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RELATIVISTIC AND RADIATIVE CORRECTIONS TO POTENTIAL MODEL LEPTONIC WIDTHS OF VECTOR-LIKE STATES

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ABSTRACT

We calculate relativistic and radiative corrections to the non-relativistic (Van Royen-Weiskopf) formula for the leptonic widths of vector-like ($n\ ^3S_1$) bound states using Q^2-duality. They are determined by the Schwinger function. This function possesses a simple, though not completely unambiguous, factorisation which allows to identify a wave function at the origin and hence to isolate the genuine radiative correction factor. This latter is similar in structure to and agrees numerically with the formula of Karplus and Klein. For the case of positronium this method constitutes an alternative and simpler derivation of the formula of Karplus and Klein. For the quarkonium states it provides a reliable estimate of QCD radiative corrections. For the ψ and Υ states these corrections are large. The same is true of the relativistic corrections.

(*) On leave of absence from Department of Physics, University of Siegen, Siegen, Germany. Partially supported by the Deutsche Forschungsgemeinschaft.

1. - INTRODUCTION

In the reconstruction of the potential between a quark and an antiquark, using inverse scattering theory[1], one needs, besides the mass locations of the $q\bar{q}$ bound states, the wave functions at the origin. In the case of the radial excitations of the ortho-quarkonium states this means knowledge of the leptonic widths. Although potential models, with simple power potentials describe quite satisfactorily the mass spectra of the heavier quark ($q \equiv c,b$) bound states[2] they are less successful in predicting their leptonic widths[3] with the help of the non-relativistic (Van Royen-Weiskopf) formula[4]

$$\Gamma(V_n \to e^+e^-) = \frac{16\pi\alpha^2 Q^2 N_c}{3(2M)^2} \, |\psi_n(0)|^2 \qquad (1)$$

$V_n(\equiv \psi/J, \psi', \psi'', ...; \Upsilon, \Upsilon', \Upsilon'',....)$ is a radially excited $q\bar{q}$ bound state. Its mass will be denoted by m_n and its coupling to the photon m_n^2/f_n, in units of the electric charge e. In the same units the charge of the quark is Q, its mass M and the number of its colours $N_c(\equiv 3)$. $\psi_n(0)$ is the wave function of the n-th bound state evaluated at the origin.

In the case of positronium[5] Eq. (1) is a good approximation, this system being eminently non-relativistic. Consequently, except perhaps in the masses, relativistic and QCD radiative effects must be more important in the properties of $q\bar{q}$ bound states than in positronium. These corrections have been extensively discussed recently[6-9] but some doubts still persist as to the reliability of their estimates. To these must be added the fact that the inadequacy of the potential model in one area casts doubts on its general validity, however impressive its success in another. The problem therefore is not just one of calculating relatively large corrections to the leptonic widths but more generally of finding a consistent approximation for the description of the structure of quarkonium systems. Even in QED the theory of relativistic bound states is not available in simple form. Secondly the extrapolation of short distance based QCD perturbation theory to long distance phenomena, such as are relevant to leptonic decays of vector mesons does not follow unquestionably from knowledge of the QCD Lagrangian. In the past[6-9] one tried to come to terms with these problems by assuming that $q\bar{q}$ bound states were approximately Coulomb-like, so that QED formulae would apply with the appropriate change in the coupling constant. A different approach has been indicated by Durand and Durand[10] who calculated relativistic corrections to the WKB approximation to the wave function at the origin with the help of duality[11,12]. The present paper is at once an extension of the calculation of these authors to include QCD radiactive corrections as well as an extension of the application of duality itself to cover hadronic and non-hadronic particle-antiparticle bound states. The necessity of this latter extension is to be able to test the validity of duality on well-known bound state systems (positronium in the case of this paper) and thereby strengthen confidence in its application to the less well known quarkonium states.

The application of duality to the calculation of radiative corrections to the decay of positronium has an intrinsic interest of its own. It constitutes an alternative and simpler derivation of the formula of Karplus and Klein[13](*). We find that apart from kinematical factors both relativistic and radiative corrections to the leptonic widths of vector-like ($n\ {}^3S_1$) bound states are determined by the Schwinger function[14].

