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MEAN FIELD THEORY FOR SPIN GLASSES

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In these recent years our understanding of spin glasses has strongly improved. In this talk I will describe the results which have been obtained in the framework of broken replica symmetry[1-13] for Ising spins.

Generally speaking in order to build a theory to describe a certain kind of phase transitions, we must go through five steps:

I) We firstly identify the physical characteristics which distinguish the two phases of the system (e. g. spontaneous magnetization).

II) We introduce an order parameter which quantifies the differences between the two phases (e. g. the value of the spontaneous magnetization).

III) We write an effective free energy as function of the order parameter; the value of the order parameter is fixed minimizing the free energy (mean field approximation) (a typical result is $m = th(\beta Jh)$).

IV) We verify that in the long range limit the mean field approximation gives the exact answer (e. g. for ferromagnetic systems this result can be rigorously proved).

V) We finally compute the corrections to the mean field approximation due to the finite range of the interaction; at this stage we can apply the whole machinery of diagrammatic expansion, renormalization group.

We shall now see how this program can be implemented for Ising spin glasses.

1. - The Glassy phase.

The main characteristics of spin glasses is the presence of irreversible effects and of very large times of approach to equilibrium. These effects show up dramatically if we compare the alternate magnetic susceptibility (χ_{LR}) with the field cooled susceptibility ($\chi_e = \frac{dM_e}{dH}$)[14-16]. It is an experimental fact that:

$$\Delta\chi = \chi_e - \chi_{LR} > 0 \tag{1.1}$$

and remanence is present in the region

$$|H| < H_c(T) \tag{1.2}$$

where the critical magnetic field $H_c(T)$ goes to zero for $T \to T_c$.

This behaviour is due to the presence of many different states in which the system may stay. These states can be characterized at the microscopical level by the value of the magnetization at each point $m_i^{[s]}$: the index i runs over the points of the lattice and s labels the different states.

When $|H| > H_c(T)$ there is only a stable state of the system; when $|H| < H_c(T)$ there are many states which are stable for small perturbations, however most of these states will be metastable states and only a small fraction will be asymptotically stable. When we change the magnetic field at fixed temperature by a very small amount, some of the previously asymptotically stable states will become metastable states and the stable will become metastable. However if the hopping time from one state to an other one is much larger than the time scale at which the magnetization M is measured, hopping can be neglected: the system remains locked in the vicinity of one state and the linear response theory can be applied to compute χ_{LR}. When we measure χ_e, we hope to be in a asymptotically stable state: the magnetization is the equilibrium one $(M_e(T,H))$ and $\chi_e = \frac{\partial}{\partial H} M_e(T,H)$.

In other words when we change H at fixed T a spin glass system should undergo a sequence of microtransitions: in a finite volume, when we hop from one state to an other state, the magnetization jumps discontinuously[9,17,18,19]; when the volume goes to infinity the distance in H between two transitions and the discontinuity in M should go together to zero: $M_e(T,H)$ should be finally a continuous curve, it is clear however that the derivative with respect to H and the infinite volume limit do not commute. Mathematically speaking the zeros of the partition function become dense near the real H axis for all values of H in the glassy region $|H| \leq H_c(T)$.

2. - The order parameter.

The meaning of the local magnetizations $m_i^{[s]}$ introduced in the previous section, is very clear: we study the evolution in time (real or computer time) of the system and we define:

$$m_i = \frac{1}{t} \int_0^t d\tau\, \sigma_i(\tau) \qquad (2.1)$$

where t is a large but not too large observation time, e.g. in a D-dimensional ferromagnetic system of size L, t must satisfy the conditions:

$$\tau_m << t << \tau_m \exp(L^{D-1}) \qquad (2.2)$$

where τ_m is the microscopic relaxation time, e.g. one Montecarlo step.

When we change the initial condition we may obtain different results for the magnetizations ; by exploring different initial conditions the full set of $m_i^{[s]}$ for different s can be obtained.

Unfortunately it is not very clear how to implement this natural definition in the framework of equilibrium statistical mechanics where all the states receive a weight $\exp(-\beta H)$. Indeed the existence of two (or more) asymptotically stable states breaks the ergodicity hypothesis (i.e. independence from the initial conditions) which is at the heart of the Gibbs-Boltzmann approach.

