ISTITUTO NAZIONALE DI FISICA NUCLEARE
Laboratori Nazionali di Frascati

LNF-81/55

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DIMENSIONAL REGULARIZATION OF THE SCHWINGER COEFFICIENT

Estratto da :
Nuovo Cimento 64A, 218 (1981)

Servizio Documentazione
dei Laboratori Nazionali di Frascati
Cas. Postale 13 - Frascati (Roma)

IL NUOVO CIMENTO VOL. 64 A, N. 2 21 Luglio 1981

Dimensional Regularization of the Schwinger Coefficient.

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(ricevuto il 19 Marzo 1981)

Summary. — We show that the dimensional regularization of the vacuum polarization amplitude contributed by fermion loops regularizes automatically the Schwinger coefficient. Contrary to what happens in the Pauli-Villars method, the regularized coefficient is not zero. It is proportional to the volume of the unit sphere in the corresponding Euclidean space-time. This result would have serious implications for gauge invariance if positivity and spectral constraints cannot be relaxed. Violations of these constraints are not necessary to achieve the convergence of the integral over the imaginary part of the vacuum polarization amplitude.

1. – Introduction.

The Schwinger term (1) is assumed to be a well-understood anomaly, of which it seems to be enough to know only that it exists. It is of concern only when it cannot be avoided (*e.g.* by subtractions), and that rarely happens. Consequently the Schwinger term and the problems connected with it have been ignored. There is another reason why it may be preferable not to get too deeply involved with the Schwinger term. It poses an almost insolvable dilemma: its presence is required by positivity and basic spectral properties (1-3).

(*) On leave of absence from Laboratori Nazionali dell'INFN, Frascati, Italia.

(1) J. SCHWINGER: *Phys. Rev. Lett.*, **3**, 296 (1959); T. GOTTO and I. IMAMURA: *Prog. Theor. Phys.*, **14**, 396 (1955).

(2) G. KÄLLÉN: lectures given at the *Winter Schools in Karpacz and Schladming* (February and March 1968) published in the *Schladming Proceedings*, edited by P. URBAN (1968).

(3) K. JOHNSON: *Nucl. Phys.*, **25**, 431 (1961).

On the other hand, the coefficient of the Schwinger term (Schwinger coefficient, for short) must be zero or at least do so through regularization in order to avoid conflict with gauge invariance (4). These two requirements are not easy to reconcile. The problem has not been much discussed, partly perhaps out of fear of entertaining heresies and partly because of the surprisingly limited number of regularization schemes adapted for the purpose. KÄLLÉN (2) discussed it extensively using the Pauli-Villars method (5); he found the regularized Schwinger coefficient to vanish. We shall show in this paper that the Schwinger coefficient is automatically and unambiguously regularized by the dimensional regularization of the propagator of vector currents of spin-$\frac{1}{2}$ fields. It turns out to be nonzero. This result may still be considered to constitute no threat to gauge invariance in those theories, like QED, in which the photon couples to pointlike fermions and the form of the current is the same everywhere. Regulators could again be used to enforce the vanishing of the overall coefficient. But is this always necessary? For instance, in the coupling of photons to hadrons the current is not everywhere a simple bilinear in quark fields. The quark structure shows up only at very high energies. At low energies the structure of the current is dominated by neutral vector mesons. An entirely different mechanism may be operating to maintain gauge invariance in this case. The vanishing of the regularized Schwinger coefficient may not be compatible with it. Such a mechanism could well be at the basis of the so-called q^2-duality (6) between the low-lying vector mesons and the free or quasi-free quarks which govern the behaviour of deep inelastic processes. A nonzero Schwinger coefficient is consistent with gauge invariance if the electromagnetic current is proportional to gauge boson fields coupled to conserved currents (7).

The hadronic electromagnetic current is peculiar in other respects. The quarks which contribute to its structure are postulated to be confined and their vacuum state is probably abnormal, so that quark condensations (8) (*i.e.* $\langle 0|\bar{q}(x)\,q(x)|0\rangle \neq 0$) may occur. The quark mass operator $M\bar{q}(x)\,q(x)$ would contribute in this case to the photon propagator. The regularization of the Schwinger coefficient for the hadronic electromagnetic current cannot then be carried out with a QED blueprint in complete disregard of these additional structures. The result of this paper is another argument for a more careful analysis of the role of the Schwinger term in hadronic quantum electrodynamics.