There is a simple, though not completely unambiguous, factorisation of this function which allows to define a wave function at the origin and hence to isolate the purely radiative correction factor. The latter is similar in structure to that of Karplus and Klein[13] and agrees with it rather well numerically. Although we do identify a wave function at the origin it is for the purpose of comparison only.

Our method is relativistic throughout and, except for the above-mentioned comparison there is no commitment to potential models. We can and have used our formulae directly to compute the leptonic widths. The results agree rather well with the data except for the $\psi''(3.768)$.

2. - DUALITY AND LEPTONIC DECAYS OF VECTOR MESONS

Q^2-duality[11,12] is the statement, based on analyticity, that averaged properties of a particle-antiparticle bound state are related to those of its constituents. Properties of the constituents which have been related in this way are the charge[11,12] and mass[15]. So far the application has been limited to vector mesons and their quark constituents. Because quarks are supposed to be confined duality has as a consequence come to be associated with what is supposed to be the dynamics of confined systems. Duality however holds also in potential models[16] and with no implication that the potentials are confining. If anything, if duality is pushed to an almost point-wise equality in the way done by Shifman Vainstein and

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(*) The idea of using duality in this way was suggested to us by Prof. J.S. Bell, to whom we are very grateful.

Zakharov[17] and implemented by Bell and Bertlmann[18] in potential models, its success becomes inexplicable in the presence of a confining potential. We shall extend it here to positronium. We wish to consider in such a framework the leptonic decays of vector-like states such as positronium and the familiar neutral vector mesons. For definiteness let us fix attention on the latter and then transcribe our formulae into the forms applicable to positronium.

According to duality the energy average of the total cross section $\sigma(e^+e^- \to V_n \to \text{had})$ for e^+e^- annihilation into hadrons is equal to a similar average of the total cross section $\sigma(e^+e^- \to q\bar{q})$ for e^+e^- annihilation into the free pair of quark and antiquark constituents of V_n. As a function of the CM energy $\sqrt{s}$ these cross sections are, respectively

$$s\sigma(e^+e^- \to V_n \to \text{had}\,;\; s) = 16\,\pi^3\alpha^2 \frac{m_n^2}{f_n^2}\,\delta(s-m_n^2) \qquad (2)$$

$$s\sigma(e^+e^- \to q\bar{q}\,;\; s) = 4\,\pi\,\alpha^2 Q^2 \left(1+\frac{2M^2}{s}\right)\left(1-\frac{4M^2}{s}\right)^{\frac{1}{2}} \left(1+\frac{4}{3}\alpha_s(s)\, h(v(s))\right) \qquad (3)$$

α is the fine structure constant, $\alpha_s(s)$ the QCD (strong) coupling constant, $v(s)=(1-(4M^2/s))^{\frac{1}{2}}$ and $h(v)$ is Schwingers function[14] defined by

$$h(v) = \frac{1}{\pi}\Big\{\left(v+\frac{1}{v}\right)\Big[L(1)+L(v^2)+2L\left(\frac{1-v}{1+v}\right)+2L\left(\frac{1+v}{2}\right)-2L\left(\frac{1-v}{2}\right)-4L(v)$$
$$+\ln\left(\frac{1+v}{2}\right)\ln\left(\frac{1+v}{1-v}\right)\Big] + \ln\left(\frac{1+v}{1-v}\right)\left[\frac{11}{8}\left(v+\frac{1}{v}\right)+\frac{v^3}{2(3-v^2)}-3\right]$$
$$+\frac{3(5-3v^2)}{4(3-v^2)}+6\ln\left(\frac{1+v}{2}\right)-4\ln(v)\Big\} \qquad (4)$$

$L(v)$ is the Spence function defined by[14,19]

$$L(v) = -\int_1^v \frac{dt}{t}\ln(1-t); \qquad 0 \leqslant x \leqslant 1$$

$$= \sum_{n=1}^{\infty} \frac{v^n}{n^2}$$

$$L(v)+L(1-v)=L(1)-\ln(v)\ln(1-v)$$

$$L(1) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \qquad (5)$$

Although h(v) is complicated its limits for $v \to 0$ and $v \to 1$ are rather simple. They are,

$$h(v) \underset{v \to 0}{\longrightarrow} \frac{\pi}{2v} - \frac{1}{\pi}(4 \ln v + 6 \ln 2 - \frac{5}{4}) \tag{6}$$

$$h(v) \underset{v \to 1}{\longrightarrow} \frac{3}{4\pi} \tag{7}$$

Schwinger[14] has also given a simple function

$$\hat{h}(v) = \frac{\pi}{2v} - \frac{3+v}{4}\left(\frac{\pi}{2} - \frac{3}{4\pi}\right) \tag{8}$$

which interpolates between these limits.

