) - \partial_\nu A_{\mu a}(x) - gf_{abc}A_{\mu b}(x)A_{\nu c}(x). \quad (1.22)$$

In eq. (1.22)  $F$  is non-linear in  $A$  which makes it difficult to solve. If there were no "matter" (i. e. no charged particles),  $\mathcal{L}_{\text{QED}}$  given by eq. (1.20) would be a free field theory. So that if you had a bag of photons, it would be a bag of free photons. (Photons carry no charge

and hence do not feel each other). Not so for QCD - it is a non-abelian theory - the gluons themselves carry (color) charge and  $\mathcal{L}_{\text{QCD}}$  is an interacting field theory. Gravitons also interact with each other (The Einstein  $\mathcal{L}$  is not given by (1.18), however).

## LECTURE 2 - SOFT RADIATION FORMULAE AND POISSON DISTRIBUTION

In this lecture, I would like to obtain the spectrum of soft radiation and show how a Poisson distribution arises generally. Since it is an important result we shall discuss it in a variety of ways.

What we would do first is solve a fixed source field theory and find that the probability that  $n$  particles are emitted is Poisson distributed<sup>(1)</sup> (For this lecture  $\hbar = c = 1$ ).

Consider a neutral scalar field of mass  $\mu$ , for which the free Lagrangian density is

$$\mathcal{L}_0(x) = \frac{1}{2} (\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2} \mu^2 \varphi^2, \quad (2.1)$$

and which is coupled to a given (c-number) source density  $\varrho(x)$  (Physically this means that the recoil of the source is being neglected, since its motion is prescribed).

Now the quantum field operator  $\Phi(x)$  satisfies

$$\Phi(x) = \Phi^{\text{in}}(x) + \int (d^4x') \varrho(x') D_r(x-x') \quad (2.2a)$$

$$= \Phi^{\text{out}}(x) + \int (d^4x') \varrho(x') D_a(x-x') \quad (2.2b)$$

where

$$(\square + \mu^2) \Phi^{\text{in/out}}(x) = 0 \quad (2.3)$$

$$\lim_{t \rightarrow -\infty} \Phi(x) = \Phi^{\text{in}}(x); \quad \lim_{t \rightarrow +\infty} \Phi(x) = \Phi^{\text{out}}(x) \quad (2.4)$$

$D_{r,a}(x)$  are the retarded and advanced propagators:

$$(\square + \mu^2) D_{r,a}(x) = \delta^4(x). \quad (2.5)$$

The  $(\pm)$  frequency components of  $\Phi$  of course obey the free field commutation relations.

E. g. ,

$$\left[ \Phi^{(+)\text{in}}(x), \Phi^{(-)\text{in}}(x') \right] = -i D(x-x') \quad (2.6)$$

$$= - \frac{1}{(2\pi)^3} \int (d^4k) \delta(k^2 - \mu^2) \varepsilon(k^0) e^{ik \cdot x}, \quad (2.7)$$

where

$$\varepsilon(k^0) = \left[ \theta(k^0) - \theta(-k^0) \right], \quad \text{and} \quad \theta(k^0) = \begin{cases} 1 & k^0 > 0 \\ 0 & k^0 < 0. \end{cases}$$

From (2.2) we get

$$\begin{aligned} \Phi^{\text{out}}(x) &= \Phi^{\text{in}}(x) + \int (d^4x') \left[ D_P(x-x') - D_A(x-x') \right] \varrho(x') \\ &= \Phi^{\text{in}}(x) - \int (d^4x') D(x-x') \varrho(x') \end{aligned} \quad (2.8)$$

$$\equiv S^{-1} \Phi^{\text{in}}(x) S, \quad (2.9)$$

where  $S$  is the (unitary)  $S$ -matrix.  $S$  must be unitary, since  $\Phi^{\text{in}}(x)$  satisfy the same C. R. Defining, the Fourier transforms

$$\Phi^{\text{out}}(x) = \frac{1}{(2\pi)^3} \int (d^4k) \delta(k^2 - \mu^2) \tilde{\Phi}^{\text{out}}(k) e^{ik \cdot x} \quad (2.10)$$

and

$$\varrho(x) = \frac{1}{(2\pi)^4} \int (d^4k) \tilde{\varrho}(k) e^{ik \cdot x}. \quad (2.11)$$

From eq. (2.8) we easily deduce that

$$\delta(k^2 - \mu^2) \left\{ \tilde{\Phi}^{\text{out}}(k) - \tilde{\Phi}^{\text{in}}(k) - i \tilde{\varrho}(k) \varepsilon(k^0) \right\} = 0.$$

We shall use the notation,  $\delta \tilde{\varphi}$  such that

$$\delta \tilde{\varphi}(k) \equiv \tilde{\Phi}^{\text{out}}(k) - \tilde{\Phi}^{\text{in}}(k) = -i \tilde{\varrho}(k) \varepsilon(k^0), \quad (2.12)$$

is a c-number.

The no. of quanta in the incoming wave is given by the corresponding no. operator

$$N^{\text{in}} = \frac{1}{(2\pi)^3} \int (d^4k) \delta(k^2 - \mu^2) \theta(k^0) \tilde{\Phi}^{(+)\text{in}}(k) \tilde{\Phi}^{(-)\text{in}}(k). \quad (2.13)$$

We want to find the probability  $P_n$  that  $n$  particles are produced if none were present in the initial state. Call, the (normalized) incoming state with no quanta  $|0\rangle_{\text{in}}$ ,

$$\tilde{\Phi}^{\text{in}}(k) |0\rangle_{\text{in}} = 0 \quad (2.14)$$

and

$${}_{\text{in}} \langle 0 | 0 \rangle_{\text{in}} = 1 \quad (2.15)$$

Let  $|0\rangle_{\text{out}}$  be the corresponding final state with no quanta. We can construct Fock space basis vectors in the usual manner

$$\begin{aligned} |k_1\rangle_{\text{out}} &= \frac{1}{\sqrt{1!}} \tilde{\Phi}^{(+)\text{out}}(k_1) |0\rangle_{\text{out}} \delta(k_1^2 - \mu^2), \\ |k_1, k_2\rangle_{\text{out}} &= \frac{1}{\sqrt{2!}} \tilde{\Phi}^{(+)\text{out}}(k_1) \tilde{\Phi}^{(+)\text{out}}(k_2) |0\rangle_{\text{out}} \delta(k_1^2 - \mu^2) \delta(k_2^2 - \mu^2), \\ |k_1, k_2, \dots, k_n\rangle_{\text{out}} &= \frac{1}{\sqrt{n!}} \prod_{i=1}^n \delta(k_i^2 - \mu^2) \tilde{\Phi}^{(+)\text{out}}(k_i) |0\rangle_{\text{out}}. \end{aligned} \quad (2.16)$$

In this basis, the most general n-particle final state vector may be written

$$|F_n\rangle = \frac{1}{(2\pi)^{3n}} \int \left[ \prod_{i=1}^n (d^4 k_i) \right] \psi_n(k_1, \dots, k_n) |k_1, \dots, k_n\rangle_{\text{out}} \quad (2.17)$$

where  $\psi_n(k_1, \dots, k_n)$  is a symmetric Schrödinger "wave function". The expectation value of  $N^{\text{out}}$  for this state is indeed n, i. e.

$$n = \langle F_n | N^{\text{out}} | F_n \rangle,$$

provided we have normalized  $\psi_n$ :

$$\frac{1}{(2\pi)^{3n}} \int \left[ \prod_{i=1}^n (d^4 k_i) \delta(k_i^2 - \mu^2) \right] |\psi_n(k_1, \dots, k_n)|^2 = 1. \quad (2.18)$$

$\psi_n$ 's also satisfy the completeness relation

$$\frac{1}{(2\pi)^{3n}} \prod_{i=1}^n \delta(k_i^2 - \mu^2) \psi_n^*(k_1, \dots, k_n) \psi_n(k'_1, \dots, k'_n) = \frac{1}{n!} \sum_{\substack{\text{Perm} \\ P}} \prod_{i=1}^n \delta^4(k_i - k'_i). \quad (2.19)$$

(The symmetrized  $\delta$ -function on the rhs may be replaced by the ordinary  $\delta$ -fn., if it is multiplied by a symmetric function of  $(k_1, \dots, k_n)$ ).

