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ON THE SINGLE POINT FIELD THEORIES

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ON THE SINGLE POINT FIELD THEORIES

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A systematic approach to reduce the dynamical degrees of freedom is presented. Certain field theory models with non-trivial large N limit defined in the infinite volume are shown to be reduced to theories defined on single point in the space-time when $N \rightarrow \infty$.

1. - INTRODUCTION

The $1/N$ expansion method has provided us a valuable approach to the understanding of gauge theories and other field theory models in the non-perturbative regions. The main reasons for this is that in the leading order of the $1/N$ approximation physics simplifies significantly and hopefully contains yet enough informations of the reality where N is finite. For example, QCD in this leading order is the sum of all the planar diagrams, confinement presumably persists to this level⁽¹⁾. People have since tried to construct this leading order contribution and

indeed was obtained in closed forms for some notable models⁽²⁾. It was advocated by Witten that for $N \rightarrow \infty$ there is a dominating configuration which he called master field, and this idea was even stated explicitly by Coleman⁽³⁾ claiming the master fields should be space-time independent and therefore there are only four matrices to be formed in an appropriate gauge. Surprisingly such a hope recently has been partially realized by Eguchi and Kawai⁽⁴⁾. They claimed the conventional $U(N)$ Wilson lattice gauge theory defined in the infinite volume

$$-S_W = \beta \sum_{\alpha} \sum_{\mu \neq \nu} \text{tr} (U_{x,\mu} U_{x+\mu,\nu} U_{x+\mu,\nu}^{\dagger} U_{x,\mu}^{\dagger}) \quad (1.1)$$

is equivalent to a much simpler model in the large N limit.

$$-S_{EK} = \beta \sum_{\mu \neq \nu} \text{tr} (U_{\mu} U_{\nu} U_{\mu}^{\dagger} U_{\nu}^{\dagger}). \quad (1.2)$$

They reduced the original model to a single point by identifying all the dynamical variables in the same directions, therefore they have implicitly employed the periodic boundary conditions

$$U_{\mu}(x + \hat{\mu}) = U_{\mu}(x) \quad (1.3)$$

However, as the authors of ref (5) pointed out that their claim does not hold in the weak coupling regions and a quenched model is proposed by them to remedy the EK model. In ref. (6) a generalized periodic boundary condition was proposed and it was shown this approach gives the same set of planar diagrams as those in the infinite volume for the scalar matrix model

$$S = \int \left\{ \text{tr}(\partial_{\mu} M \partial_{\mu} M^{\dagger}) + m^2 \text{tr} (MM^{\dagger}) + \frac{g}{4N} \text{tr} (MM^{\dagger} MM^{\dagger}) \right\} d^D x$$

The following rule was adopted to reduce the theory

$$M(x + \hat{\mu}) = e^{i\theta^\mu} M(x) e^{-i\theta^\mu}$$
$$(e^{i\theta^\mu})_{i,j} = \delta_{ij} e^{i\theta_j^\mu} \quad i,j = 1, \dots, N \quad (1.4)$$

This approach was subsequently applied to gauge theories in ref. (7). The main conclusion there was for $O(N)$ gauge theory and it was shown that no spontaneous symmetry breaking can occur in any coupling region.

In the present paper we intend to describe a more complete account of the approach. We will follow consistently the methods employed in refs. (6,7) by more examples and analysis, some points for $U(N)$ gauge theory will be further clarified.

The layout of the paper goes as follows. In Sec. 2 we interpret the generalized periodic boundary conditions with heuristical arguments which serve to explain the physical mechanism of the rule (6). Sec.3 discussed the basic applications with single model. Sec. 4 discusses the non-linear matrix model or the chiral spin model which serves to further continue the approach. Sec. 5 discusses problems in the gauge theories and Sec. 6 is devoted to various applications of approach, especially for the numerical simulations. In the conclusion we also try to argue the possible connections between the gauge theory and chiral spin model at large N limit.

2. - THE GENERALIZED PERIODIC BOUNDARY CONDITIONS

Quenched background gauge field method was suggested in ref. (6) and was also employed to gauge theories⁽⁷⁾. Here we would explain its physical working by illustrating it with intuitive argument.

It should be recognized that as the size of the system becomes smaller and

smaller, the boundary effects become more and more important, in other words, an appropriate prescription of boundary conditions is getting more and more relevant. In order to resemble the physics of the infinite volume on a smaller size space-time, the finite size effects must be somehow maximally eliminated.