Which is the standard solution in the ferromagnetic case ? When H is different from zero the system may stay only in one possible state ; the difficulties arise at $H = 0$: two states of opposite magnetization are present below the critical temperature. We can define the spontaneous magnetization at $H = 0$ by adding to the Hamiltonian a small term $\Sigma_i \varepsilon h_i \sigma_i$ where h_i are the values of an external magnetic field which may be space dependent : we define the h dependent spontaneous magnetization as

$$m_i^{[h]} = \lim_{\varepsilon \to 0^+} m_i(\varepsilon h) \tag{2.3}$$

where $m_i(\varepsilon h)$ is the magnetization in presence of the perturbation.

If in a ferromagnetic case we consider a staggered h field, $m_i^{[h]} = 0$; on the contrary if h is translational invariant there are only two possibilites $m_i = \pm m$, m being the spontaneous magnetization ; s takes only two values.

The value of the spontaneous magnetization can be obtained also by considering the correlation function $\langle \sigma_i \sigma_k \rangle$ at large distance $|i-k|$. In each of the possible states of the system we must have

$$\overline{\sigma_i \sigma_k} \to m_i^{[s]} m_k^{[s]} , \qquad |i-k| \to \infty . \tag{2.4}$$

We obtain that the thermodynamic average is given by :

$$\langle \sigma_i \sigma_k \rangle \to \Sigma_s p_s m_i^{[s]} m_k^{[s]} \tag{2.5}$$

p_s being the weight of the s^{th} state in the thermodynamic average. At zero external field

$$p_1 = p_2 = \frac{1}{2} , \qquad m^{[1]} = m^{[2]} = m , \qquad \langle \sigma_i \sigma_k \rangle \to m^2 . \tag{2.6}$$

The spontaneous breaking of the symmetry shows up as a violation of the clustering decomposition : the spontaneous magnetization can be computed using the

classical theorems on the decomposition of a non clustering state into a set of clustering pure states.

I am describing these two procedures in details because they cannot be applied to spin glasses: in the infinite volume limit the number S of possible microscopic states $m_i^{[s]}$ may become infinite and quantities like

$$\langle \sigma_i \sigma_k \rangle , \qquad \langle \sigma_i \sigma_k \rangle^2 \tag{2.7}$$

may go to zero exponentially without giving precise informations on the values of the $m_i^{[s]}$; often the onset of spin glass order is characterized by a non zero value for the order parameter $q_{EA}^{[s]}$ defined by

$$q_{EA}^{[s]} = \lim_{V \to \infty} \frac{\Sigma_i (m_i^{[s]})^2}{V} \tag{2.8}$$

V being the volume of the system. It should be clear that q_{EA} cannot be extracted from the knowledge of the spin correlation functions at large distance.

We face the same problem if we try to compute the $m_i^{[s]}$ by adding a magnetic field; from the point of view of the spin glass a constant magnetic field will look like a staggered magnetic field; for a generic choice of the h_i we will not be able to split different states and also in presence of a magnetic field we have:

$$m_i^{[h]} = \quad p_s \, m_i^{[s]} \tag{2.9}$$

where an infinite number of p_s may be different from zero also at finite h. However if we take the h_i proportional to one of the possible magnetizations, $h_i \propto m_i^{[s]}$ only the state labelled by s will contribute to the sum in eq. (2.9). In this way we obtain the bootstrap equation:

$$m_i^{[h]} = m_i^{[s]}, \qquad h_i \propto m_i^{[s]}. \tag{2.10}$$

For example in the ferromagnetic case previously described, the magnetic field is taken to be point independent as well as the magnetization.

In a spin glass we don't know the direction in which the magnetization points: eq. (2.10) seems to be useless and we face a dead end.