Now, we are ready to calculate the probability amplitude for the creation of n particles: Using eqs.(2.12) and (2.19) we have:

$$\begin{aligned} \langle F_n | 0 \rangle_{\text{in}} &= \frac{1}{(2\pi)^{3n}} \frac{1}{\sqrt{n!}} \int \left[ \prod_i (d^4 k_i) \delta(k_i^2 - \mu^2) \right] \psi_n^*(k_1, \dots, k_n) \cdot \\ &\quad \cdot \text{out} \langle 0 | (\tilde{\Phi}^{\text{in}}(k_1) + \delta\tilde{\varphi}(k_1)) (\tilde{\Phi}^{\text{in}}(k_2) + \delta\tilde{\varphi}(k_2)) \dots | 0 \rangle_{\text{in}}. \end{aligned} \quad (2.20)$$

using eq. (2.14),



$$\langle F_n | 0 \rangle_{in} = \frac{1}{(2\pi)^{3n}} \frac{1}{\sqrt{n!}} \int \left[ \prod_i (d^4 k_i) \delta(k_i^2 - \mu^2) \delta \tilde{\varphi}(k_i) \right] \psi_n^*(k_1, \dots, k_n)_{out} \langle 0 | 0 \rangle_{in}.$$

The probability

$$P_n = \left| \langle F_n | 0 \rangle_{in} \right|^2$$

is easily computed using the completeness relation (2.19). One finds

$$P_n = \frac{1}{n!} (\bar{n})^n \left| \langle 0 | 0 \rangle_{in} \right|^2, \quad (2.21)$$

where the mean number  $\bar{n}$  is given by

$$\bar{n} = \frac{1}{(2\pi)^3} \int (d^4 k) \delta(k^2 - \mu^2) \left| \delta \tilde{\varphi}(k) \right|^2. \quad (2.22)$$

Since the total probability must be 1, i. e.  $\sum_n P_n = 1$ , we must have

$$\left| \langle 0 | 0 \rangle_{in} \right|^2 = e^{-\bar{n}}. \quad (2.23)$$

Thus, the normalized probability for  $n$  particle production is given by the Poisson distribution

$$P_n = \frac{1}{n!} (\bar{n})^n e^{-\bar{n}}. \quad (2.24)$$

The multiplicity (or mean number)

$$\bar{n} = \sum_n n P_n.$$

We may also obtain<sup>(2)</sup> the total probability per unit frequency interval,  $(dP/d\omega)$ , if the maximum energy carried by a single particle is restricted to be  $E$ . Using the F. T., we have

$$\left( \frac{dP_E}{d\omega} \right) = \int_{-\infty}^{\infty} \left( \frac{dt}{2\pi} \right) e^{i\omega t} \sum_n \frac{1}{n!} e^{-\bar{n}_E} \prod_{i=1}^n \left[ \int_E \frac{(d^4 k_i)}{(2\pi)^3} \delta(k_i^2 - \mu^2) \left| \delta \tilde{\varphi}(k_i) \right|^2 e^{-ik_i^0 t} \right], \quad (2.25)$$

where subscript  $E$  in any integral denotes integrating the corresponding energy variable up to  $E$ . Thus,  $\bar{n}_E$  is given by eq. (2.22) with energy  $k^0 \leq E$ . A little algebra then leads to

$$\left( \frac{dP_E}{d\omega} \right) = \int_{-\infty}^{\infty} \left( \frac{dt}{2\pi} \right) e^{i\omega t} \exp \left\{ - \int_E \frac{(d^4 k)}{(2\pi)^3} \delta(k^2 - \mu^2) \left| \delta \tilde{\varphi}(k) \right|^2 \left[ 1 - e^{-ik^0 t} \right] \right\}. \quad (2.26)$$

Let us apply to a (point) source particle, say an electron, in QED. The scalar source function  $q(x)$  gets replaced by the vector current  $j_\mu(x)$ . The classical current is given by

$$j_{\mu}(x) = e \int_{-\infty}^{\infty} d\tau p_{\mu}(\tau) \delta(x^{\nu} - \tau p^{\nu}(\tau)). \quad (2.27)$$

Consider an electron to suddenly change its momentum (through scattering):

$$p_{\mu}(\tau) = \begin{cases} p_{\mu} & \tau < 0 \\ p'_{\mu} & \tau > 0. \end{cases} \quad (2.28)$$

In a real scattering process, this imposes the restriction that the admissible frequency range must be  $\ll 1/\tau^*$ , where  $\tau^*$  is the effective collision time. The F. T. of (2.27) gives

$$\tilde{j}_{\mu}(k) = e \left( \frac{p_{\mu}}{pk} - \frac{p'_{\mu}}{p'k} \right). \quad (2.29)$$

The average number of quanta (photons) is determined to be

$$\bar{n}_E = \int_E \frac{(d^4k)}{(2\pi)^3} \delta(k^2) \left| \tilde{j}_{\mu}^*(k) \tilde{j}^{\mu}(k) \right|. \quad (2.30)$$

The angular integrations in (2.26) can be performed and we obtain

$$\left( \frac{dP_E}{d\omega} \right) = \int_{-\infty}^{\infty} \left( \frac{dt}{2\pi} \right) e^{i\omega t} - \beta \int_0^E \left( \frac{dk}{k} \right) \left[ 1 - e^{-ikt} \right], \quad (2.31)$$

where the infra-red factor  $\beta$ , for this example is given by

$$\beta(Q^2) = \left( \frac{2\alpha}{\pi} \right) \left[ \frac{(4m^2 + 2Q^2)}{\sqrt{Q^2(4m^2 + Q^2)}} \ln \left( \sqrt{1 + \frac{Q^2}{4m^2}} + \sqrt{\frac{Q^2}{4m^2}} \right) - 1 \right], \quad (2.32)$$

where  $Q$  is the 3-momentum transfer,  $\underline{Q} = (\underline{p} - \underline{p}')$ ,  $m$  is the electron mass and  $\alpha$  is the fine structure constant  $\alpha = e^2/4\pi \approx 1/137$ . In the two extreme limits, we have

$$\beta(Q^2) \begin{cases} \xrightarrow[\text{N. R.}]{Q^2 \ll 4m^2} \left( \frac{8\alpha}{3\pi} \right) \left( \frac{Q^2}{4m^2} \right) \end{cases} \quad (2.33a)$$

$$\beta(Q^2) \begin{cases} \xrightarrow[\text{Ext. Rel.}]{Q^2 \gg 4m^2} \left( \frac{2\alpha}{\pi} \right) \ln \left( \frac{Q^2}{4m^2} \right). \end{cases} \quad (2.33b)$$

The first limit reflects that as  $Q \rightarrow 0$  (no acceleration), there should be no radiation. The second limit, valid for large angle scattering at high energies, shows that there are indeed large radiative corrections. This expression is a typical "leading log" of gauge theories.

It would be useful, for later purposes to ask now: where do these "logs" come from? They come from an angular integration. When we shall apply similar techniques to condensed matter phenomena with lumped circuits say, where sources become macroscopic, spatial degrees of freedom get frozen. Thus, time and its conjugate, energy are the only relevant variables left. In those cases there would appear no logs in  $\beta$ . But more about this later.