This goal can be achieved by using the method of quenched gauge fields. For the convenience of presentation, we take the continuum case and concentrate on one of the directions in the space-time. The range of this direction is taken to be finite while others infinite.

We require for this direction

$$A_{ab}^\nu(x + \hat{\mu}) = A_{ab}^\nu(x) e^{i(\theta_a^\mu - \theta_b^\mu)} \quad (2.1)$$

$$\psi_a(x + \hat{\mu}) = \psi_a(x) e^{i\theta_a^\mu} \quad (2.2)$$

The A and ψ field can be thought of gluon and quark fields on more generic fields in the regular and fundamental representations, respectively. The boundary conditions say that the field variables experience the background gauge fields and the latter is to be quenched.

The system is depicted in Fig. 1. We use the wave function and Feynman path integral language. Let us consider a particle or any other object at point A, we are interested in evaluating the total probability of arriving at point B. Suppose we have

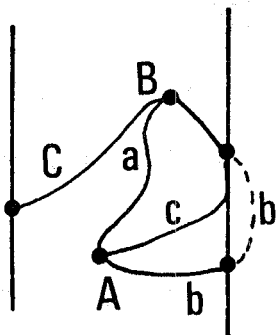


FIG. 1 - The heuristical diagram.

used the usual periodic boundary conditions where the left boundary and the right one are identified. Typically the object would follow two kinds of paths a and b in the infinite volume as shown in Fig. 1. However due to the finiteness in that direction, we get some extra paths such like path c in Fig. 1. In order to describe correctly the physics of the infinite volume in a finite region, the contributions arising from such like path c must be eliminated. This can be achieved by averaging over the randomly distributed background gauge fields. In this way an object annihilated at the right hand side boundary and created again at the right hand side boundary along the path b in Fig. 1 will be associated with opposite path, after averaging over θ 's such a contribution survives, while an object following path c will necessarily have non trivial phase dependence and the contributions so obtained will vanish often integrating over the θ 's.

However there is a hole in the above argument. Let us take for example a quark anti-quark combination $\bar{\psi}_a \psi_b$ or a gluon field A_{ab}^μ , there are N^2 of them. The above argument would not hold when the two indices coincide, in such case that object does not have θ -dependence and it would still follow the "wrong paths" as before. But there are only N such objects, so we have to send N to infinity to suppress those unwanted contributions.

Moreover, sending N to infinity also serves to make the phases distributed continuously, since they would play the rôle of momenta. In this way we have reduced maximally the finite volume effects in the large N limit.

3. - THE BASIC APPROACH

In ref. (6) it was shown that the exact set of planar diagrams can be obtained on a single point. The proof was via the perturbation theory proceeding order by order. The proof could also be done using non-perturbative means such as Dyson-Schwinger equations. For the case of presentation, we will use perturbation for the linear model

and D-S equations for its nonlinear generalization in next section. For completeness we summarize the results got in ref. (6). The model discussed was a linear scalar matrix model in D-dimensional space-time

$$S = \int dx \left\{ \text{tr}(\partial_\mu M \partial_\mu M^\dagger) + m^2 \text{tr}(MM^\dagger) + \frac{g}{4N} \text{tr}(MM^\dagger MM^\dagger) \right\} \quad (3.1)$$

where M is an arbitrary complex matrix. We use the double line representation for the Feynman diagrams in which we do not have to write down indices for Feynman values (Fig. 2).

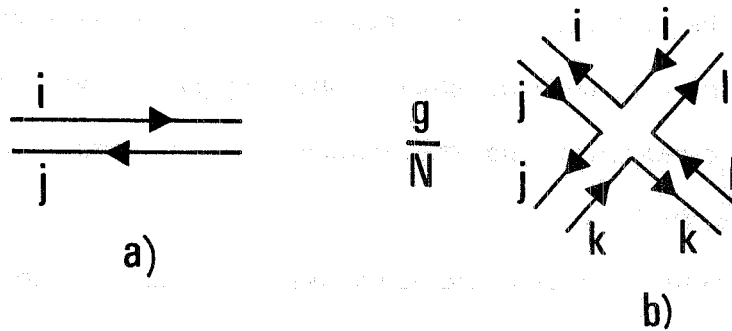


FIG. 2 - The propagator and vertex in double-line representation.

The Feynman rules for the propagator and vertex are extremely simple (see Fig. 2).

$$\begin{aligned} \text{propagator} & \quad \frac{1}{p^2 + m^2} \\ \text{vertex} & \quad \frac{g}{N} \end{aligned} \quad (3.2)$$

where no summation over repeated indices.