The way out can be found by introducing two real identical weakly coupled replicas of the same system[1-4]; the global Hamiltonian is:

$$H_2 = \sum_{i,k} J_{i,k} (\sigma_i^1 \sigma_k^1 + \sigma_i^2 \sigma_k^2) - 2\varepsilon \Sigma_i \sigma_i^1 \sigma_i^2 \tag{2.11}$$

where the $J_{i,k}$ are the random quenched coupling among the spins and the index $a = 1, 2$ labels the two replicas of the system ; when $\varepsilon = 0$ the replicas are decoupled. For positive ε each of the two replicas acts as an external magnetic field on the other one and both must be locked in the same state. In the limit $\varepsilon \to 0$ we find :

$$\Sigma_s p_s (m_i^{[s]})^2 = \langle \sigma_i^1 \sigma_i^2 \rangle \ . \tag{2.12}$$

If we assume that the quantity $q_{EA}^{[s]}$ is not s-dependent in the infinite volume limit and we introduce the quenched free energy density F_2 of the two coupled replicas we obtain :

$$\begin{aligned} q_{EA} &= - \frac{d}{d\varepsilon} F_2(\varepsilon)\Big|_{\varepsilon = 0} \ , \\ F_2(\varepsilon) &= \lim_{V \to \infty} - \frac{1}{2\beta V} \ln \left[\Sigma_{\{\sigma\}} \exp(- \beta H_2(\varepsilon) \right] . \end{aligned} \tag{2.13}$$

The volume V is the total number of spins in each replica. This apparently baroque construction is the only way at our disposal to compute q_{EA} in the framework of equilibrium statistical mechanics when an infinite number of possible states is present.

It is convenient to generalize this construction by introducing r replicas :

$$\begin{aligned} H_r(\varepsilon) &= \Sigma_{i,k} J_{i,k} \Big(\sum_1^r{}_a \sigma_i^a \sigma_k^a \Big) - \varepsilon \Sigma_i \Big(\Sigma_{a,b} \sigma_i^a \sigma_i^b - r \Big) \ , \\ F_r(\varepsilon) &= - \frac{1}{\beta r V} \ln \left[\Sigma_{\{\sigma\}} \exp(- \beta H_r(\varepsilon) \right] . \end{aligned} \tag{2.14}$$

The same arguments give :

$$Q(r) = - \frac{d}{d\varepsilon} F_r(\varepsilon)\Big|_{\varepsilon = 0} = (r - 1) q_{EA} \ . \tag{2.15}$$

In this way we define the function $Q(r)$ for integer values of r ; the definition can be extended to non integer values in a natural way :

$$\begin{aligned} F_r(\varepsilon) &= - \frac{1}{\beta r V} \ln \int dh_i \exp\Big(- \frac{\beta}{4} \Sigma_i h_i^2 \Big) \Big[Z(\varepsilon^{1/2} h_i) \Big]^r \ , \\ Z(\varepsilon^{1/2} h_i) &= \Sigma_{\{\sigma\}} \exp \Big[- \beta H(\varepsilon^{1/2} h_i) \Big] \ , \end{aligned} \tag{2.16}$$

$$H(\varepsilon^{1/2} h_i) = \sum_{i,k} J_{i,k} \sigma^i \sigma^k + \Sigma_i \left[\varepsilon - \varepsilon^{1/2} h_i \sigma_i \right] ,$$

$$Q(r) = - \frac{d}{d\varepsilon} F_r(\varepsilon) \Big|_{\varepsilon = 0} . \qquad (2.16)$$

It easy to chek that for integer r eq. (2.16) coincide with eq. (2.14).

Let us see which informations are contained in the function Q(r): if the linear response theory is assumed to be valid we get:

$$Q(r) = - 1 + \chi_{LR}/\beta + r q_{EA} . \qquad (2.17)$$

From the relation $Q(1) = 0$, we obtain the Fischer result:

$$\chi_{LR} = \beta(1 - q_{EA}) \qquad (2.18)$$

which is at zero magnetic field a consequence of the linear responce theory. On the other hand under reasonable hypothesis we can show that:

$$\chi_e = \beta(1 - Q(0)) . \qquad (2.19)$$

We stay in the glassy phase if and only if the function Q(r) is not a linear function of r; it is natural to study the function $q(r) = \frac{d}{dr} Q(r)$. As far as the function Q(r) is linear on the integers for $r \geq 1$, it is reasonable to assume that

$$q(r) = q_{EA} , \qquad r \geq r_c , \qquad r_c \leq 1 . \qquad (2.20)$$

The final results in terms of the function q(r) are:

$$\chi_e = \beta \int_0^1 dr\,(1 - q(r)) ,$$

$$\chi_{LR} = \beta(1 - q(1)) ,$$

$$\Delta\chi = \beta \int_0^1 dr\,(q(1) - q(r)) , \qquad (2.21)$$

$$q(1) = q_{EA} .$$

At low temperatures q(r, T) becomes a function of only $r/T = y$ [9,11]

