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IMPROVING THE LATTICE ACTION NEAR THE CONTINUUM LIMIT

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We propose a general method to partially compensate for the effects of the regulator for a lattice quantum field theory close to the continuum limit (i.e., the lattice spacing $a \sim 0$). An explicit example is given.

In the last few years it has become fashionable to study euclidean quantum field theories (qft) on the lattice. The properties of the exact qft are supposed to be recovered in the limit where the lattice spacing a goes to zero. There are many actions which can be written on the lattice which formally converge to the same action in the continuum as a consequence of the freedom in adding arbitrary operators which vanish when $a \to 0$. For example, for SU(N) gauge theories various different actions [1] have been studied which approach the renormalization group behaviour expected in the weak-coupling regime ($a \to 0$) at different rates [2]. In this letter we propose to write an action for which the effects of the lattice regularization are minimized.

We work in the framework of the paper by Symanzik [3] where the following general statement is made. For small lattice spacing, a given lattice approximation of a continuum renormalizable field theory in d dimensions is equivalent to a theory on the continuum, having an action

$$S_d + a^2 S_{d+2} + O(a^4), \tag{1}$$

where S_d is the ordinary continuum action; S_{d+2} contains operators of dimension $d + 2$ which are the remnants of the lattice regularization and are perturbatively computable. We will give a general criterium for eliminating operators of dimensions $d + 2$ up to two-loop corrections. We illustrate our method with an example of the O(N) non-linear sigma model in two dimensions: the generalization to any renormalizable theory is straightforward.

For the O(N) non-linear sigma model, the action on the continuum is

$$S = (\beta/2) \int d^2x \, [\partial_\mu s(x)]^2, \tag{2}$$

where $s(x)$ is a real N-component field constrained by

$$s(x) \cdot s(x) = 1 \tag{3}$$

and β is the inverse coupling constant. We define on a square lattice the discretized action as follows:

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$$S=\sum_x \{(-\tfrac{4}{3}\beta + A_1)[s(x+\hat{\mu}_x)+s(x+\hat{\mu}_y)]\cdot s(x)$$

$$+(-\beta/12 + A_2)[s(x+2\hat{\mu}_x)+s(x+2\hat{\mu}_y)]\cdot s(x) + A_3[s(x+\hat{\mu}_x+\hat{\mu}_y)+s(x+\hat{\mu}_x-\hat{\mu}_y)]\cdot s(x)\}, \quad (4)$$

$\hat{\mu}_{x,y}$ are the unit vectors in the $x-y$ directions. The terms proportional to β in eq. (4) have been chosen to give, for $a\to 0$, a bare propagator $G_0(k)$:

$$G_0(k)\approx [k^2 + O(a^4k^6)]^{-1}. \quad (5)$$

In other words, we used a discretized form of the lagrangian which is correct up to $O(a^4)$.

We fix A_i as functions of β in such a way as to cancel order by order S_{d+2}. (At orders greater than $1/\beta$ one should also add four spin couplings.) A_i have a perturbative expansion in powers of $1/\beta$:

$$A_i = A_i^0 + (1/\beta)A_i^1 + \dots . \quad (6)$$

We computed A_i^0 which appear at one loop by computing the spin–spin correlation function on the lattice [4].

Even if we are interested in an $O(N)$ invariant quantity such as the spin–spin correlation function, to make a perturbative computation we have to break the $O(N)$ symmetry by defining

$$s(x) = ((1-\pi^2(x))^{1/2}, \pi(x)), \qquad \pi^2(x)\equiv \pi(x)\cdot\pi(x) \quad (7)$$

$\pi(x)$ is a $(N-1)$ component field which is of $O(1/\sqrt{\beta})$. Some care must be taken because the $\pi(x)$ correlation function is infrared divergent while infrared divergences cancel for invariant quantities (analogously, for gauge theories one should compute a gauge invariant quantity such as the Wilson loop).

We define

$$G(x)\equiv \langle s(x)\cdot s(0)\rangle_c \approx \langle \pi(x)\cdot\pi(0)\rangle_c + \tfrac{1}{4}\langle \pi^2(x)\pi^2(0)\rangle_c + O(1/\beta^3), \quad (8)$$

c means connected.

Using standard perturbation theory one finds:

$$G(k)=\frac{N-1}{\beta}\left\{G_0(k)-\frac{G_0^2(k)}{\beta}\left[\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi} d^2p\left(\frac{F(p-k)}{F(p)}-1-\frac{1}{2}\frac{F(k)^2}{F(p)F(p-k)}\right)+2A_1(2-\cos k_x-\cos k_y)\right.\right.$$

$$\left.\left.+2A_2(2-\cos 2k_x-\cos 2k_y)+2A_3[2-\cos(k_x+k_y)-\cos(k_x-k_y)]\right]\right\}, \quad (9)$$

$$G_0(k)\equiv 1/2F(k); \qquad F(k)=\tfrac{7}{3}-\tfrac{4}{3}(\cos k_x+\cos k_y)+\tfrac{1}{6}(\cos^2 k_x+\cos^2 k_y).$$

The second term in the integral, coming from the constraint [eq. (3)] cancels the ultraviolet divergence of the first integrand. The third term cancels infrared divergences. We chose A_i^0 to cancel terms of $O((1/\beta)a^2k^4)$ in eq. (9).

The number of independent constants corresponds to the number of independent operators of dimension 4:

$$\sum_\mu [\partial_\mu^2 S]^2, \qquad \sum_{\mu,\nu} [\partial_\mu\partial_\nu S]^2. \quad (10)$$

By integrating eq. (9) numerically, one obtains:

$$A_2 = -0.0147, \qquad A_3 = -0.0046, \quad (11)$$

with an accuracy of less than 5%. In the perturbative region ($\beta \gtrsim 1$) we found very small corrections at the one loop level: most of the $O(a^2)$ effects are already given by the lowest-order first-neighbour and next-to-neighbour couplings of eq. (4).

We observe that by using Manton or Villain type actions one finds, at the one-loop level, the same counterterms we have found for the Wilson-like action. Indeed, the difference between these various actions is of the form

$$\beta \sum_{\mu} \{[s(x+\mu) - s(x)]^2\}^2 \approx O(a^2/\beta). \tag{12}$$

This term only provides a wave function renormalization for the π field without changing the $O(a^2)$ corrections to the two-point function we have considered. In general, these kinds of terms modify the $O(a^2)$ corrections to the four-point function.

The method we have proposed in this paper is quite general. The size of the corrections we find for the $O(n)$ non-linear sigma model in two dimensions is small: however, this might be peculiar to the model we considered and analogous computations should be repeated for the interesting case of lattice gauge theories.

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