On a single point according to the generalized periodic boundary condition

$$M_{ij}(x + \hat{\mu}) = M_{ij}(x) e^{i(\theta_i^\mu - \theta_j^\mu)}, \quad (3.4)$$

we have the following action

$$S = \sum_{i,j} (\theta_i - \theta_j)^2 |M_{ij}|^2 + m^2 \text{tr} (MM^+) + \frac{g}{4N} \text{tr} (MM^+ MM^+). \quad (3.5)$$

Now the Feynman rules are almost the same as (3.2, 3.3) except the propogator is replaced by

$$\frac{1}{(\theta_i - \theta_j)^2 + m^2} \quad (3.6)$$

The identifications between the correlation function in the infinite volume and on a single point were proposed to be, for example, for a two point function

$$\langle M_{ij}(0) M_{ij}^+(x) \rangle = \int \prod_{\mu,i} \frac{d\theta_i^\mu}{2\pi} e^{i(\theta_i - \theta_j) \cdot x} \langle M_{ij} M_{ij}^+ \rangle_\theta \quad (3.7)$$

where the left hand sides is evaluated using (3.1) and the right hand side using (3.5). This relation is obvious in the free case.

It is easily seen that the above relation persists to highen orders in perturbation. Consider a graph shown in Fig.3, the contribution is

$$\begin{aligned} \langle M_{ij} M_{ij}^+ \rangle_\theta &= \sum_{l,k} \left(\frac{g}{N}\right)^2 \frac{1}{[(\theta_i - \theta_j)^2 + m^2]^2} \frac{1}{(\theta_i - \theta_k)^2 + m^2} \\ &\quad \times \frac{1}{(\theta_l - \theta_j)^2 + m^2} \times \frac{1}{(\theta_k - \theta_l)^2 + m^2}, \end{aligned} \quad (3.8)$$

identifying $\theta_i - \theta_j = P$, $\theta_i - \theta_k = k$, $\theta_l - \theta_j = l$,

$$\theta_k - \theta_l = P - k - l \quad \text{and}$$

$$\frac{1}{N^2} \sum_{k,l} = \int \left(\frac{d^D l}{2\pi}\right)^D \int \left(\frac{d^D k}{2\pi}\right)^D. \quad (3.9)$$

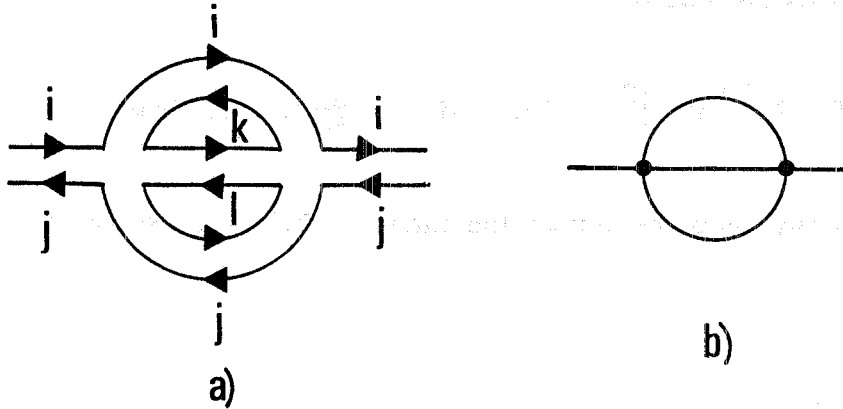


FIG. 3 - Diagrams for the renormalization arguments.

which is exactly the usual Feynman diagram in Fig. 3b.

It should be noted that the result would not be true when the two free indices k and l happen to coincide, but such events occur rarely for they are higher orders in $1/N$. Therefore in the leading order of $1/N$ we do not have to bother.

There is a sharp cut-off in the theory, we can remove it by extending the limit of integration from $(-\pi, \pi)$ to $(-\infty, \infty)$. The usual renormalization can be discussed in the general way. Consider two diagrams shown in Fig. 4, where the mass