$$q(r, T) \simeq q_s(r/T) \qquad (2.22)$$

and it can be computed from the formula:

$$q_s(y) = -\frac{1}{V}\frac{d}{dy}\frac{1}{y}\frac{d}{d\varepsilon}\ln\int dh_i \exp - G(\varepsilon, h_i) \ ,$$
$$G(\varepsilon, h_i) = \Sigma_i \frac{h_i^2}{4} + y\, U(\varepsilon^{1/2}, h_i) \qquad (2.23)$$

where $U(\varepsilon^{1/2}, h_i)$ is the zero temperature internal energy of the Hamiltonian H of eq. (2.16); this last expression can be easily computed using the standard Montecarlo techniques. The shape of the function q(r) contains many informations on the breaking of the linear response theory: it cannot be a constant inside the glassy phase. We propose to take it as order parameter for a spin glass: in this way the macroscopic state of a spin glass is characterized by a function defined on the interval 0-1 [7-9]. We will see in the next Section that q(r) can be computed in the replica formalism and that q(r) is not a constant only if the replica symmetry is broken. The spontaneous breaking of the replica symmetry should be identified with the onset of irreversible behaviour.

3. - The mean field approximation.

In the replica theory the expectation value of the free energy is:

$$F = -\frac{1}{\beta V}\frac{d}{dn} Z^n\Big|_{n=0} \qquad (3.1)$$

and the n^{th} power of the partition function Z is written as [2]

$$Z^n = \sum_{\{\sigma_i^a\}} \exp\left[-\beta \sum_{i,k} \sum_{1,a}^{n} \sigma_i^a \sigma_k^a J_{i,k}\right]. \qquad (3.2)$$

It is usual to introduce as order parameter the expectation value of the product of two spin in different replicas

$$Q_i^{a,b} = \langle \sigma_i^a \sigma_i^b \rangle . \qquad (3.3)$$

In the mean field approximation the free energy is written as function of Q; e. g.

$$F = -\frac{d}{dn}\left[\frac{\beta}{2}\sum_{1}^{n} {}_{a,b}\, Q_{a,b}^2 + \frac{1}{\beta}\ln\left(\sum_{\{a\}} \exp\left(\sigma^a \sigma^b Q_{a,b}\right)\right)\right] . \qquad (3.4)$$

The free energy is obviously invariant under the group of permutations P_n which exchange the labels of the different replicas. The value of $Q^{a,b}$ is given by

a stable stationary point of the free energy. It was realized by Thouless and de Almeida[3] that the replica symmetric solution (all $Q^{a,b}$ equal) is unstable for $|H| < H_r(T)$ and in this region the replica symmetry is broken. I was suggested in ref. 4 that the actual pattern of symmetry breaking is

$$P_n \longrightarrow P_{n/m} \otimes (P_m)^{n/m} \tag{3.5}$$

which in the $n \to 0$ limit becomes

$$P_0 \longrightarrow P_0 \otimes (P_m)^0 \ . \tag{3.6}$$

P_0 is the group of permutations of zero elements and $(P_m)^0$ denotes the product of P_m with itself zero times ($\ln P_m$ would be more appropriate).

It was shown in ref. 7 that eq. (3.6) may be generalized to

$$P_0 \longrightarrow (P_{m_1})^0 \otimes (P_{m_r})^0 \otimes \cdots\cdots \otimes P_0 \equiv G_{\{m\}} \ . \tag{3.7}$$

In this scheme P_0 always contains as unbroken subgroup P_0 so that we can have an infinite number of m_i. For integer n the m_i s must also be integers however when n is non integer also the m_i's may become non integers. If some technical conditions are satisfied one can associate to a matrice Q which is invariant under the group $G_{\{m\}}$ a function $q_R(x)$ on the interval 0-1.

It is very interesting to note that if we compute the function Q(s) defined in Section 2, we find

$$Q(s) = \int_0^s q_R(x)\,dx \ . \tag{3.8}$$

Eq. (3.8) implies

$$q_R(x) = q(x) \ . \tag{3.9}$$

This result can be proved by noticing that the free energy (2.16) is invariant under a group P_0 which is explicitely broken when $\varepsilon \neq 0$ to $P_0 \otimes (P_s)^0$. In presence of this symmetry breaking term, the matrix Q will orient itself in the direction of the external force as far as possible.