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(4) G. WENTZEL: *Phys. Rev.*, **74**, 1070 (1948); L. S. BROWN: *Phys. Rev. Sect. B*, **150**, 1338 (1966).
(5) W. PAULI and F. VILLARS: *Rev. Mod. Phys.*, **21**, 434 (1949).
(6) A. BRAMON, E. ETIM and M. GRECO: *Phys. Lett. B*, **41**, 609 (1972); J. J. SAKURAI: *Phys. Lett. B*, **46**, 207 (1973).
(7) N. M. KROLL, T. D. LEE and B. ZUMINO: *Phys. Rev.*, **157**, 1376 (1967).
(8) M. A. SCHIFMAN, A. I. VAINSTEIN and V. I. ZAKHOROV: *Nucl. Phys. B*, **147**, 385 (1979).

2. – Gauge invariance and the Schwinger coefficient.

The usual definition of regularization as a procedure for eliminating infinities with the proviso that such a procedure should not leave behind finite effects is very restrictive. The famous Adler-Bell-Jackiw anomaly [9] is a good example of a measurable effect which comes entirely from regularization. Furthermore, it is not true that regularization is required only for taming infinities. By regularization we shall mean any consistent procedure which guarantees in a theory those properties which it is supposed to exemplify. For instance, replacing the axial vector current

$$j^5_\mu(x) = \bar{\psi}(x)\gamma_\mu\gamma_5\psi(x) \tag{1}$$

by the nonlocal one [10]

$$j^5_\mu(x, \eta) = \bar{\psi}\left(x + \frac{\eta}{2}\right)\gamma_\mu\gamma_5\psi\left(x - \frac{\eta}{2}\right)\exp\left[ieQ\int\limits_{x-\eta/2}^{x+\eta/2} dy_\nu\, A_\nu(y)\right] \tag{2}$$

and evaluating its divergence as the limit for $\eta \to 0$ of that of $j^5_\mu(x, \eta)$ amounts effectively to a regularization but not immediately in the sense of eliminating an infinity. It gives the same result as follows from the Pauli-Villars regularization of the axial vector–two-vector current vertex [11], *viz.*

$$\partial_\mu j^5_\mu(x) = \lim_{\eta\to 0}\partial_\mu j^5_\mu(x, \eta) = 2iMj^5(x) + \frac{e^2Q^2}{16\pi^2}\varepsilon_{\mu\nu\lambda\tau}F_{\mu\nu}(x)F_{\lambda\tau}(x)\,. \tag{3}$$

Q is the charge in units of e carried by the field $\psi(x)$, M is the mass, $j^5(x) = \bar{\psi}(x)\gamma_5\psi(x)$, $A_\mu(x)$ the electromagnetic potential and $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$. The exponential in eq. (2) protects the invariance of $j^5_\mu(x, \eta)$ under the gauge transformations

$$\psi(x) \to \psi(x)\exp[ieQf(x)]\,, \qquad A_\mu(x) \to A_\mu(x) + \partial_\mu f(x)\,. \tag{4}$$

The Schwinger term may be generated by the same kind of procedure from the commutator

$$[j_0(x_0, \boldsymbol{x}), j_k(x_0, \boldsymbol{y})] = -iC_k(\boldsymbol{x}, \boldsymbol{y})\,, \qquad k = 1, 2, 3, \tag{5}$$

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[9] S. L. Adler: *Phys. Rev.*, **177**, 2426 (1969); J. S. Bell and R. Jackiw: *Nuovo Cimento A*, **60**, 47 (1969).
[10] S. L. Adler: lectures at the *Brandeis Summer School*, Vol. **1**, edited by S. Deser, M. Grisaru and H. Pendleton (1970).
[11] D. Barua and S. N. Gupta: *Phys. Rev. D*, **13**, 3240 (1976).

where

$$j_\mu(x) = eQ\bar{\psi}(x)\gamma_\mu\psi(x)\,. \tag{6}$$

This too is a form of regularization required not to eliminate an infinity, even though the point splitting does later introduce one, but rather to correct for the inconsistency of the two sides of eq. (5) present in the canonical formalism. It is, therefore, to be expected that other procedures of regularization may not yield the same result as the point-splitting method.