It is compared with the exact function h(v) in Fig. (1). The difference between the two is less than 5% for v in the interval $0 \leq v \leq 1$. The behaviour of h(v) in the non-relativistic limit ($v \to 0$) is compatible with the effect that the probability for

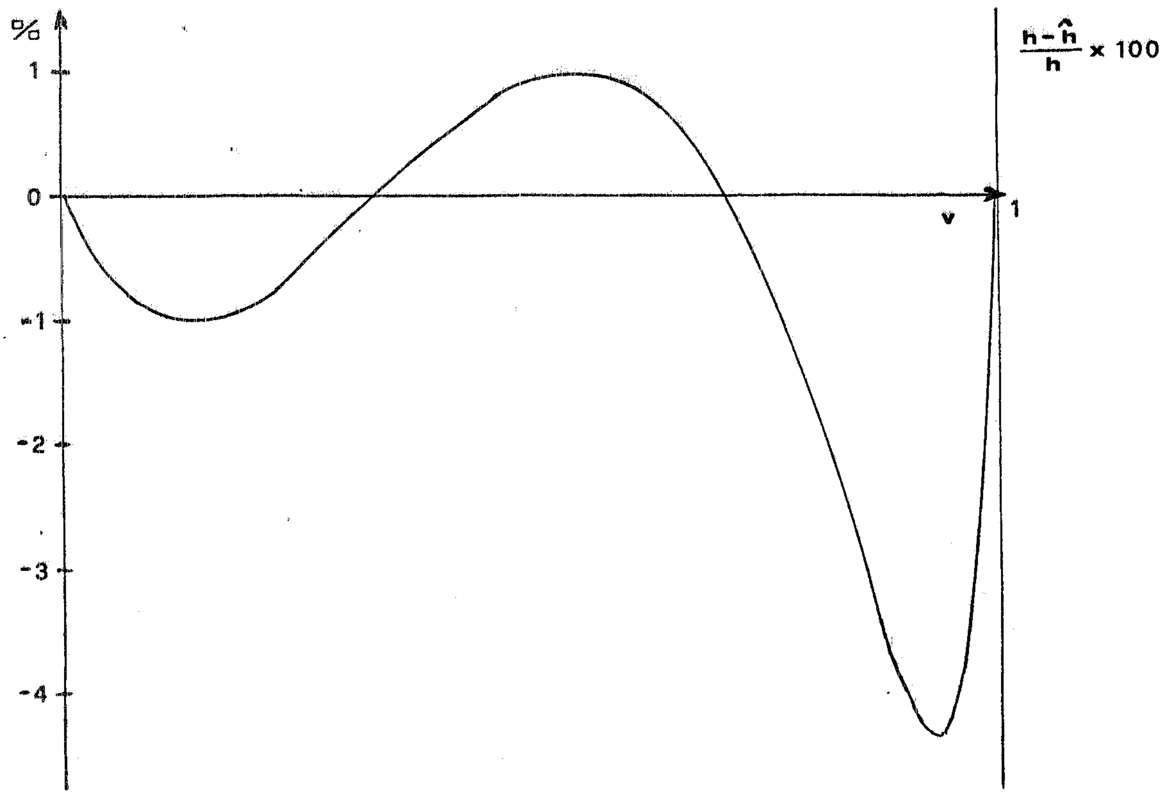

FIG. 1 - Comparison of the interpolating function $\hat{h}(v)$, Eq. (8), with the exact Schwinger function h(v), Eq. (4).