An x dependent q(x) function implies that the replica symmetry is broken together with the linear responce theory.

4. - The infinite range model.

In the infinite range model[2] (Sherrington-Kirpatrick) the Hamiltonian is given

by:

$$H = \sum_{1}^{N}{}_{i,k} J_{i,k}\, \sigma_i \sigma_k - H \sum_{1}^{N} i\, \sigma_i \tag{4.1}$$

H being the magnetic field, $J_{i,k}$ random gaussian variables $\langle J^2_{i,k} \rangle = 1/N$.

In the thermodynamic limit one finds[7]

$$F(\beta, H) = \max_{q(x)} F_{ef}(\beta, H | q) \tag{4.2}$$

where the effective free energy is given by:

$$F_{ef}(\beta, H\ q) = -\frac{\beta}{4}\left(1 + \int_0^1 q^2(x)\,dx - 2\,q(1)\right) - \frac{1}{\sqrt{2\pi}\,\beta} \int_{-\infty}^{+\infty} dZ \exp(-Z^2/2)\, g(0, H + Z\sqrt{q(0)}) \tag{4.3}$$

where the function $g(x, H)$ satisfies the q dependent differential equation[7,23]

$$\frac{\partial g}{\partial x} = -\frac{1}{2}\frac{\partial q}{\partial x}\left[\frac{\partial^2 g}{\partial h^2} + x\left(\frac{\partial g}{\partial h}\right)^2\right] \tag{4.4}$$

with the boundary condition

$$g(1, h) = \ln\left[\mathrm{ch}(\beta h)\right] .$$

In this way one finds results for the thermodynamic quantities which are in excellent agreement with the Montecarlo data.

One finds a line of transitions at $H = H_c(T)$ where

$$H_c(T) \simeq (T_c - T)^{3/2}, \qquad T \sim T_c ,$$
$$H_c \sim (2 \ln 1/T)^{1/2}, \qquad T \longrightarrow 0 . \tag{4.5}$$

The transitions from a constant to a non constant function q(x) (onset of irreversibility) is a third order transition at $H = 0$ while it becomes fourth order at $H \neq 0$. We find also

$$\Delta\chi\Big|_{H=0} \propto (T - T_c)^2 , \qquad \Delta\chi\Big|_{T\neq 0} \propto (H_c^{(T)} - H) , \tag{4.6}$$
$$\chi = 1 - h^{4/3} + O(h^2) , \qquad T < T_c .$$

The results of the analytic approach can be recovered with very high approximation from the **PaT** hypothesis[10]:

$$\frac{\partial S}{\partial H} = \frac{\partial M}{\partial T} = 0 , \qquad \frac{\partial q_{EA}}{\partial H} = 0 , \qquad \text{for } |H| < H_c(T) \tag{4.7}$$

This hypothesis is exact in the random energy model of Derrida[20] where the entropy of the glassy phase is zero.

According to eq. (4.7) the glassy phase is characterized by temperature independent magnetization. Eq. (4.7) seems to suggest that the system has at its disposal many equivalent states and, when a magnetic field is added, it goes from one state to an other equivalent state having the same entropy.

A more detailed analysis of the predictions of the PaThypothesis can be found in the original literature[10-13].

5. - The real case.

In principle the method here exposed allows us a systematic study of the corrections to the mean field approximation. In the three dimensional case these corrections should be very small for the long range forces (as diluited alloys) and should be more sizable for short range forces. Also in this last case both computer simulations[21] and real experiments[15] shows small deviations from mean field theory, expecially for quantities like χ_e.

The impressive success of mean field theory in predicting χ_e should not be considered as an undoubtable evidence for the existence of a transitions: there are many cases for which the corrections to the mean field destroy the transition, but they are small far from the would be transition and experimentally it is very difficult to distinguish a transition from a "quasitransition"[22].

The theoretical situation is rather promising, it would be very important to have a definitive experimental confirmation of the correctness of these ideas. A precise measurement of the critical line $H_c(T)$ and of $\Delta\chi$ near this line would be crucial.

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