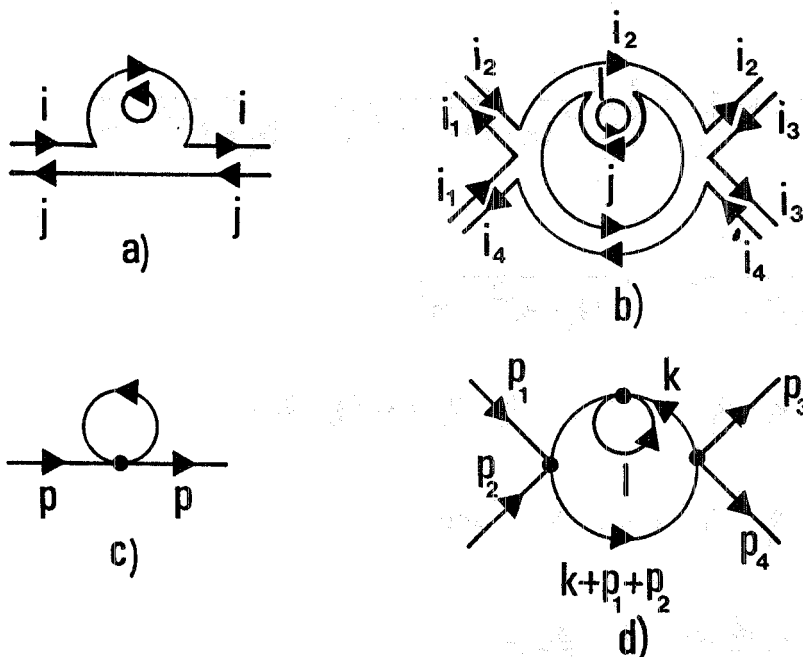


FIG. 4 - Diagrams for the renormalization arguments.

renormalization is required. The contributions are respectively

$$\left(\frac{g}{N}\right) \sum_k \frac{1}{\left[(\theta - \theta_j)^2 + m^2\right]^2} - \frac{1}{(\theta_i - \theta_k)^2 + m^2} \quad (3.10)$$

and

$$\begin{aligned} & \left(\frac{g}{N}\right)^3 \frac{1}{(\theta_{i1} - \theta_{i2})^2 + m^2} \frac{1}{(\theta_{i4} - \theta_{i1})^2 + m^2} \frac{1}{(\theta_{i2} - \theta_{i3})^2 + m^2} \frac{1}{(\theta_{i3} - \theta_{i4})^2 + m^2} \\ & \cdot \sum_{j,l} \frac{1}{\left[(\theta_{i2} - \theta_j)^2 + m^2\right]^2} - \frac{1}{(\theta_j - \theta_l)^2 + m^2} - \frac{1}{(\theta_{i4} - \theta_j)^2 + m^2} \end{aligned} \quad (3.11)$$

Making the identifications similar to (3.9) and are indicated in Fig. 4c) and Fig. 4d). In this way all the references on particular θ 's are only in the momentum forms, which are appropriate to the corresponding Feynman diagrams in the usual sense. If we subtract the quadratic divergent part in Fig. 4a, the subtraction will be same for Fig. 4b. Therefore it is renormalizable in the common sense.

In the infinite volume we have the general factorization property at large N limit

$$\langle \text{tr A tr B} \rangle = \langle \text{tr A} \rangle \langle \text{tr B} \rangle, \quad (3.12)$$

where A and B are products of M's and are not necessarily local objects. To be consistent, on a single point, we have to have the factorization property of the type

$$\int \prod_{\mu,a} \frac{d\theta_a^\mu}{2\pi} \langle \text{tr A}(\theta) \text{tr B}(\theta) \rangle_\theta = \int \prod_{\mu,a} \frac{d\theta_a^\mu}{2\pi} \langle \text{tr A}(\theta) \rangle_\theta \int \prod_{\mu,a} \frac{d\theta_a^\mu}{2\pi} \langle \text{tr B}(\theta) \rangle_\theta \quad (3.13)$$

which was implied in refs. (6,7). This can be viewed as the generalization of (3.12) on a single point in the presence of background gauge fields. Its validity can be checked using perturbation theories and can be also understood by inspection. Since in $\langle \text{tr A}(\theta) \text{tr B}(\theta) \rangle_\theta$ there is no free index summation joining two M's belonging separately to tr

$A(\theta)$ and $\text{tr } B(\theta)$, there is no θ -phases being assigned identically from $\text{tr } A(\theta)$ to $\text{tr } B(\theta)$. While the index summations are running independently, the indices belonging to $\text{tr } A(\theta)$ and $\text{tr } B(\theta)$ can run across each other accidentally, which causes the connected correlations in θ 's. But this is again of higher order in $1/N$, in accordance with the observations made in Sec. 2.