An alternative derivation based on Feynman diagrams is more suitable for discussing evolution and Renormalization Group (R. G.) equations. Consider a basic process ( $i \rightarrow f$ ) without soft photons, to be characterized by the matrix element  $M_{if}$ . Now, consider the emission of  $n$  independent (uncorrelated) soft photons of 4-momenta  $k_1, \dots, k_n$ , with a classical current  $\tilde{j}_\mu(k)$ , given by eq. (2.29). Neglect of recoil and correlations lead us to the matrix element<sup>(3)</sup>

$$M_{\mu_1, \dots, \mu_n}^{(n)} \approx \tilde{j}_{\mu_1}(k_1) \tilde{j}_{\mu_2}(k_2) \dots \tilde{j}_{\mu_n}(k_n) M_{if} = \quad (2.34)$$

$$= \int d\omega \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{i\omega t - i \sum_{j=1}^n k_j^0 \tau} \tilde{j}_{\mu_1}(k_1) \dots \tilde{j}_{\mu_n}(k_n) M_{if}. \quad (2.34a)$$

Rate  $\Gamma_n$  (or cross-section  $\sigma_n$ ) for production of  $n$  real photons of total energy  $\omega$  would be given by

$$\frac{1}{\Gamma_0} \left( \frac{d\Gamma_n}{d\omega} \right) = \frac{1}{n!} \int_{-\infty}^{\infty} \left( \frac{dt}{2\pi} \right) e^{i\omega t} \left[ \int_0^E \left( \frac{d^3k}{2k^0} \right) \tilde{j}_\mu^*(k) \tilde{j}_\mu^n(k) e^{-ik^0 t} \right]^n, \quad (2.35)$$

where  $\Gamma_0$  would be the rate without the emission of soft photons. Thus, the inclusive rate is given by

$$\frac{1}{\Gamma_0} \left( \frac{d\Gamma}{d\omega} \right) = \frac{1}{n} \left( \frac{d\Gamma_n}{d\omega} \right) \frac{1}{\Gamma_0} = \int_{-\infty}^{\infty} \left( \frac{dt}{2\pi} \right) e^{i\omega t + \beta \int_0^E \left( \frac{dk}{k} \right) e^{i\omega t}} \quad (2.36)$$

where

$$\beta = \frac{1}{2} k^2 \int d\Omega_k \left| \tilde{j}_\mu^*(k) \tilde{j}^\mu(k) \right|.$$

Eq. (2.36) has the typical IR divergence as  $k \rightarrow 0$ . This is compensated for by another divergence in  $\Gamma_0$  which is due to virtual photons. It is most easily found by defining the left side as a probability:

$$\left( \frac{dP}{d\omega} \right) = P_0 \int_{-\infty}^{\infty} \left( \frac{dt}{2\pi} \right) e^{i\omega t + \beta \int_0^E \left( \frac{dk}{k} \right) e^{i\omega t}}. \quad (2.37)$$

Requiring that eq. (2.37) be properly normalized, we find

$$\left(\frac{dP_E}{d\omega}\right) = \int_{-\infty}^{\infty} \left(\frac{dt}{2\pi}\right) e^{i\omega t - \beta \int_0^E \left(\frac{dk}{k}\right) [1 - e^{-ikt}]} , \quad (2.38)$$

identical to eq. (2.31).

Amazingly enough, for  $\omega \leq E$ , eq. (2.38) can be solved exactly. The answer is

$$\left(\frac{dP_E}{d\omega}\right) = \left[ \frac{\beta}{\gamma^\beta (1+\beta)} \right] \left(\frac{1}{\omega}\right) \left(\frac{\omega}{E}\right)^\beta , \quad (2.39)$$

where  $\ln \gamma = 0.5772$  is Euler's constant.

The factor  $(1/\omega)$  is the celebrated IR divergence - which is softened by the factor  $(\omega/E)$  due to the virtual exchange and real emission of an infinite number of soft quanta. The factor  $\gamma^\beta \Gamma(1+\beta)$  is a normalization factor, which can be usually ignored for small  $\beta$ , since then this term is  $\simeq 1 + \frac{\pi^2}{12} \beta^2 + \dots$ .

#### Footnotes and References :

- (1) - This presentation is based on W. Thirring and B. Touschek, Philosophical Mag. 42, 244 (1951).
- (2) - E. Etim, G. Pancheri and B. Touschek, Nuovo Cimento 51, 276 (1967).
- (3) - M. Greco, G. Pancheri and Y. Srivastava, Nuclear Phys. B101, 234 (1975).

### LECTURE 3 - VARIOUS EQUATIONS - MASTER, EVOLUTION, RENORMALIZATION GROUP AND BOLTZMANN

I would now like to obtain some equations through the summed soft radiation formulae of the type (2.38). Their relationship to R. G. equations and Altarelli-Parisi equations in QCD will be explored. Similarity with the Boltzmann and other master equations will also be indicated.

Consider the scaled variable  $x = \omega/E$  (fraction of energy  $E$  carried by the radiation) and define the probability density

$$\pi(x, \beta) \equiv E \left(\frac{dP_E}{d\omega}\right) . \quad (3.1)$$

Then, we may write

$$\pi(x, \beta) = \int_{-\infty}^{\infty} \left(\frac{d\tau}{2\pi}\right) e^{ix\tau - \beta \int_0^\tau \left(\frac{dy}{y}\right) [1 - e^{-iy}]} . \quad (3.2)$$

Taking derivative w. r. t.  $x$  and performing some algebra, we find<sup>(1, 2)</sup>

$$\frac{d\pi(x, \beta)}{dx} = \frac{(\beta - 1)}{x} \pi(x, \beta) + \frac{\beta}{x} \pi(x - 1). \quad (3.3)$$

$\pi$  is a probability density non-vanishing only for  $x > 0$ ,

$$\pi(x) = 0 \quad \text{for} \quad x \leq 0. \quad (3.4)$$

The analyticity of eq. (3.2) guarantees this. Eq. (3.3) leads to a simple equation for  $x \leq 1$ :

$$\frac{d\pi(x, \beta)}{dx} = \frac{\beta - 1}{x} \pi(x, \beta) \quad (x \leq 1). \quad (3.5)$$

Its solution is  $\pi(x, \beta) \sim x^{-1+\beta}$ . The coefficient of proportionality is determined by normalization. This leads to the result quoted in eq. (2.39).

To discuss the relationship with R. G. equations consider the variation of eq. (3.2) with :

$$\frac{\partial \pi(x, \beta)}{\partial \beta} = \int_0^1 \left( \frac{dy}{y} \right) [\pi(x-y, \beta) - \pi(x, \beta)]. \quad (3.6)$$

Eq. (3.6) shows how the IR divergence as ( $y \rightarrow 0$ ) on the right hand side is cancelled. Let us assume (without proof here) that eq. (3.6) describes the soft gluon spectrum in QCD<sup>(3)</sup>. Since QCD has two components, quarks and gluons, it follows that the quark density  $Q(x, \beta)$  should be

$$Q(x, \beta) = \pi(1 - x, \beta) \quad (x \leq 1). \quad (3.7)$$

Using eqs. (3.4) and (3.6), we may write

$$\frac{\partial Q(x, \beta)}{\partial \beta} = \int_0^{1-x} \frac{dy}{y^{1-\epsilon}} Q(x+y, \beta) - \int_0^1 \frac{dy}{y^{1-\epsilon}} Q(x, \beta), \quad (3.8)$$

where we have introduced a small parameter  $\epsilon$  to make the two integrals separately convergent. After the integrations are performed,  $\epsilon \rightarrow 0^+$ . Now consider the moments of eq. (3.8):

$$\frac{\partial M_n(\beta)}{\partial \beta} = \int_0^1 \frac{dz z^{n-1}}{(1-z)^{1-\epsilon}} \int_0^1 dy y^{n+1-\epsilon} Q(y, \beta) - \int_0^1 \frac{dz}{(1-z)^{1-\epsilon}} M_n(\beta), \quad (3.9)$$

where

$$M_n(\beta) = \int_0^1 \left( \frac{dy}{y} \right) y^n Q(y, \beta). \quad (3.10)$$

The limit  $\epsilon \rightarrow 0^+$  should be handled carefully, as the following example will illustrate. If we let  $\epsilon \rightarrow 0$  and take the difference of the two integrals, we find

$$\frac{\partial M_n(\beta)}{\partial \beta} = \int_0^1 \frac{dz}{(1-z)} [z^{n-1} - 1] M_n(\beta) = [\psi(1) - \psi(n)] M_n(\beta), \quad (3.11)$$

$$\equiv \bar{A}_n M_n(\beta), \quad (3.12)$$

which is the one loop R. G. equation result if we substitute for  $\bar{A}_n$  the anomalous dimension.