$C_k(\boldsymbol{x}, \boldsymbol{y})$ is zero if the commutator is evaluated by means of the canonical anticommutation relation for the field $\psi(x)$. This result is inconsistent with positivity and spectral properties for, if one takes the vacuum expectation value of both sides of (5) and inserts a complete set of intermediate states between the currents, one finds

$$\langle 0|C_k(\boldsymbol{x}, \boldsymbol{y})|0\rangle = C\frac{\partial}{\partial x_k}\delta^3(\boldsymbol{x}-\boldsymbol{y})\,, \tag{7}$$

where

$$C = \frac{1}{\pi}\int\limits_0^\infty \mathrm{d}s\,\mathrm{Im}\,\pi(s)\,. \tag{8}$$

$\mathrm{Im}\,\pi(q^2)$ is the imaginary part of the vacuum polarization amplitude $\pi(q^2)$ defined by

$$\begin{aligned}\pi_{\mu\nu}(q) &= -i\int \mathrm{d}^4x \exp[iqx]\langle 0|T\{j_\mu(x), j_\nu(0)\}|0\rangle = \\ &= (q^2 g_{\mu\nu} - q_\mu q_\nu)\pi(q^2) + (g_{\mu\nu} - g_{\mu 0}g_{\nu 0})\,C\,.\end{aligned} \tag{9}$$

Because of the last term in eq. (9), which comes from the noncovariance of the T-product, $\pi_{\mu\nu}(q)$ fails to be gauge invariant. It is through this term, which represents the coupling of two photons to a charged particle at the same space-time point (corresponding to the so-called sea-gull graph), that the Schwinger coefficient contributes to the photon mass. Indeed with eq. (9) the photon propagator, to second order in eQ, is

$$\begin{aligned}D_{\mu\nu}(q) &= \frac{g_{\mu\nu}}{q^2} + \frac{g_{\mu\lambda}}{q^2}\pi_{\lambda\tau}(q)\frac{g_{\tau\nu}}{q^2} = \\ &= \frac{g_{\mu\nu}}{q^2}\left[1 + \left(\pi(q^2) + \frac{C}{q^2}\right)\right] - \frac{q_\mu q_\nu}{q^4}\pi(q^2) - \frac{C}{q^4}g_{\mu 0}g_{\nu 0}\,.\end{aligned} \tag{10}$$

If we iterate this, by summing the diagrams in fig. 1, we find for small q^2

$$D'_{\mu\nu}(q) = \frac{g_{\mu\nu}}{q^2 - C} + O(q^2)\,. \tag{11}$$

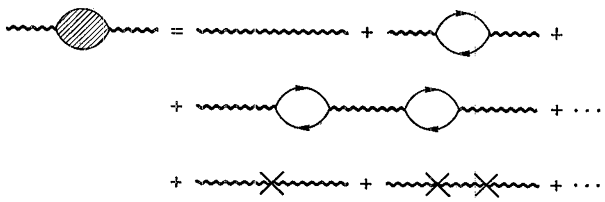

Fig. 1.

We insist that it is this undesirable result which regularization has to repair consistently with positivity and spectral conditions. The use of regulators violates the latter requirements and it is, therefore, not surprising that with it one succeeds in regularizing C to zero. This is, however, a roundabout way of saying that the vanishing of $C_k(\boldsymbol{x}, \boldsymbol{y})$ in eq. (5) was no problem in the first place.