establishing a bound state of a particle and an antiparticle in mutual Coulomb (or Coulomb-like) attraction and in relative motion with (non-relativistic) velocity $u=2v$ increases by the factor $(1+\frac{\pi\alpha}{2v})$ or $(1+\frac{\pi}{2v}\cdot\frac{4}{3}\alpha_s)$ in the case of QCD [14,20]. This means that in the non-relativistic limit, where potential model descriptions are expected to be valid, one can incorporate $(1+\frac{\pi}{2v}\cdot\frac{4}{3}\alpha_s)^{\frac{1}{2}}$ or $(1+\frac{\pi\alpha}{2v})^{\frac{1}{2}}$ as the case may be, in the definition of the wave function. It is convenient then to factor $(1+\frac{\pi}{2v}\cdot\frac{4}{3}\alpha_s)$ or more generally the $v \to 0$ limit of $\hat{h}(v)$ (or $h(v)$) out of the entire function.

The factorisation is less ambiguous in terms of $\hat{h}(v)$; one finds

$$1+\frac{4\alpha_s}{3}\hat{h}(v) = \left(1-\frac{4\alpha_s}{3}\hat{h}_-(v)\right)\left[1+\frac{4\alpha_s}{3}\hat{h}_+(v)+\left(\frac{4\alpha_s}{3}\right)^2\frac{\hat{h}_-(v)\hat{h}_+(v)}{1-\frac{4\alpha_s}{3}\hat{h}_-(v)}\right] \tag{9}$$

with

$$\hat{h}_+(v) = \frac{\pi}{2v}$$
$$\hat{h}_-(v) = \frac{3+v}{4}\left(\frac{\pi}{2}-\frac{3}{4\pi}\right) \tag{10}$$

Guided by Eqs. (6) and (9) we operate an analogous factorisation of h(v) with

$$h_+(v) = \frac{\pi}{2}+\frac{1}{\pi}\Big\{\left(v+\frac{1}{v}\right)\Big[L(1)+L(v^2)+2L\left(\frac{1-v}{1+v}\right)+2L\left(\frac{1+v}{2}\right)-2L\left(\frac{1-v}{2}\right)-4L(v)+$$
$$+\ln\left(\frac{1+v}{2}\right)\ln\left(\frac{1+v}{1-v}\right)\Big]+6\ln\left(\frac{1+v}{2}\right)-4\ln(v)\Big\} \tag{11}$$

and

$$h_-(v) = \frac{\pi}{2}-\frac{1}{\pi}\left\{\frac{3(5-3v^2)}{4(3-v^2)}+\ln\left(\frac{1+v}{1-v}\right)\left[\frac{11}{8}\left(v+\frac{1}{v}\right)+\frac{v^3}{2(3-v^2)}-3\right]\right\} \tag{12}$$

There is a little, but not serious ambiguity in the distribution of the non-leading terms in the limits $v \to 0$ and $v \to 1$ between the functions $h_\pm(v)$. To show the effect of these terms in $h_\pm(v)$ compared to the corresponding functions in Eq. (9) we have plotted the ratios $a_\pm(v)=h_\pm(v)/\hat{h}_\pm(v)$ in Fig. (2). $a_+(v)$ remains everywhere close to one while $a_-(v)$ departs from unity only for $v \lesssim 0.4$.

Let us now turn to the implementation of duality. Averaging the cross sections in Eqs. (2) and (3) over the energy interval $m_n-\Delta m_n/2 \leq \sqrt{s} \leq m_n+\Delta m_n/2$ one finds

$$\frac{m_n^2}{f_n^2} = \frac{Q^2}{4\pi^2}\left(1+\frac{2M^2}{m_n^2}\right)\left(1-\frac{4M^2}{m_n^2}\right)^{\frac{1}{2}}\frac{dm_n^2}{dn}\left(1+\frac{4\alpha_s(m_n^2)}{3}h(v(m_n^2))\right) \tag{13}$$

