To be submitted to
Lett. al Nuovo Cimento

ISTITUTO NAZIONALE DI FISICA NUCLEARE
Laboratori Nazionali di Frascati

LNF-82/43(P)
28 Giugno 1982

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ON PREHISTORY OF SPIN KINEMATICS

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Every physicist is aware of the fact that Hamilton's quaternion theory (1843) played a major role in the early history of spinor theory[1]. On the other hand, however, it is remarkable that the current physics literature fails to appreciate the fact that Darboux's spinor picture of rigid body kinematics[2] does contain in itself the same mathematical formalism as that used to day in spin kinematics. It seems worthwhile therefore to close this information gap, now that there is active interest in spin kinematics, owing to its convenience in the description of polarization effects for relativistic particles trapped in the field of a circular accelerator or storage ring[3-8].

In 1887, Vol. I of Darboux's Leçons[2] on theory of surfaces appeared. This Vol. I is based on a course he had given at the Sorbonne during the 1882-1885 academic years. In Part I, using kinematical approach, he made fundamental contributions to geometry.

For the purposes of the present note it will suffice to restrict our attention at what is embodied in the first three Chapters of Part I, Vol. I.

In Chapt. I he formulated the three-dimensional real vector picture of the kinematics of a rigid body constrained by being pivoted at one fixed point, namely the equation

$$\dot{\bar{s}} = \bar{s} \times \bar{\omega} , \tag{1}$$

which can be expressed in terms of the components $(\alpha, \beta, \gamma) \equiv \bar{s}$ and $(p, q, r) \equiv \bar{\omega}$ as

$$\dot{\alpha} = \beta r - \gamma q , \qquad \dot{\beta} = \gamma p - \alpha r , \qquad \dot{\gamma} = \alpha q - \beta p \tag{1a}$$

(his Eqs. (8), page 5). Here, $\bar{\omega}(t)$ specifies the (known) angular velocity of rotation of the body about its istantaneous axis and $\bar{s}$ denotes the position-vector of any (selected) point which lies on the surface of the unit sphere fixed in the body and with center at the fixed point.

Now, the point that has not till now attracted particular notice from physicists is the following: On Chapt. II, Darboux was able to write eqs. (1a) in spinor form, namely

$$\dot{\nu} = -c\mu - (b-h)\nu \,, \qquad \dot{\mu} = a\nu + (b+h)\mu \tag{2}$$

(his Eqs. (16), pag. 25), where $(\nu, \mu) \equiv \psi$ is a spinor; a, b and c are given by

$$2a = -(q-ip) \,, \qquad 2b = -ir \,, \qquad 2c = -(q+ip) \tag{3}$$

and h denotes any arbitrary function.

More closely related to our interests is the case $h = 0$, which had been considered by Darboux in the second edition[9] of his Leçons, i. e. the eqs.

$$\dot{\nu} = -c\mu - b\nu \,, \qquad \dot{\mu} = a\nu + b\mu \tag{4}$$

(his[9] Eq. 22, page 35). These eqs. are, in fact, identical with

$$\dot{\psi} = (i/2)\,\bar{\omega}\cdot\bar{\sigma}\,\psi \,, \tag{4a}$$

i. e., formally identical with the time-dependent Schrödinger equation for the spin motion of a point-like electron interacting with an external magnetic field. In eq. (4a), the components of the vector $\bar{\sigma}$ are given by the Pauli spin matrices

$$\sigma_p = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \,, \qquad \sigma_q = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} \,, \qquad \sigma_r = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \,. \tag{5}$$

Once the close formal similarity between the rigid-body kinematics and the spin kinematics is recognized, it should be not surprising that the result above has relation to the precession of a magnetic dipole in a magnetic field. In fact, the basic eqs. (1) can be reinterpreted, quite generally, as if they were the Thomas-BMT[10] equations governing the precession of the spin vector, $\bar{s}$, of a relativistic point-like particle moving in an electromagnetic field (eq. (3) of Ref. (10)). (Readers interested in knowing the general expression of $\omega(t) \equiv (p, q, r)$ in this case are referred, e. g. to Ref. (4)).

So it was Darboux who first recognized the potentialities of the spinor formalism. Moreover, the spinor picture of spin motion, eqs. (4), central to analyses of polarization phenomena in high-energy synchrotrons and storage rings, arises directly and naturally from Darboux's original treatment.

Because of both the great elegance of the Darboux's formulation[2], as well as its historical interest, it will be given in detail in the subsequent remaining part of the present note.

Darboux first observes that eqs. (1a) have a conservation law associated with them. By multiplying $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$ by α, β, γ respectively and adding the three equations, it follows that

$$\alpha^2 + \beta^2 + \gamma^2 = \text{constant} \tag{6}$$

(his[2] eq. (2), page 19). Since he assumes that $\bar{s}$ traces out an orbit on the unit sphere, then necessarily

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \tag{7}$$

(his eq. (7), page 21).

The key to the reduction of eqs. (1a) to the pair of eqs. (2) is the parametrization of the unit-sphere eq. (7), namely

$$\alpha = (1-xy)/(x-y)\,, \qquad \beta = i(1+xy)/(x-y)\,, \qquad \gamma = -(x+y)/(x-y)\,, \tag{8}$$

so that x = constant, $-1/y$ = constant are the two conjugate imaginary systems of generators (of course, the point α, β, γ is real if and only if x, $-1/y$ are conjugate complex). Clearly, as remarked incidentally by Darboux on Chap. III, pages 36-37, the same mapping of the complex variable $x = (\alpha + i\beta)/(1+\gamma)$ on the unit sphere can be obtained by the well-known stereographic projection of the sphere to the complex plane.

Apart from the change $\gamma \to -\gamma$, our eqs. (8) coincide with Darboux's[2] Eq. (9), page 22. The replacement $\gamma \to -\gamma$ permits one to express $\bar{\sigma}$ (in our eqs. (4a)) in terms of the customary Pauli matrices, eqs. (5). We feel that the advantage of this slight change outweighs that of historical accuracy.

Combining the parametric eqs. (8) with (1a), Darboux obtains a Riccati differential equation for x, namely

$$\dot{x} = a + 2bx + cx^2 \tag{9}$$

(his eq. (13), page 23). The corresponding differential equation for y is the same as eq. (9). (Sign reversal of a and c gives Darboux's eqs. (10), page 22).

Finally, on introduction of two new functions μ and ν by the substitution

$$x = \mu/\nu\,, \tag{10}$$

he obtains the equation

$$\nu\dot{\mu} - \mu\dot{\nu} = a\nu^2 + 2b\mu\nu + c\mu^2 , \qquad (11)$$

and notes that it splits into the two coupled eqs. (2).

Acknowledgment

The author is indebted to Mrs. Prof. M. A. Snaider for kindly making the facilities of the Library of Istituto Matematico G. Castelnuovo available to him.

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