However, eq. (3.11) is only an approximation since we can easily check, from the explicit expression (2.39), that  $M_n(\beta)$  obeys the equation

$$\frac{\partial M_n(\beta)}{\partial \beta} = [\psi(1) - \psi(n+\beta)] M_n(\beta) \quad (3.13)$$

(In the above equations,  $\psi(Z)$  is the logarithmic derivative of the gamma function).

The correct procedure thus consists in making a small  $\varepsilon$ -expansion of eq. (3.9) and performing the integrals after cancelling the  $1/\varepsilon$  term. What one obtains is

$$\frac{\partial M_n(\beta)}{\partial \beta} = [\psi(1) - \psi(n)] M_n(\beta) + \int_0^1 dx x^{n-1} (\ln x) Q(x, \beta), \quad (3.14)$$

which coincides with eq. (3.13) when the explicit form for  $Q(x, \beta)$  is used.

In R. G. or Altarelli-Parisi equations, the complete anomalous dimension  $A_n$  enters, which includes the (the hitherto neglected) "hard" collinear gluons and reads

$$A_n = \int_0^1 \frac{dz}{(1-z)} [z^{n-1} - 1] \left( \frac{1+z^2}{z} \right). \quad (3.15)$$

In this sense RG equation is "superior". However, our formula is "superior" in that it sums soft radiation to all orders whereas in RG equation the IR divergence is introduced and cancelled only to first order (See discussion after eq. (3.11)).

We have a conjecture for an IR improved RG equation for the moments based on the following observation. In eq. (3.14), if we consider  $n$  as a continuous variable, we obtain

$$\left( \frac{\partial}{\partial \beta} - \frac{\partial}{\partial n} \right) M_n(\beta) = \bar{A}_n M_n(\beta). \quad (3.16)$$

It is the summation of soft quanta which makes eq. (3.16) non-local in  $n$ . Our suggestion is to replace  $\bar{A}_n$  by  $A_n$ . It has all the correct limits: for large  $n$  and first order in  $\beta$  it is certainly correct. Perhaps one of you may want to check it in the two loop approximation!

We have done a large amount of high-energy phenomenology based on the above results for which I refer you to refs. (3-7). Incidentally for SU(3) color, and 4 "flavours" of quarks;

$$\beta \simeq \left(\frac{16}{25}\right) \ln\left(\frac{\ln Q^2/\Lambda^2}{\ln Q_0^2/\Lambda^2}\right), \quad (3.17)$$

where  $Q_0$  is a reference point and  $\Lambda$  is a mass parameter related to the definition of the QCD coupling constant  $\alpha_{\text{strong}}$ .

Now let us discuss the evolution equations (Altarelli-Parisi<sup>(8)</sup>) in QED. Consider an electron of momentum  $P$  which distributes itself into photons carrying momentum  $xP$  and electrons of momentum  $(1-x)P$ . The equivalent number of photons is given by

$$N_\gamma = \left(\frac{\alpha}{2\pi}\right)\left(\frac{4}{x}\right)(\ln P/m_e). \quad (3.18)$$

Let  $\tau = -2 \ln(m_e/P)$ , so that to first order

$$\frac{dN_\gamma(x)}{d\tau} = \left(\frac{\alpha}{\pi}\right)\frac{1}{x}. \quad (3.19)$$

Thus, we can say, the number of electrons inside an electron

$$N_e(x, \tau) \simeq \delta(1-x) + \frac{\alpha}{\pi} \frac{1}{(1-x)} \tau + C(\alpha^2). \quad (3.20)$$

Of course, this equation has an IR divergence at  $x=1$ . Since the total number of electrons is not changed by the interaction, we must require

$$\int N_e(x, \tau) dx = 1, \quad (3.21)$$

which is a statement of current conservation. Thus, we modify (3.20) to read

$$\frac{dN_e^{(1)}}{d\tau} = \left(\frac{\alpha}{2\pi}\right) \left[ \frac{2}{1-x} - C \delta(1-x) \right] \equiv \left(\frac{\alpha}{2\pi}\right) p_{ee}(x), \quad (3.22)$$

where  $C$  (an infinite constant) is chosen such that

$$\int dx p_{ee}(x) = 0. \quad (3.23)$$

Now, the first order transition probability (T. P.)  $p_{ee}(x)$  is independent of "time"  $\tau$ , but the complete electron distribution is not. Its time development is given by a product of the T. P.  $p_{ee}(x)$  and the actual electron distribution at time  $\tau$ . That is

$$\begin{aligned}
 \frac{\partial N_e(x, \tau)}{\partial \tau} &= \left(\frac{\alpha}{2\pi}\right) \int_x^1 \left(\frac{dy}{y}\right) N_e(y, \tau) p_{ee}(x) \\
 &= \left(\frac{\alpha}{2\pi}\right) \int_x^1 \left(\frac{dy}{y}\right) N_e(y, \tau) \left[ \frac{2}{1-x/y} - C \delta(1-x/y) \right] = \\
 &= \left(\frac{\alpha}{2\pi}\right) \left\{ -C N_e(x, \tau) + 2 \int_x^1 \frac{dy}{y-x} N_e(y, \tau) \right\} .
 \end{aligned} \tag{3.24}$$

Eq. (3.24) has a simple physical interpretation. The first term on the right arises from a decrease in  $N_e$  due to the bremsstrahlung of electrons of fraction  $x$ : it is naturally therefore proportional to  $N_e(x, \tau)$ . The second term represents an increase, due to those electrons with momentum fraction  $y > x$ , which are "bremming" into  $x$ . The relative loss of momentum is  $x/y$ .

Interpreted as above, eq. (3.24) is quite similar to the Boltzmann transport equation (in the absence of external fields)<sup>(9)</sup>:

$$\begin{aligned}
 \left. \frac{\partial f_{\underline{k}}}{\partial t} \right|_{\text{scatt.}} &= \int (f_{\underline{k}'} - f_{\underline{k}}) Q(\underline{k}', \underline{k}) (d\underline{k}') \\
 &= -f_{\underline{k}} \int Q(\underline{k}, \underline{k}') (d\underline{k}') + \int f_{\underline{k}'} Q(\underline{k}, \underline{k}') (d\underline{k}') ,
 \end{aligned} \tag{3.25}$$

where the first term is the decrease due to electrons leaving the state  $\underline{k}$  and the second gives the increase due to electrons scattering from  $\underline{k}'$  into  $\underline{k}$ . Here  $\int Q(\underline{k}, \underline{k}') (d\underline{k}')$  plays the role of the (infinite) constant  $C$  earlier.