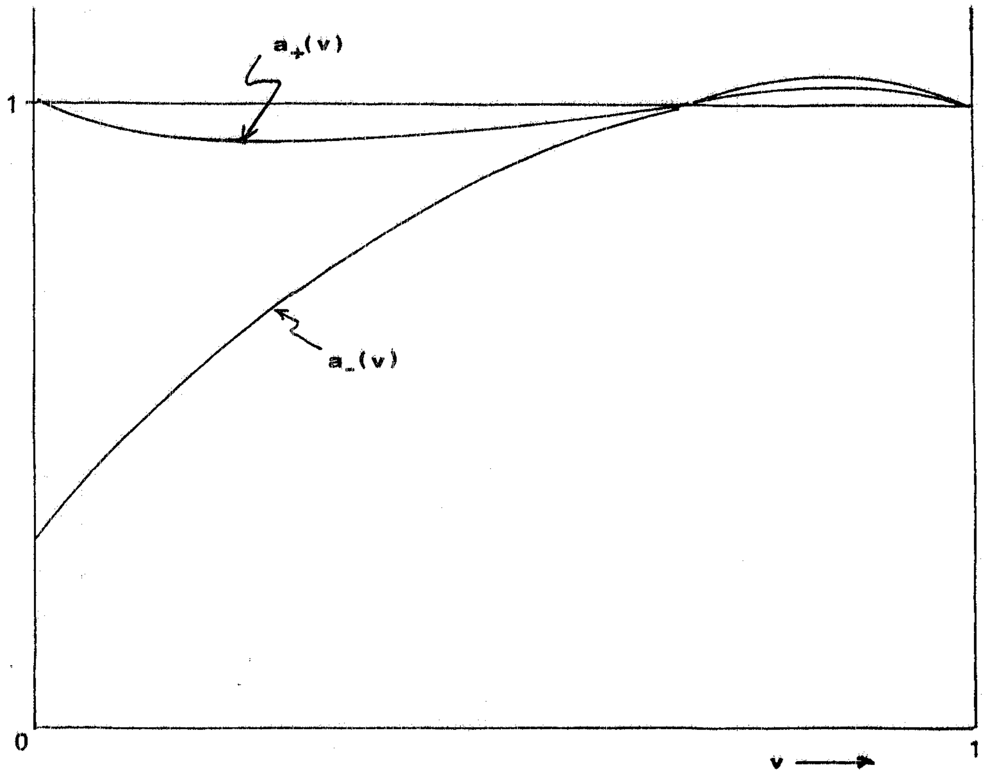

FIG. 2 - Ratio of the factorised parts $h_{\pm}(v)$ of the Schwinger function, Eqs. (11,12), to the corresponding interpolating functions $\hat{h}_{\pm}(v)$; Eq. (10). $a_{\pm}:=h_{\pm}/\hat{h}_{\pm}(v)$.

The significance of Eq. (13) is that if the mass and charge of the quark are given together with the meson mass spectrum then the leptonic widths of these mesons are also known. The relativistic formula for the leptonic width in terms of f_n and the mass m_n is

$$\Gamma\,(V_n \longrightarrow e^+e^-) = \frac{4\,\pi\alpha^2}{3}\,\frac{m_n}{f_n^2} \qquad (14)$$

Substituting for f_n from Eq. (13) into (14) one finds, indeed, that duality relates the leptonic widths to the mass spectrum. For $N_c=3$ one gets

$$\Gamma\,(V_n \longrightarrow e^+e^-) = \frac{16\,\pi\alpha^2 Q^2}{(2M)^2}\left[\frac{M^{3/2}}{4\pi^2}\, m_n^{\frac{1}{2}}\, \frac{dm_n}{dn}\, \frac{2}{3}\left(\frac{M}{m_n}\right)^{\frac{1}{2}} .\left(1+\frac{2M^2}{m_n^2}\right)\right.$$

$$\left.\left(1-\frac{4M^2}{m_n^2}\right)^{\frac{1}{2}} \left\{ 1+\frac{4\alpha_s}{3}\, h_+(v)+\left(\frac{4\alpha_s}{3}\right)^2 \frac{h_-(v)h_+(v)}{1-\frac{4\alpha_s}{3}\, h_-(v)}\right\}\right] . \qquad (15)$$

$$.\left(1-\frac{4\alpha_s}{3}\, h_-(v)\right).$$

Eq. (9) has also been used. Although in the present approach we are not committed to potential models we wish nevertheless, for purposes of comparison, to estimate the relativistic and radiative corrections to the leptonic widths computable from such models. To this end we compare Eq. (15) with (1) and observe that the WKB approximation for the wave function at the origin in terms of the spectrum of eigenvalues of the Schrödinger equation is

$$|\psi_n^{(o)}(0)|^2_{WKB} \simeq \frac{M^{3/2}}{4\pi^2} m_n^{\frac{1}{2}} (1-\frac{2M}{m_n})^{\frac{1}{2}} \frac{dm_n}{dn} \tag{16}$$