4. - NON-LINEAR MATRIX MODEL

The non-linear generalization of (3.1) will be discussed in this section, which happens to be called chiral spin model in the literature (8). It can be reviewed as the non-linear generalization of (3.1) in the following way. Take the theory to be

$$S = \beta \int_{\mathbf{x}} \text{tr} (\partial_{\mu} M \partial_{\mu} M^{\dagger}) d^D \mathbf{x} \quad (4.1)$$

where M is arbitrary complex matrix. The interaction is induced by the constraint on the integral measure in the same sense usually done for non-linear σ model

$$\delta(MM^{\dagger} - 1) \quad (4.2)$$

at every space time point.

This model may be one of the simplest to give planar diagrams at large N , as can be seen by introducing a dummy hermitean matrix field $\alpha(x)$. The partition function is now

$$Z = \int \mathcal{D}M \mathcal{D}\alpha e^{-\beta \int \{ \text{tr} (\partial_{\mu} M \partial_{\mu} M^{\dagger}) + i \text{tr} \alpha (MM^{\dagger} - 1) \}} \quad (4.3)$$

The planar diagram interpretation at large N follows obviously when we put M and α into the double line representation⁽¹⁾.

At zero dimension the planar diagrams are readily summed

$$Z \propto \int \mathcal{D}a (\det a)^{-N} e^{-i \text{tr} a} = \int \mathcal{D}a e^{-\text{tr} a - N \ln \det a} \quad (4.4)$$

It is clear the expression depends only on eigenvalues of the hermitean matrix a , the integral can be evaluated by following closely the method of ref. (2). At one dimension on a lattice the theory is exact 2-dimensional lattice gauge theory⁽⁷⁾.

Here we are mainly concerned with its lattice version, which is $U(N) \times U(N)$ chiral spin model

$$-S = \beta \sum_{x, \mu} \text{tr} U(x) U^+(x + \hat{\mu}) + \text{C.C} \quad (4.5)$$

Using the generalized boundary conditions (1.4), we reduce the chiral spin model to a single point

$$-S = \beta \sum_{\mu} \text{tr} U e^{i\theta\mu} U^+ e^{-i\theta\mu} + \text{C.C} \quad (4.6)$$

In the following we will show the two theories are equivalent. We will proceed with Dyson Schwinger equations and show that theories produce the same set of D-S equations and assume that the Dyson-Schwinger equations of all kinds correlation functions are sufficient to specify the theory⁽⁴⁾. Thereby we establish the announced equivalence.

We start with a simple expression

$$\langle \text{tr} U(k) U^+(\ell) T^j \rangle \quad (4.7)$$

where T^j s are $U(N)$ generators. We get a D-S equation by making a shift at position ℓ , $U(\ell) \rightarrow (1 + i \varepsilon T^j) U(\ell)$

$$N \langle \text{tr} U(k) U^+(\ell) \rangle + \beta \sum_{\hat{Q}} \langle \text{tr} U(k) U^+(\ell) U(\ell + \hat{Q}) U^+(\ell) \rangle - \quad (4.8)$$

$$- \beta \sum_{\hat{Q}} \langle \text{tr} U(k) U^+(\ell + \hat{Q}) \rangle = 0.$$

Like in a gauge theory we have to consider the correlations with more than one operator for site, for example

$$\langle \text{tr } U(k) U^+(\ell) U(m) U^+(\ell) \rangle \quad (4.9)$$

is related to higher order correlations via

$$\begin{aligned} N \langle \text{tr } U(k) U^+(\ell) U(m) U^+(\ell) \rangle + \langle \text{tr } U(k) U^+(\ell) \rangle \langle \text{tr } U(m) U^+(\ell) \rangle + \\ + \beta \sum_{\hat{Q}} \langle \text{tr } U(k) U^+(\ell) U(\ell + \hat{Q}) U^+(\ell) U(m) U^+(\ell) \rangle - \\ - \beta \sum_{\hat{Q}} \langle \text{tr } U(k) U^+(\ell + \hat{Q}) U(m) U^+(\ell) \rangle = 0. \end{aligned} \quad (4.10)$$

Here we have used the factorization assumption for the second term at large N , which ensures the closeness of the infinite set of D-S equations.

As in gauge theories we can go on to consider all complicated correlations by using the well known flipping - switching methods⁽¹⁰⁾. Note that since there is no local gauge invariance, we will have the D-S equations of the type

$$\sum_{\hat{Q}} \langle \text{tr } U(R) U^+(\ell) U(m) U^+(m + \hat{Q}) \rangle = \sum_{\hat{Q}} \langle \text{tr } U(k) U^+(\ell) U(m + \hat{Q}) U^+(m) \rangle \quad (4.11)$$

by shifting a variable at an arbitrary position m instead of two ends.