Now, consider moments of  $N_e(x, \tau)$ :

$$\begin{aligned}
 M_e^{(N)}(\tau) &= \int_0^1 \left(\frac{dx}{x}\right) x^N N_e(x, \tau) \\
 M_e^{(1)}(\tau) &= \int_0^1 dx N_e(x, \tau) = \text{constant, indep. of } \tau \\
 M_e^{(2)}(\tau) &= \int_0^1 x dx N_e(x, \tau) = \text{total momentum (in P-units) carried} \\
 &\hspace{15em} \text{by the electrons}
 \end{aligned} \tag{3.26}$$

etc.

Moments of eq. (3.24) are found to be

$$\frac{dM_e^{(N)}(\tau)}{d\tau} = -\left(\frac{\alpha}{2\pi}\right) A_{ee}^{(N)} M_e^{(N)}(\tau) , \tag{3.27}$$



where

$$A_{ee}^{(N)} = \int_0^1 \frac{dz}{z} z^N p_{ee}(z). \quad (3.28)$$

Eq. (3.27) can be trivially solved:

$$M_e^{(N)}(\tau) = M_e^{(N)}(\tau_0) e^{-\left(\frac{\alpha}{2\pi}\right) A_{ee}^{(N)} (\tau - \tau_0)}. \quad (3.29)$$

Consider  $N=2$  moment:

$$M_e^{(2)}(\tau) = M_e^{(2)}(0) e^{-\left(\frac{\alpha}{2\pi}\right) A_{ee}^{(2)} \tau}.$$

Thus, we find that the total momentum carried by the electrons goes exponentially (as a function of  $\tau$ ) to zero: the whole momentum is transferred from the electrons to the photon system. Clearly, then  $\frac{1}{\left(\frac{\alpha}{2\pi}\right) A_{ee}^{(2)}} = \tau^*$ , plays the role of a "relaxation time".

What is the corresponding situation in the transport equation? <sup>(10)</sup> Consider the first moment:

$$\frac{\partial}{\partial t} \int f_{\underline{k}}(\underline{dk}) = \int(\underline{dk}) \int(\underline{dk}') \left[ f_{\underline{k}'} - f_{\underline{k}} \right] Q(\underline{k}, \underline{k}') = 0, \quad (3.30)$$

just as before.

For the second moment, we have

$$\begin{aligned} \frac{\partial}{\partial t} e \int v_{\underline{k}} f_{\underline{k}}(\underline{dk}) &= e \int v_{\underline{k}}(\underline{dk}) \left[ f_{\underline{k}'} - f_{\underline{k}} \right] Q(\underline{k}, \underline{k}')(\underline{dk}') \\ &= e \int f_{\underline{k}'}(\underline{dk}') \int (v_{\underline{k}} - v_{\underline{k}'}) Q(\underline{k}, \underline{k}')(\underline{dk}'). \end{aligned} \quad (3.31)$$

In the simplified, sharp Fermi level approximation,

$$\int (v_{\underline{k}} - v_{\underline{k}'}) Q(\underline{k}, \underline{k}')(\underline{dk}) = -\frac{v_{\underline{k}'}}{\tau^*}, \quad (3.32)$$

defines the relaxation time  $\tau^*$ . Then eq. (3.31) leads to

$$\frac{\partial J(t)}{\partial t} = -\frac{1}{\tau^*} e \int f_{\underline{k}} v_{\underline{k}}(\underline{dk}) = -\frac{1}{\tau^*} J(t)$$

and we find for the current

$$I(t) = I(0) e^{-t/\tau^*} \quad (3.33)$$

in exact analogy with the AP result.

Now the significance of all this for the noise problem can be stated simply. Consider current-current correlations:

$$S_I(\omega) = \int dt e^{i\omega t} \overline{I(t)I(0)}. \quad (3.34)$$

If we use eq. (3.33) into eq. (3.34) we find the typical Lorentzian spectrum:

$$S_I^{\text{Lorentz}}(\omega) \sim \frac{1}{1 + (\omega\tau^*)^2} \quad (3.35)$$

which is devoid of any  $1/\omega$  noise. In our picture, it is the quantum electrodynamic fluctuations to eq. (3.33) which introduce the interesting  $(1/\omega)^{1-\beta}$  spectrum. To these matters therefore we turn our attention now.

Footnotes and References:

- (1) - E. Etim, G. Pancheri and B. Touschek, *Nuovo Cimento* 51, 276 (1967).
- (2) - G. Pancheri-Srivastava and Y. Srivastava, *Phys. Rev.* D15, 2915 (1977).
- (3) - G. Pancheri-Srivastava, Y. Srivastava and M. Ramòn Medrano, *Phys. Rev.* D23, 2533 (1981).
- (4) - G. Pancheri-Srivastava and Y. Srivastava, *Phys. Rev. Letters* 43, 11 (1979); G. Curci, M. Greco and Y. Srivastava, *Phys. Rev. Letters* 43, 834 (1979); *Nuclear Phys.* B159, 451 (1979).
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- (8) - G. Altarelli and G. Parisi, *Nuclear Phys.* B126, 298 (1977); G. Parisi, *Proceedings 11th Rencontre de Moriond on Weak Interactions and Neutrino Physics*, ed. by J. Tran Thanh Van (1976).
- (9) - See, for example, J. Ziman, *Principles of the Theory of Solids* (Cambridge University Press, 1964), Chapter 7.
- (10) - This analysis is due to G. Pancheri (unpublished).

LECTURE 4 - SOFT NOISE FOR CONDENSED MATTER SYSTEMS

Now we wish to consider modifications due to soft QED radiation (soft photons) for condensed matter systems<sup>(1, 2, 3)</sup>. I shall illustrate here, the general method<sup>(3)</sup> through a specific electron tunneling model of a shot noise device - which is commonly used for engineering purposes.

In the temporal gauge, the electric field  $\underline{\mathcal{E}}$  is given by

$$\underline{\mathcal{E}} = - \frac{1}{c} \left( \frac{\partial \underline{A}}{\partial t} \right) \quad (4.1)$$

where  $\underline{A}$  is the vector potential. The voltage  $V$  across a path  $P$  taken by an electron as it moves through the device

$$V = - \int_P \underline{\mathcal{E}}(\underline{dr}) , \quad (4.2)$$

is described by Faraday's law

$$\Phi = \int_P \underline{A}(\underline{dr}) , \quad (4.3)$$

$$V = - \frac{1}{c} \left( \frac{d\Phi}{dt} \right) . \quad (4.4)$$

Thus, the electromagnetic action is

$$\Delta S = \left( \frac{e}{c} \right) \int_P \underline{A} \cdot (\underline{dx}) = \frac{e}{c} \Phi . \quad (4.5)$$

Let  $\nu_{r\ell}(\Phi)$  be the amplitude (in frequency units) for an electron (to the left of the device) in state  $\ell$  to tunnel into a state  $r$  (to the right of the device). In terms of electron creation and annihilation operators, the tunneling Hamiltonian reads<sup>(4)</sup>

$$H_T = \hbar \sum_{r\ell} \left( \nu_{r\ell}(\Phi) C_r^+ C_\ell + \nu_{r\ell}^*(\Phi) C_\ell^+ C_r \right) . \quad (4.6)$$

From the path integral action principle, we may incorporate the effect of the EM action as follows: Given an amplitude for a process  $P$  and an action modification  $\Delta S$  due to an interaction, the amplitude is phase modulated to read

$$\text{Amp}(P, \Delta S) = e^{\frac{i}{\hbar}(\Delta S)} \text{Amp}(P, 0) . \quad (4.7)$$

Thus, for the above electron we get

$$\nu_{r\ell}(\Phi) = \nu_{r\ell}(0) \exp\left(\frac{ie\Phi}{\hbar c}\right) . \quad (4.8)$$