The superscript in $\psi_n^{(o)}$ means that the wave function is considered without radiative corrections. Taking into account the argument about the effect of $h_+(v)$ in the limit $v \to 0$ we identify the terms within square brackets in Eq. (15) with the modulus squared of the relativistically corrected wave function at the origin

$$|\psi_n(0)|^2 = |\psi_n^{(1)}(0)|^2_{WKB}\left[\frac{2}{3}(\frac{M}{m_n})^{\frac{1}{2}}(1+\frac{2M^2}{m_n^2})(1+\frac{2M}{m_n})^{\frac{1}{2}}\right] . \tag{17a}$$

where

$$|\psi_n^{(1)}(0)|^2_{WKB} = |\psi_n^{(o)}(0)|^2_{WKB}\left[1+\frac{4\alpha_s}{3} h_+(v) + (\frac{4\alpha_s}{3})^2 \frac{h_-(v)h_+(v)}{1-\frac{4\alpha_s}{3} h_-(v)}\right] \tag{17b}$$

The remaining factor

$$C_n(v;\alpha_s) = 1 - \frac{4\alpha_s(m_n^2)}{3} h_-(v\,(m_n^2)) \tag{18}$$

is then, for this state, the genuine QCD radiative correction factor.

These formulae apply also to positronium if one makes the substitution(*) $4\alpha_s/3 \to \alpha$. In this case the corresponding radiative correction factor

$$C(v;\alpha) = 1 - \alpha h_-(v \simeq (E/M)^{\frac{1}{2}}) \simeq 1-\alpha \frac{3+v}{4}(\frac{\pi}{2} - \frac{3}{4\pi}), \tag{19}$$

(*) The correction leading to the Schwinger function were originally calculated in QED.

where E is the ground state energy of ortho-positronium and M the electron mass, is to be compared with the formula of Karplus and Klein.

$$C_{KK}(v;\alpha) = 1 - \frac{4\alpha}{\pi} \qquad (20)$$

Both formulae are similar in structure and, as will be seen in connection with Tables I and II, they also agree rather well numerically. By itself this is already an interesting result: we have here an alternative and definitely simpler derivation of the Karplus-Klein formula[13] as well as an indication that duality is applicable also to bound systems, not made up of quarks, in which the binding potential is not confining. Furthermore in connection with the calculation of leptonic widths of vector mesons this result assures us that the estimate of QCD radiative corrections with the help of duality will be reliable.

We have computed the relativistic and radiative correction factors $r_n = |\psi_n(0)|^2 / |\psi_n(0)|^2_{WKB}$ and $C_n(v;\alpha_s)$ respectively from Eqs. (17) and (18) for the ψ and Υ states. The results are shown in Tables I and II for three different values of quark masses. We have used for these calculations Schwinger's interpolating function $\hat{h}(v)$ and the factorisation in Eq. (9). Also shown in the Tables is the Karplus-Klein correction factor $C(v;\alpha_s)_{KK} = 1 - 16\,\alpha_s(m_n^2)/3\pi$ with the QCD fine structure constant. The radiative corrections for the ψ and Υ states are thus consistently of the order of 50% and 40% respectively. No model can therefore afford to neglect them(*).

The relativistic corrections on the other hand vary a little bit more from state to state and for a fixed state, with the quark mass. They are on the whole larger for the ψ than for the Υ states and are comparable to the radiative corrections. Note that r_n is essentially a product of kinematical factors. It is unity for positronium. A non-relativistic potential which tries to simulate the effect of r_n (and C_n) will have problems with the mass spectrum unless it reckons with a complete breakdown of the WKB approximation for $\psi_n^{(o)}(0)$. This is difficult to conceive with power-law potentials.

(*) Note the good agreement between C_n and $C_n(KK)$ in these Tables. The agreement holds for all values of the coupling constant $\alpha \to \frac{4\alpha_s}{3}$ and hence also in the case of positroinum.