To second order in eQ corresponding to the second graph on the right-hand side of fig. 1, the imaginary part of $\pi(q^2)$ is

$$\text{(12)} \qquad \operatorname{Im}\pi(s) = \frac{e^2Q^2}{12\pi}\,\Theta(s-4M^2)\left(1+\frac{2M^2}{s}\right)\left(1-\frac{4M^2}{s}\right)^{\frac{1}{2}}.$$

The integral in eq. (8) is, therefore, infinite. If, in addition to the field $\psi(x)$ carrying charge eQ and mass M, we introduce the auxiliary fields $\psi_i(x)$ $(i = 1, 2, \ldots)$ whose charges eQ_i and masses M_i satisfy the conditions [5]

$$\text{(13a)} \qquad Q^2 + \sum_{i=1}^{\infty} Q_i^2 = 0\,,$$

$$\text{(13b)} \qquad Q^2M^2 + \sum_{i=1}^{\infty} Q_i^2M_i^2 = 0\,,$$

then, with $Q_{i=0} = Q$, $M_{i=0} = M$,

$$\text{(14)} \qquad \operatorname{Reg} C = \sum_{i=0}^{\infty}\frac{e^2Q_i^2}{12\pi^2}\int\limits_{4M_i^2}^{\infty}\mathrm{d}s\left(1+\frac{2M_i^2}{s}\right)\left(1-\frac{4M_i^2}{s}\right)^{\frac{1}{2}} =$$

$$= \sum_{i=0}^{\infty}\frac{e^2Q_i^2}{12\pi^2}\int\limits_{4M_i^2}^{\infty}\mathrm{d}s\left[\left(1+\frac{2M_i^2}{s}\right)\left(1-\frac{4M_i^2}{s}\right)^{\frac{1}{2}}-1\right] =$$

$$= \sum_{i=0}^{\infty}\frac{e^2Q_i^2}{12\pi^2}\lim_{\varepsilon\to 0}\int\limits_{4M_i^2}^{\infty}\mathrm{d}s\left(\frac{s}{4M_i^2}\right)^{-\varepsilon}\left[\left(1+\frac{2M_i^2}{s}\right)\left(1-\frac{4M_i^2}{s}\right)^{\frac{1}{2}}-1\right] = -\sum_{i=0}^{\infty}\frac{e^2Q_i^2M_i^2}{6\pi^2}$$

is indeed zero (2). But eqs. (13) violate positivity and spectral conditions. Of course, difficulties with gauge invariance could be temporarily avoided by using in eq. (9) the covariant T-product

$$T^*(j_\mu(x), j_\nu(0)) = T(j_\mu(x), j_\nu(0)) - iC(g_{\mu\nu} - g_{\mu 0} g_{\nu 0})\, \delta^4(x)\,. \tag{15}$$

The problem is, however, that such a subtraction must be consistent with the whole renormalization program. There is an outstanding infinity in the theory which is only evaded by the subtraction but not satisfactorily eliminated.

3. – Dimensional regularization.

Quite apart from its possible implications, it is interesting that the Schwinger coefficient is automatically regularized by the dimensional regularization of the vacuum polarization tensor (see fig. 2)

$$\pi_{\mu\nu}(q) = -i\frac{e^2Q^2}{(2\pi)^4}\int \mathrm{d}^4p\, \mathrm{Tr}\left\{\frac{1}{\not p + M}\gamma_\mu \frac{1}{\not p + \not q + M}\gamma_\nu\right\}. \tag{16}$$

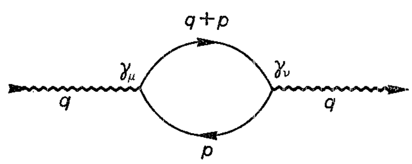

Fig. 2.

Now, if we replace the integral over p in (16) by a n-dimensional Euclidean integral, we get upon evaluating traces and integrating (12)

$$\pi_{\mu\nu}(q, n) = [q^2 g_{\mu\nu} - q_\mu q_\nu]\frac{2e^2Q^2\pi^{n/2}n}{(2\pi)^n}\Gamma(2-n/2)\cdot \int_0^1 \mathrm{d}x\, x(1-x)[q^2x(1-x) - M^2]^{n/2-2}\,. \tag{17}$$

The imaginary part of the polarization amplitude is, therefore,

$$\mathrm{Im}\,\pi(s) = \frac{2e^2Q^2}{(2\pi)^n} n\pi^{n/2}\Gamma\left(2-\frac{n}{2}\right)\sin\frac{n\pi}{2}\cdot \int_0^1 \mathrm{d}x\, x(1-x)\Theta\big(M^2 - sx(1-x)\big)[M^2 - sx(1-x)]^{n/2-2}\,. \tag{18}$$

(12) G. 'T HOOFT and M. VELTMAN: DIAGRAMMAR, CERN Report No. 73-9 (1973).