To compute the electronic quantum shot noise current operator

$$J = -c \frac{dH_T(\Phi)}{d\Phi}, \quad (4.9)$$

we may use eqs. (4.6), (4.8) to obtain

$$J = J^+ \exp\left(\frac{ie\Phi}{\hbar c}\right) + J^- \exp\left(-\frac{ie\Phi}{\hbar c}\right), \quad (4.10)$$

where

$$J^+ = -ie \sum_{r\ell} v_{r\ell}(0) C_r^+ C_\ell, \quad J^- = +ie \sum_{r\ell} v_{r\ell}^*(0) C_\ell^+ C_r.$$

Ideally, one would like to compute the nth order current correlation functions:

$$W_n(t_1, t_2, \dots, t_n; I) = \langle J(t_1), \dots, J(t_n) \rangle_I, \quad (4.11)$$

where the averages  $\langle B \rangle_I$  are defined as

$$\langle B \rangle_I \equiv \frac{\langle \psi_0 | \left| B e^{\left(\frac{i}{\hbar c}\right) \int I(t) \Phi(t) dt} \right|_+ | \psi_0 \rangle}{\langle \psi_0 | \left| e^{\left(\frac{i}{\hbar c}\right) \int I(t) \Phi(t) dt} \right|_+ | \psi_0 \rangle}. \quad (4.12)$$

In eq. (4.12) I is an external (c-number) current source, through which the Schwinger action S(I) is defined as the ground state ( $\psi_0$ ) persistence amplitude<sup>(5)</sup>:

$$e^{\frac{i}{\hbar} S(I)} = \langle \psi_0 | \left| e^{\left(\frac{i}{\hbar c}\right) \int I(t) \Phi(t) dt} \right|_+ | \psi_0 \rangle. \quad (4.13)$$

The subscript (+) in the above denotes time ordering.

The irreducible photon propagators of the device are determined by the action given by eq. (4.13) through functional differentiations:

$$D_n(t_1, t_2, \dots, t_n; I) = c \frac{\delta^n S(I)}{\delta I(t_1), \dots, \delta I(t_n)} \quad (4.14)$$

For example,

$$D_1(t, I) = c \frac{\delta S}{\delta I(t)} = \langle \Phi \rangle_I \quad (4.15)$$

and

$$D_2(t, s, I) = c \frac{\delta^2 S}{\delta I(t) \delta I(s)} = \frac{\delta \langle \Phi \rangle_I(t)}{\delta I(s)}. \quad (4.16)$$

As an example of eq. (4.16), let us observe that Faraday's law (4.4) along with (4.16) tells us (after a Fourier transformation) that the photon propagator (for  $I \rightarrow 0$ )

$$D(\omega) = ic \frac{Z(\omega)}{\omega} \quad (4.17)$$

where  $Z$  is the radiative impedance.

Proof of eq. (4.17):

From eq. (4.4)

$$\tilde{V}(\omega) = -\frac{1}{c} i \omega \tilde{\Phi}(\omega) = \tilde{I}(\omega) Z(\omega)$$

eq. (4.16) gives

$$\tilde{D}(\omega) = -\frac{c}{i\omega} Z(\omega) = \frac{ic}{\omega} Z(\omega). \quad \text{Q. E. D.}$$

Also, the two point current correlation function

$$K(t_1 - t_2) = \frac{i}{\hbar} \lim_{I \rightarrow 0} W_2(t_1, t_2; I) \quad (4.18)$$

determines the Kubo formula for admittance  $Y(\omega)$  via

$$Y(\omega) = \left(\frac{2}{\omega}\right) \int_0^{\infty} dt e^{i\omega t} \text{Re}K(t). \quad (4.19)$$

Higher order engineering functions are determined in a similar fashion by the set of correlation functions  $\{W_n\}$ . From eqs. (4.10) and (4.11) we have

$$W_n(t_1, \dots, t_n; I) = \sum_{\sigma_1, \dots, \sigma_n = \pm 1} F_n(t_1 \sigma_1, t_2 \sigma_2, \dots, t_n \sigma_n; I), \quad (4.20)$$

$$F_n(t_1 \sigma_1; \dots; t_n \sigma_n; I) = \left\langle \prod_{j=1}^n \sigma_j j(t_j) e^{\left[\frac{ie}{\hbar c} \sigma_j \Phi(t_j)\right]} \right\rangle_I. \quad (4.21)$$

In eq. (4.21) we may remove the phase factors from the amplitude average on the right hand side using the factorization theorem. For any functional  $A(\Phi)$ , we have

$$\langle A(\Phi) B \rangle_I = A(\langle \Phi \rangle_I - i\hbar c \frac{\partial}{\partial I}) \langle B \rangle_I. \quad (4.22)$$

Using (4.22), (4.21) may be written as

$$F_n(t_1 \sigma_1; \dots t_n \sigma_n; I) = e^{-\frac{i}{\hbar} S(I)} \left( \prod_{j=1}^n e^{\sigma_j \frac{\delta}{\delta I(t_j)}} \right) e^{+\frac{i}{\hbar} S(I)} \left( \prod_{\ell=1}^n J^{\sigma_\ell}(t_\ell) \right)_I. \quad (4.23)$$

Eq. (4.23) is an exact starting point for evaluating the shot noise current correlation functions.

Now it is quite generally valid that any charged particle tunneling event may be considered as instanton "flash disturbances"<sup>(2, 3, 6)</sup>. Thus, the current of  $n$  "instanton" normal electronic shots in directions  $\sigma_1 \dots \sigma_n$  at times  $t_1 \dots t_n$  has the evident form

$$I_n(t; t_1 \sigma_1 \dots t_n \sigma_n) = e \sum_{j=1}^n \sigma_j \delta(t - t_j). \quad (4.24)$$

The complex phase modulation  $\vartheta_n(t_1 \sigma_1 \dots t_n \sigma_n; I)$  of the ground state persistence amplitude due to  $n$  instanton shots can be defined in terms of the Schwinger action  $S(I)$  as

$$\hbar \vartheta_n(t_1 \sigma_1 \dots t_n \sigma_n; I) = S(I + I_n) - S(I). \quad (4.25)$$

Eqs. (4.23) and (4.24) then give us

$$F_n(t_1 \sigma_1 \dots t_n \sigma_n; I) = e^{i \vartheta_n(t_1 \sigma_1 \dots t_n \sigma_n; I)} \left\langle \prod_{j=1}^n J^{\sigma_j}(t_j) \right\rangle_{I+I_n}. \quad (4.26)$$

The physical implications of eq. (4.26) are worthy of note: (i) the instanton phases  $\{\vartheta_n\}$  determine the modulation produced by the multiple photon emission. This renormalizes the shot noise current fluctuation functions  $\{W_n\}$ . (ii) The instanton current sources  $\{I_n\}$  determine the recoil effects on the shot events due to such emissions.

The soft-photon ("no-recoil") approximation consists in neglecting the back reaction of the radiation on the sources. (That is,  $I_n$  is dropped in computing the second factor in eq. (4.26)).