Next we work on a single point. The mapping between the theory defined in the infinite volume and on single point is proposed to be

$$\langle \text{tr } U(k) U^+(\ell) \rangle = \int \prod_{\mu, a} \frac{d\theta_{\mu}^a}{2\pi} \langle \text{tr } U e^{i \sum_{\mu} n_{\mu} \theta_{\mu}} U^+ e^{-i \sum_{\mu} n_{\mu} \theta_{\mu}} \rangle_{\theta} \quad (4.12)$$

where n_{μ} specifies the minimal number of steps needed to go from point k to ℓ in

μ -th direction and which can be negative in the anti-direction, $\langle \rangle_{\theta}$ denotes average using the action (4.6).

Repeat the above steps, for example, we start with

$$\left\langle \text{tr } U e^{i \sum_{\mu} n_{\mu} \theta_{\mu}} U^{\dagger} T^j e^{-i \sum_{\mu} n_{\mu} \theta_{\mu}} \right\rangle_{\theta} \quad (4.13)$$

we get same equation as (4.8) except one more term

$$\left\langle \text{tr } e^{-i \sum_{\mu} n_{\mu} \theta_{\mu}} \text{tr } U e^{i \sum_{\mu} n_{\mu} \theta_{\mu}} U^{\dagger} \right\rangle_{\theta} \quad (4.14)$$

After averaging over the background gauge fields and using the factorization property discussed in Sec. 3

$$\int \prod_{\mu} \frac{d\theta_{\mu}}{2\pi} \langle \text{tr } A \text{tr } B \rangle_{\theta} = \int \prod_{\mu} \frac{d\theta_{\mu}}{2\pi} \langle \text{tr } A \rangle_{\theta} \int \prod_{\mu} \frac{d\theta_{\mu}}{2\pi} \langle \text{tr } B \rangle_{\theta} ,$$

this contribution vanishes.

The phases θ 's are taking care that just the correct source terms such as the second term in (4.10) is kept intact. One can get on to convince oneself that we have the one to one correspondance of D-S equations between the theory defined on a single point and that in the infinite volume. For the equations of the type of (4.11), we simply start with the expression

$$\left\langle \text{tr } U e^{i \sum_{\mu} n_{\mu} \theta_{\mu}} U^{\dagger} e^{-i \sum_{\mu} n_{\mu} \theta_{\mu}} e^{i \sum_{\mu} m_{\mu} \theta_{\mu}} T^j e^{-i \sum_{\mu} m_{\mu} \theta_{\mu}} \right\rangle \quad (4.15)$$

Therefore, we have established the equivalence of the two theories in the large N limit.

It is interesting to check if there is any kind of spontaneous symmetry breaking in the weak coupling regions, where is seen that U matrix is fluctuating around

diagonal form

$$U = V e^{i\theta\varphi} V^\dagger \quad V = e^{ia} \quad (4.16)$$

$$a^\dagger = a \sim 0$$

where φ is another diagonal matrix of eigenvalues of U.

Let us look at the free energy

$$F = \int \prod_{\mu,a} \frac{d\theta_\mu^a}{2\pi} \ln \int dV \prod_a d\varphi_a \prod_{a>b} \sin^2 \frac{\varphi_a - \varphi_b}{2} \quad (4.17)$$

$$e^{-\beta \sum_\mu \text{tr} V e^{i\varphi} V^\dagger e^{i\theta_\mu V} e^{-i\varphi} V^\dagger e^{-i\theta_\mu}}$$

Carrying out the interaction over a which expanded to second order is gaussian, we notice that the repulsion between the eigenvalues φ_a 's is just cancelled out by the attraction induced in the exponent, therefore the eigenvalues are perfectly randomly distributed even in the weak coupling regions and there is no spontaneous symmetry breaking observed for gauge theory in ref. (5).

5. GAUGE THEORIES

In ref. (7) we discussed the gauge theories at large N and the main conclusion there is for O(N) gauge group. Some difficulty was pointed out for U(N). It is that if we do not specify a gauge condition, when reduced to a single point the θ 's dependence can be transformed away completely, which is again the Eguchi-Kawai model. This fact suggests that the trouble with the gauge theories lies in its gauge freedom in the infinite volume. To be consistent, we have to strip its freedom by

fixing the gauge condition before reducing to a single point, stating equivalently, we have to require that only physical variables can experience the background gauges on a single point. For simplicity, we choose the axial gauge where $U_t=1$

The reduced action now is

$$-S = \beta \sum_{i \neq j} \text{tr} U_i U_j U_i^\dagger U_j^\dagger + (\beta \sum_j \text{tr} U_j e^{i\theta_t} U_j^\dagger e^{-\theta_t} + \text{C.C.}) \quad (5.1)$$

where i and j run from 1 to 3.