First consider values of $n \geqslant 2$ such that $\mathrm{Im}\,\pi(s)$ can be written in the form

$$\mathrm{Im}\,\pi(s) = -\Theta(s-4M^2)\frac{4e^2Q^2}{(2\pi)^n(n/2-1)}\,n\,\frac{\pi^{1+n/2}}{\Gamma(n/2)}\cdot \cdot\frac{\mathrm{d}}{\mathrm{d}s}\left\{s^{n/2-1}\int\limits_0^{x_-(s)}\mathrm{d}x\,[(x-x_-(s))(x-x_+(s))]^{n/2-1}\right\}, \tag{19}$$

where

$$x_\pm(s) = \frac{1}{2}\left[1 \pm \left(1-\frac{4M^2}{s}\right)^{\frac{1}{2}}\right] \tag{20}$$

are the solutions of the equation

$$x^2 - x + \frac{M^2}{s} = 0\,. \tag{21}$$

For $n<2$ the representation (19) is not valid, since it neglects an infinite surface term arising from the limit

$$\lim_{x\to x_-}(x-x_-)^{n/2-1}\,. \tag{22}$$

Dropping this infinite term would allow one naively to use eq. (19) in order to continue the Schwinger term in n dimensions to all $n<2$ except for $n=1$, where it has a simple pole. This is easily seen by performing the integral over x in (19), which gives

$$\int\limits_0^{x_-(s)}\mathrm{d}x\,[(x-x_-(s))(x-x_+(s))]^{n/2-1} = = \frac{2(x_+x_-)^{n/2}}{nx_+}\,{}_2F_1\left(1, 1-n/2; 1+n/2; \frac{x_-}{x_+}\right), \tag{23}$$

and then evaluating

$$C(n) = \frac{1}{\pi}\int\limits_0^\infty \mathrm{d}s\,\mathrm{Im}\,\pi(s, n) \tag{24}$$

with the result

$$C(n) = \frac{-4e^2Q^2M^{n-2}}{(2\pi)^n}\,\frac{\pi^{n/2}}{\Gamma(n/2)}\,n\,{}_2F_1(1, 1-n/2; 1+n/2; 1); \tag{25}$$

${}_2F_1(a, b; c; z)$ is the hypergeometric function. The failure of the naive continuation from $n\geqslant 2$ to $n=1$ arises from the breakdown of the identity

$${}_2F_1(1, 1-n/2; 1+n/2; 1) = \frac{\Gamma(1+n/2)\Gamma(n-1)}{\Gamma(n/2)\Gamma(n)} = \frac{n}{2(n-1)} \tag{26}$$

at $n=1$. Since the point $n=4$ at which we shall use eq. (25) is outside of the range of n in which (19) is not applicable, we shall not discuss the significance of the singularity of $C(n)$ at $n=1$ any further.

Making use of (26) in (25), we finally have

$$C(n) = -\frac{2e^2Q^2M^{n-2}}{(2\pi)^n}\frac{\pi^{n/2}}{\Gamma(n/2)}\frac{n}{n-1}, \qquad n\neq 1. \tag{27}$$

By the usual prescription, the regularized Schwinger coefficient in real space-time is obtained by setting $n=4$ in eq. (27), that is

$$\operatorname{Reg} C = -\frac{2\alpha Q^2 M^2}{3\pi}, \tag{28}$$

where $\alpha = e^2/4\pi$.

We stress that this result is unambiguous for, even though $C(n)$ would be formally divergent for all n, if we used Kummer's transformation

$${}_2F_1\left(1, 1-n/2; 1+n/2; \frac{x_-}{x_+}\right) = \frac{1}{1-x_-/x_+}\,{}_2F_1\left(1, n; 1+n/2; \frac{x_-}{x_- - x_+}\right) \tag{29}$$

to express the integral in eq. (23) in the form

$$\int_0^{x_-(s)} dx\,[(x-x_-(s))(x_+ - x(s))]^{n/2-1} = \tag{30}$$

$$= (2/n)\frac{(x_+x_-)^{n/2}}{x_+ - x_-}\,{}_2F_1\left(1, n; 1+n/2; \frac{x_-}{x_- - x_+}\right),$$