To illustrate, consider the admittance of the shot current device to be computed using eq. (4.19). We may employ eqs. (4.18), (4.20), and (4.26) to obtain

$$K(t_1 - t_2) = \frac{i}{\hbar} \sum_{\sigma_1, \sigma_2 = \pm 1} e^{i \vartheta_2(t_1 \sigma_1; t_2 \sigma_2; 0)} \left\langle J^{\sigma_1}(t_1) J^{\sigma_2}(t_2) \right\rangle_{I_2}. \quad (4.27)$$

The instanton source  $I_2$  on the rhs yields current recoil effects. If these are neglected, then from the admittance  $y(\omega)$ , in the absence of radiation effects

$$y(\omega) = \left( \frac{2}{\omega} \right) \int_0^\infty dt e^{i\omega t} \text{Re}k(t), \quad (4.28)$$

defining

$$i\chi(|t_1 - t_2|) \equiv \lim_{I \rightarrow 0} \vartheta_2(t_1, -\sigma; t_2, +\sigma; I), \quad (4.29)$$

we get

$$K(t) \simeq e^{-\chi(t)} k(t). \quad (4.30)$$

To compute  $\chi(t)$ , we expand  $\vartheta_n$  in terms of the irreducible photon propagators  $D_n$  as

$$\begin{aligned} \hbar c \vartheta_n(t_1 \sigma_1 \dots t_n \sigma_n; I) &= \sum_{N=1}^{\infty} \frac{1}{N!} \int ds_1 \dots \int ds_N D_N(s_1 \dots s_N; I) \cdot \\ &\cdot \prod_{j=1}^n I_n(s_j; t_1 \sigma_1 \dots t_n \sigma_n). \end{aligned} \quad (4.31)$$

To order  $\alpha = e^2/\hbar c$  (the fine structure constant), we find

$$\chi(t) = -i\alpha [D(0) - D(t)] + O(\alpha^2). \quad (4.32)$$

Proof of eq. (4.23):

$$i\chi(t_1 - t_2) = \lim_{I \rightarrow 0} \vartheta_2(t_1, -\sigma; t_2, \sigma; I),$$

$$\begin{aligned} i\chi(t_1 - t_2) &\simeq \frac{1}{\hbar c} \sum_{N=1}^{\infty} \frac{e^N}{N!} \int ds_1 \dots \int ds_N D_N(s_1 \dots s_N; 0) \cdot \\ &\cdot \prod_{j=1}^N (\delta(s_j - t_1) - \delta(s_j - t_2)), \end{aligned}$$

$$i\hbar c \chi^{(1)}(t_1, t_2) = e \int ds D_1(s) [\delta(s - t_1) - \delta(s - t_2)] = 0,$$

$$\begin{aligned} i\hbar c \chi^{(2)}(t_1, t_2) &= \frac{e^2}{2} \int ds_1 \int ds_2 D_2(s_1, s_2) [\delta(s_1 - t_1) - \delta(s_1 - t_2)] [\delta(s_2 - t_1) - \\ &- \delta(s_2 - t_2)] = \frac{e^2}{2} \left\{ D_2(t_1, t_1) - D_2(t_1, t_2) - D_2(t_2, t_1) + D_2(t_2, t_2) \right\} = \\ &= e^2 [D(0) - D(t_1 - t_2)], \end{aligned}$$

$$\chi^{(2)}(t) = -i\alpha [D(0) - D(t)].$$

QED

The normalized soft photon energy distribution  $dP(\omega)$  is defined as

$$\left(\frac{dP}{d\omega}\right) = \int_{-\infty}^{\infty} \left(\frac{dt}{2\pi}\right) e^{i\omega t} e^{-\chi(t)} . \quad (4.33)$$

Since  $\chi(0) \equiv 0$  to all orders in  $\alpha$ , eq. (4.33) rigorously defines a normalized distribution. If we substitute for  $D(t)$  the impedance formula (eq. (4.17))

$$D(t) = \left(\frac{ic}{\pi}\right) \int_0^{\infty} \left(\frac{d\omega}{\omega}\right) e^{-i\omega|t|} \text{Re}Z(\omega + i0^+) , \quad (4.34)$$

to order  $\alpha$ ,  $\chi$  becomes

$$\chi(t) = \left(\frac{\alpha c}{\pi}\right) \int_0^{\infty} \left(\frac{d\omega}{\omega}\right) \text{Re}Z(\omega + i0^+) [1 - e^{-i\omega t}] \quad (4.35)$$

$$= (4\alpha) \int_0^{\infty} \left(\frac{d\omega}{\omega}\right) \left[\frac{\text{Re}Z(\omega + i0^+)}{R_0}\right] [1 - e^{-i\omega t}] , \quad (4.36)$$

where  $R_0 = 4\pi/c$  is the vacuum radiation impedance.

It is instructive to compare eq. (4.36) with its corresponding "high-energy" counterpart derived earlier (cf. eqs. (2.31) et seq.). If we define an effective  $\beta$  by considering  $\omega\tau^* \ll 1$ , where  $\tau^*$  is the relaxation time,

$$\beta \approx (4\alpha)(R/R_0) , \quad (4.37)$$

valid for  $\beta$  small. The resistance  $R = \lim_{\omega \rightarrow 0} \text{Re}Z(\omega)$ .

#### Footnotes and References:

- (1) - P. Handel was the first to emphasize the importance of soft photons in  $1/f$  noise (P. Handel, Phys. Rev. Letters 34, 1492 and 1495 (1975)). He and his collaborators have utilized the above idea for an impressive series of applications. See e.g. P. Handel, Phys. Rev. 22, 745 (1980); K. van Vliet, P. Handel and A. vander Ziel, Physica 108A, 511 (1981); T. Sherif and P. Handel, Phys. Rev. 26, no. 1 (1982).
- (2) - "Soft-Photon Emission from Electron Pair Tunneling in small Josephson Junctions", by A. Widom, G. Pancheri, Y. Srivastava, G. Megaloudis, T. D. Clark, H. Prance and R. Prance, Phys. Rev. B26, (1982).
- (3) - "Quantum Electrodynamical Circuit Soft Photon Renormalization of the Conductance in Electronic Shot Noise Devices", by A. Widom, G. Pancheri, Y. Srivastava, G. Megaloudis, T. Clark, H. Prance and R. Prance (June 1982 - to be published).
- (4) - See, e.g. C. Duke, "Tunneling in Solids" (Academic Press, New York, 1969).
- (5) - J. Schwinger, "Quantum Kinematics and Dynamics" (W. A. Benjamin Inc., New York, 1970).
- (6) - A. Widom, Phys. Rev. B21, 5166 (1980).



LECTURE 5 - DIVERSE TOPICS

Let me very briefly mention soft-graviton emission in QCD. An accelerated electron radiates in QED. An accelerated quark radiates gluons in QCD. Thus, it should come as no surprise that an accelerated mass radiates soft gravitons in QGD.

The latter can be computed in the no recoil approximation using for the classical graviton source the stress-energy tensor<sup>(1, 2)</sup>

$$T^{\mu\nu}(x) = \sum_{n, \sigma_n = \pm 1} \sigma_n m_n v_n^\mu v_n^\nu \int_0^{\sigma_n \infty} d\tau \delta(x - v_n \tau), \quad (5.1)$$

where for the nth particle  $m_n$  is its mass and  $v_n^\mu$  the 4-velocity, the sign factor  $\sigma_n = \pm 1$  tells us whether it is incoming or outgoing. The energy-momentum conservation reads

$$\sum_n \sigma_n m_n v_n^\mu = 0. \quad (5.2)$$

Once again, one finds a Poisson distribution for the graviton radiation and in the earlier notation, we may write an analogous result

$$\left(\frac{dP}{d\omega}\right) = \int \left(\frac{dt}{2\pi}\right) e^{i\omega t - \beta_G} \int_0^\Lambda \left(\frac{dk}{k}\right) [1 - e^{-ikt}], \quad (5.3)$$

where  $\Lambda$  is a suitable energy separating "soft" from "hard" gravitons.  $\beta_G$  is given by

$$\beta_G = \left(\frac{G}{2\pi}\right) \sum_{m,n} \frac{1+v_{nm}^2}{\sqrt{1-v_{nm}^2}} \frac{\sigma_n \sigma_m m_m m_n}{v_{mn}} \ln\left(\frac{1+v_{mn}}{1-v_{mn}}\right), \quad (5.4)$$

where

$$v_{nm} = \sqrt{1 - \left(\frac{m_n m_m}{p_n p_m}\right)^2}. \quad (5.5)$$

Clearly

$$\left(\frac{dP}{d\omega}\right) = \frac{\beta_G}{\gamma^{\beta_G} \Gamma(1+\beta_G)} \frac{1}{\Lambda} \left(\frac{\omega}{\Lambda}\right)^{-1+\beta_G}, \quad (5.6)$$

as before.