We would like first to study the problematic weak coupling region, in which U_i 's are fluctuating around its diagonal form (5). Note also that now there is no a $U(N)$ global gauge transformation which leaves the eigenvalues of U_i 's invariant, which may suggest that the eigenvalues of U_i 's do not play particular rôle for the physical observables.

We calculated the eigenvalue distribution for large β following ref. (5), it is found

$$F \propto \int \prod_a \frac{d\theta_a^t}{2\pi} \ln \int \prod_{i,a} d\theta_a^i e^{a \sum_b \sum_i \left\{ \ln \sin^2 \frac{\theta_t^a - \theta_t^b}{2} - \ln \sum_{\nu}^{(i)} \sin^2 \frac{\theta_\nu^a - \theta_\nu^b}{2} \right\}} \quad (5.2)$$

where ν runs from 1 to D and the D th θ is θ_t which is quenched by assumption, (i) denote summation with i th term missing. It is easily seen that it is consistent with the observations made in ref. (5) that for $D > 2$ the θ^i 's will attract each other at weak coupling regions and degeneracy in eigenvalues θ^i 's may occur.

To the first order, we assume all θ_a are equal to one value α^i as was done in ref. (5).

$$U_i = e^{i\alpha^i} e^{ib_i} \quad b_i = b_i^\dagger \quad \text{tr} b_i = 0 \quad (5.3)$$

where b_i 's are small. When expanding in b_i we see in this case the first term in (5.1)

gives a quartic term while the second term in (5.1) gives a quadratic one. In other words, in the weak coupling region, where degeneracy in the eigenvalues occurs, the second term in (5.1) is dominating, therefore the internal energy is correctly obtained in the first order in $1/\beta$, which is $1 - \frac{1}{2D} \frac{N}{\beta}$.

Since the second term in (5.1) is dominating in the weak coupling region, the eigenvalues of b 's are randomly distributed as the case of the chiral spin, therefore, we expect that open Wilson loops in ref. 4) will have vanishing contribution even at weak coupling region. We have not shown this explicitly. It seems that only a Montecarlo check can decide if (5.1) is identical to the Wilson's theory in the intermediate coupling regions.

6. - MONTECARLO SIMULATIONS

It is tempting to apply these results for simulating QCD on a lattice. The most naive proposal would be to take our formulation for the $O(N)$ group ($SO(N)$ and $O(N)$ are equal at $N \rightarrow \infty$ limit) and implement for $N=10 \sim 20$. What could we gain from numerical calculation point of view (we suppose $1/N$ corrections are negligible)?

It is clear that different Montecarlo runs must be done for different values of the θ 's. However the number of the θ 's which are extracted (N_θ) seems to be the critical parameter. It is easy to understand that the single point system (at $N \gg 1$) will simulate a system of volume proportional to N_θ .

This observation implies that at the end we cannot avoid to do a number of extractions proportional to the effective volume we consider.

The computation will become very long with increasing N .

If the main limitation is CPU time, nothing is gained. If the main limitation is memory (and this starts to be the case in the future), something important can be done.

Indeed in an SU(N) theory we expect that the finite size effects should be reduced by quenching up to a factor $\frac{\alpha N}{N} = \alpha$. This means that in a typical Montecarlo simulation for SU(3) we could work on a large lattice (say, 6^4) and reduce of a factor 10-20% the finite volume effects by quenching the θ 's. This operation would correspond to an increase of the volume by a small factor ($2 \sim 3?$), and therefore it seems that it would not be necessary to perform many different choices of the θ 's.

For this purpose we need an algorithm working for SU(3) and it is easy to implement on a computer. As discussed in the preceding section that the main difficulty with gauge theories is that it is not so simple to superimpose a background gauge field. However if gauge fields are composite as in the CP^{n-1} model, this difficulty disappears. It was shown by Bars that the SU(N) gauge theory can be written in terms of "corner operators" $W_\mu(x)$ on a lattice in such a way that

$$U_\mu(i) = W_\mu(i) W_\mu^+(i + \hat{\mu}),$$

and the action is the standard one in terms of the U's. Now the theory is obviously invariant under the transformation

$$W_\mu \longrightarrow W_\mu R,$$

R is a SU(N) rotation and

$$U'_\mu = W'_\mu(i) W'_\mu(i + \hat{\mu}) = U_\mu.$$

We can now put the background gauge fields acting on the W_μ 's from right hand side, the argument of preceding sections should apply here too. The final theory is more complicated than the O(N) theory which should be preferred for analytic computations but it is more suited for numerical simulations. It is easy to see that this choice leads

to the form at the action in the ref. (5).