$C(n)$, defined in this case as the limit

$$C(n) = \lim_{\lambda\to 1}\int_{4M^2\lambda^2}^{\infty} ds\,\operatorname{Im}\pi(s; \lambda M; n), \tag{31}$$

would still exist and be given by eq. (27). Equation (29) is an analytic continuation which is valid for all $x_-/x_+ \neq 1$. At $x_- = x_+$ the right-hand side of (29) is formally divergent, while the left-hand side exists. Replacing M by λM keeps $x_-(s) \neq x_+(s)$ for all s and allows one to use eq. (29) over the whole range of the s-integration. The limit $\lambda \to 1$, taken at the end of all operations, avoids the difficulty of using eq. (29), in which it is not valid. There is, therefore, no ambiguity in the determination of $C(n)$.

The rather straightforward way of getting eq. (28) from a method of regularization we now like to believe in indicates that the problem of reconciling the presence of the Schwinger term with gauge invariance and current conservation has not been completely resolved if we insist in maintaining positivity and standard spectral properties.

It is important in this connection to observe how dimensional regularization achieves finiteness of C $(n = 4)$. It amounts effectively to obtaining the integrand in the second line of eq. (14), which vanishes for $s \to \infty$, without using the Pauli-Villars conditions, eqs. (13). We have in fact (*)

$$(32)\qquad \lim_{n\to 4} \operatorname{Im} \pi(s, n) = \frac{e^2Q^2}{12\pi}\,\Theta(s-4M^2)\left\{\left(1+\frac{2M^2}{s}\right)\left(1-\frac{4M^2}{s}\right)^{\frac{1}{2}}-1\right\} \neq$$

$$\neq \operatorname{Im} \lim_{n\to 4} \pi(s, n) = \frac{e^2Q^2}{12\pi^2}\,\Theta(s-4M^2)\left(1+\frac{2M^2}{s}\right)\left(1-\frac{4M^2}{s}\right)^{\frac{1}{2}},$$

from which it is clear that conditions similar to those in eqs. (13) are necessary to achieve the convergence of the integral of the last term. The vanishing of Reg C in this case is the price paid to maintain consistency. This is not necessary for the integral of the second term of (29), which is automatically finite.

(*) The first equality can most easily be obtained by using the identity

$$\int_0^{x_-} dx\,[\ldots] = \tfrac{1}{2}\left\{\int_0^{x_-} dx\,[\ldots] + \int_{x_+}^{1} dx\,[\ldots]\right\}$$

in eq. (19).

● RIASSUNTO (*)

Si mostra che la regolarizzazione dimensionale dell'ampiezza di polarizzazione del vuoto fornita da cappi di fermioni regolarizza automaticamente il coefficiente di Schwinger. Contrariamente a ciò che accade per il metodo di Pauli-Villars, il coefficiente regolarizzato è non nullo. È inoltre proporzionale al volume della sfera unitaria nello spazio-tempo euclidiano corrispondente. Questo risultato avrebbe importanti implicazioni nell'invarianza di gauge se la positività e i vincoli spettrali non si potessero rendere meno rigidi. La violazione di questi vincoli non è necessaria per ottenere la convergenza dell'integrale della parte immaginaria dell'ampiezza di polarizzazione del vuoto.

(*) *Traduzione a cura della Redazione.*

Размерная регуляризация коэффициента Швингера.

Резюме (*). — Мы показываем, что размерная регуляризация амплитуды поляризации вакуума, в которую дают вклад фермионные петли, автоматически регуляризует коэффициент Швингера. В противоположность этоми, в методе Паули-Вилларса регуляризованный коэффициент не равен нулю. Этот коэффициент пропорционален объему единичной сферы в соответствующем эвклидовом пространстве-времени. Этот результат будет иметь серьезные следствия для калибровочной инвариантности, если ограничения положительности и спектральные ограничения не могут быть ослаблены. Нарушения этих ограничений не являются необходимыми для достижения сходимости интеграла от мнимой части амплитуды поляризации вакуума.

(*) *Переведено редакцией.*