There are differences, however. The forward peaking present in the EM case (for relativistic speeds) is not present here. In the non-relativistic limit ( $v \ll c$ )

$$\beta_G \simeq \left(\frac{3Gm^2}{5\pi}\right) \left(\frac{v}{c}\right)^2 \sin^2 \vartheta, \quad (5.7)$$

where  $\vartheta$  is the scattering angle. This should be compared with  $\beta_{EM}$ , which in the same

limit reads

$$\beta_{EM} \approx \left(\frac{8\alpha}{3\pi}\right) v^2 . \quad (5.8)$$

For possible applications to condensed matter systems (stars and other massive objects undergoing accelerations on the one hand and gravitational wave detector noise on the other), we should formulate the problem in terms of macroscopic stress and strains just as we had to relate the photon propagator to lumped circuit impedance in the EM case. This problem is presently under study and I shall not comment on it here any further.

I would now like to close these lectures with two physical applications relating to EM noise.

(i) Josephson junctions<sup>(3)</sup>

A simple Josephson junction carries an electron pair (Cooper pair,  $q = 2e$ ) current

$$J = -q\nu \sin\left(\frac{q\Phi}{\hbar c}\right) = \left(\frac{iq}{2}\right) \left[ \nu \exp\left(\frac{iq\Phi}{\hbar c}\right) - \nu \exp\left(-\frac{iq\Phi}{\hbar c}\right) \right] , \quad (5.9)$$

where  $\Phi$  is the magnetic flux coordinate.

Now the situation is identical to the electron tunneling case (with  $e \rightarrow q = 2e$  and  $J^- = -J^+$ ). See Section 4. Thus, for a weakly voltage carrying state of the junction, we find

$$\beta \approx \frac{q^2}{\pi\hbar} R . \quad (5.10)$$

Now we can convert this into an engineering formula for current fluctuations  $S_J(\omega)$ , in a junction radiating into a cold environment (e. g. the vacuum):

$$S_J(\omega) = \left(\frac{q^2\nu^2}{4}\right) \left[ \frac{dP(\omega)}{d\omega} \right] . \quad (5.11)$$

For small  $\beta$  and  $\omega\tau^* \ll 1$ , the current fluctuation spectral function becomes

$$S_J(\omega) \approx \left(\frac{q^2\nu^2}{4}\right) \beta (\tau^*\omega)^\beta \left(\frac{1}{\omega}\right) \quad (5.12)$$

with  $\tau^*$  an internal time scale (relaxation time).

This is our crucial result which shows that weak voltage carrying states of a small Josephson junction exhibit " $(1/\omega)^{(1-\beta)}$  - noise" in the spectral fluctuations of Cooper pair currents. At the one loop level of QED perturbation theory, the critical exponent  $\beta$  is given by eq. (5.10).

(ii) Resonant Line Shape Corrections<sup>(4)</sup>

Consider a specific circuit element containing a single electron tunneling device. In the linearized regime, we can incorporate the soft-radiation renormalization into the engineering conductance of the circuit element.

As in any electrical engineering problem, the impedance of the electron tunneling device is computed by adding the reciprocal of the impedances of the resistor (R) and Capacitor (C), and then inverting

$$Z(\omega) = \frac{R}{(1 - i\omega\tau)} \quad , \quad (5.13)$$

where

$$\tau = RC \quad . \quad (5.14)$$

The propagator  $D(t)$  corresponding to an impedance  $Z$  is given by

$$D(t) = \left(\frac{ic}{\pi}\right) \int_0^{\infty} \left(\frac{d\omega}{\omega}\right) e^{-i\omega t} \operatorname{Re} Z(\omega + i0^+) \quad , \quad (5.15)$$

which contains an infrared (IR) divergence at  $\omega = 0$ . It is precisely this IR divergence which is eliminated in measured quantities by QED soft photon summations. Explicitly, to lowest order in  $\alpha$ , using eqs. (4.36) and (5.13), IR finite  $\chi(t)$  reads (in the linearized regime)

$$\chi(t) = (4\alpha) \left(\frac{R}{R_0}\right) \int_0^{\infty} \frac{d\omega}{\omega [1 + \omega^2 \tau^2]} \left[1 - e^{-i\omega t}\right] \quad . \quad (5.16)$$

Now, consider renormalization of the conductance functions. Let  $y(\omega)$  denote purely condensed matter admittance and  $Y(\omega)$  photon renormalized admittance. Calling

$$G(\omega) = \operatorname{Re} Y(\omega + i0^+) \quad (5.17)$$

and

$$g(\omega) = \operatorname{Re} y(\omega + i0^+) \quad . \quad (5.18)$$

The conductance of the circuit is simply computed from the admittance  $y(\omega)$ . The resonance form is

$$\operatorname{Rg}(\omega) = \frac{\omega_0^4}{\left[(\omega^2 - \omega_0^2)^2 + \omega^2 \gamma^2\right]} \quad (5.19)$$

where the resonant frequency

$$\omega_0 = \frac{c}{\sqrt{LC}} \quad (5.20)$$

and the width

$$\gamma = \frac{1}{\tau} = \frac{1}{RC} . \quad (5.21)$$

Folding the soft radiation probability  $dP(\omega)$ , we have

$$G(\omega) = \int_0^{\omega} dP(\omega') \left[ \left( \frac{\omega}{\omega'} \right) - 1 \right] g(\omega - \omega') . \quad (5.22)$$

Using eqs. (5.16), (5.19) and (5.22), the problem of computing soft radiation corrections to the line shape is now reduced to quadratures

$$\frac{dP}{d\omega} = \int_{-\infty}^{\infty} \left( \frac{dt}{2\pi} \right) \exp \left\{ i\omega t - \int_0^{\infty} \left( \frac{dk}{k} \right) \frac{(1 - e^{-i\omega t})}{(1 + k^2 \tau^2)} \right\} , \quad (5.23)$$

$$RG(\omega) = \int_0^{\omega} dP(\omega') \left[ \left( \frac{\omega}{\omega'} \right) - 1 \right] \frac{\omega_0^2}{\left\{ \left[ (\omega - \omega')^2 - \omega_0^2 \right]^2 + (\omega - \omega')^2 \gamma^2 \right\}^2} . \quad (5.24)$$

Footnotes and References:

- (1) - S. Weinberg, Phys. Rev. 140B, 516 (1965).
- (2) - B. de Witt, Phys. Rev. 162, 1239 (1967).
- (3) - See ref. (2) after Section 4.
- (4) - See ref. (3) after Section 4.

### GENERAL CONCLUSIONS AND OUTLOOK

In these lectures I have tried to expose the unifying features of soft radiation spectrum in QED, QCD and QGD. These spectra have been linked in condensed matter physics to the low frequency noise and in high energy physics to a variety of phenomena (broadening of jets, low transverse momentum reactions etc.).

I find it amazing (and satisfying) that QED field theoretic methods are directly applicable to problems of lumped circuits in electrical engineering. Accurate low noise electronics measurements would allow us in the very near future to check the reliability of general quantum field theoretic approximation techniques. An exciting prospect !

I take this opportunity to thank the Organizers of this School for allowing me to speak about matters which have occupied much of my time and energy recently. To my listeners, I thank them for their enthusiasm and interest.