In practice the prescription on a lattice L^4 amount to doing the standard computation for internal links and to extract the link which are responsible of the boundary conditions with a different measure, e.g.

$$U_1(L, y, z, t) = L^{\mathbf{x}}, (y, z, t) W(1, y, z, t) D_1(\theta) W^{\mathbf{x}}(1, y, z, t)$$

where

$$L_1(y, z, t) = \prod_x^{L-1} U_1(x, y, z, t),$$

W is a $SU(3)$ matrix extracted with the Haar measure and $D_x(\theta)$ is a diagonal $U(3)$ matrix containing the θ 's.

Another possibility, at least in the scalar case discussed before, would be to try to use the replica trick. We write

$$\langle \ln Z \rangle_{\theta} = \frac{d}{dn} \langle Z^n \rangle_{\theta} \Big|_{n=0}$$

$$Z = \int dM \exp - S(M, \theta)$$

where $S(M, \theta)$ is given in 3.5) and $\langle \rangle_{\theta}$ stands for

$$\int \prod_i \frac{d\theta_i}{2\pi} \exp - \frac{1}{2} \sum_i \frac{\theta_i^2}{\Lambda^2},$$

and Λ plays the rôle of an ultraviolet cutoff.

For integer n we have that

$$\langle Z^n \rangle_{\theta} = \int \prod_{\alpha=1}^n dM^{\alpha} \exp - \left[\sum_{\alpha=1}^n S(M^{\alpha}, \theta) \right]$$

where the index α runs over the different replicas.

The integral over the θ is now trivial (gaussian) and it gives

$$\langle Z^n \rangle_\theta = \int \prod_{\alpha=1}^n dM^\alpha \frac{1}{P(M)^{D/2}} \exp - \sum_{\alpha=1}^n S_R(M^\alpha)$$

where $S_R(M) = m^2 \text{tr} MM^+ + \frac{g}{4N} \text{tr}(MM^+ MM^+)$ and $P(M)$ is a polynomial of order $2N$ in the $M_{a,b}^\alpha$.

If we know the value of $\langle Z^n \rangle_\theta$ for integral n we can extrapolate it at $n=0$. This can be done analytically (that would solve the problem) in the limit $N \rightarrow \infty$ or numerically by Montecarlo technics. The advantage here would be to bypass the problem of integrating over the θ 's numerically. It is clear that a lot of work has still to be done in this direction.

7. - CONCLUSIONS AND DISCUSSIONS

We have been following a systematic approach to reduce some field theories from infinite volume to a single point. As a result the original models are now few matrix problems; the linear and non linear matrix models in D -dimensional space time now is a problem of integrating one matrix coupled to d gauge phases. $U(N)$ gauge theory in our description now is a problem of integrating $D-1$ $U(N)$ matrices coupled to one gauge phases. Most interesting and challenging problem would be to find exact solutions. The relations between gauge theory and chiral spin model, if there is any, could be discussed on a single point if we are only interested in the large N limit. We discuss one plausible relation between the $U(N)$ gauge theory and the chiral spin model in the following. Let us consider $U(N)$ lattice gauge theory on a single point for D dimensions (5.1). Let us concentrate on the weak coupling region $\beta \gg 1$, we know that the U matrices are restricted to fluctuate around their diagonal form. The following parametrization might be possible; we choose one of U matrices to be fast

fluctuating variable which is a $N \times N$ unitary matrix while the other $D-2$ to be slow variables which are already in their diagonal forms (11). But the fluctuations of these diagonal matrices can be still react back to the 'fast' variable, the N by N unitary matrix. In order to make the diagonal matrices really slow variables we have to immobilize the eigenvalues of those $D-2$ directions, that is the quenching. The free energy obtained in this way is almost the one of the chiral spin model in $D-1$ directions, except some extra measure factors for eigenvalues, which are to make them repeling each other, since those variables are already quenched, these effects are not important.

After completing this paper, several works arrived us⁽¹³⁻¹⁶⁾, in which the author discuss the EK model or modifications from various points of view, and arise many interesting questions.

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