3. - CONCLUSIONS

According to Eq. (15) leptonic widths of vector mesons are computable if, besides quark charge and mass, the mass spectrum of these mesons is given. Non-relativistic potential models have been remarkably successful in fitting the spectra of the ψ and Υ states. The large relativistic and radiative corrections to the non-relativistic leptonic widths in Tables I and II imply that these models will be equally remarkably unsuccessful in predicting the correct leptonic widths with the help of Eqs. (1) and (16). This fact can be surprising only at first sight. The leptonic widths are proportional to the modulus squared of the wave function at the origin. There is no theorem which guarantees that if a certain potential reproduces a specified spectrum of eigenvalues then its associated set of eigenfunctions will necessarily be the same as that of the given eigenvalue spectrum(*). For this to be the case one has to impose that these eigenfunctions satisfy the same boundary conditions. This in turn would say that the potential is unique. This is certainly not the case for the quarkonium states where completely different potentials are knwon[21] to reproduce equally well the ψ and Υ spectra. The sets of eigenfunctions corresponding to these potentials, that is to say, the predictions of the corresponding models for the leptonic widths, are different.

This is where duality becomes indispensible. It establishes directly a correspondence between the mass spectrum and the leptonic widths, which is unique within certain limits. One has now only to check if the correspondence (Eq. (13)) is valid. We have done so by fitting the ψ and Υ spectra with a mass formula of the form

$$m_n^2 = m_o^2\,(1+bn)^{1+\lambda} \qquad ; \; n=0,1,2,.. \tag{21}$$

and then used it in Eqs. (13) and (14). The values of the parameters (b, λ) for the ψ and Υ states are, respectively, (1.67,-0.7) and (0.82,-0.81). The predicted leptonic widths are shown in the last columns of Tables I and II. Except for the $\psi''(3.768)$

(*) As an example we note that the parameters of the potentials $V_1(x) = \frac{1}{2} M\,\omega^2 x^2$ and $V_2(x) = A + B(\frac{x}{a} - \frac{a}{x})^2$, $x > 0$, can be chosen so that their energy spectra match. The associated eigenfunctions are however different; they are proportional respectively to the Hermite and Laguere polynominals.

TABLE I - Relativistic (r_n) and radiative (C_n) correction factors to the Van Royen-Weiskopf formula for the leptonic widths of the members of the ψ-family. Three different values for the quark mass M have been chosen. The leptonic widths in the last column have been calculated from Eq. (15) using the mass formula in Eq. (21) to fit the experimental ψ-spectrum. For C_n(KK) and ρ_n consult the text.

V_n	m_n	M	C_n	C_n(KK)	r_n	ρ_n	$\Gamma(V_n \to e^+e^-)$/keV
ψ_0	3.097	1.0	0.500	0.492	0.587	1.420	4.741
		1.2	0.518	0.492	0.719	1.614	4.824
		1.4	0.545	0.492	0.871	1.722	4.709
ψ_1	3.685	1.0	0.520	0.522	0.495	1.144	2.044
		1.2	0.531	0.522	0.593	1.349	2.087
		1.4	0.544	0.522	0.702	1.510	2.109
ψ_2	3.768	1.0	0.523	0.526	0.485	1.111	1.281
		1.2	0.533	0.526	0.579	1.316	1.308
		1.4	0.545	0.526	0.685	1.479	1.324
ψ_3	4.030	1.0	0.531	0.537	0.456	1.016	0.979
		1.2	0.539	0.537	0.541	1.216	0.998
		1.4	0.549	0.537	0.635	1.384	1.012
ψ_4	4.159	1.0	0.535	0.542	0.444	0.974	0.787
		1.2	0.542	0.542	0.525	1.170	0.802
		1.4	0.552	0.542	0.614	1.339	0.814
ψ_5	4.415	1.0	0.542	0.550	0.422	0.897	0.680
		1.2	0.549	0.550	0.496	1.087	0.692
		1.4	0.556	0.550	0.576	1.254	0.703

TABLE II - The same as Table I, but for the members of the Υ-family.

V_n	m_n	M	C_n	C_n	r_n	ρ_n	$\Gamma(V_n \to e^+e^-)$/keV
Υ_0	9.458	3.5	0.653	0.639	0.681	1.230	1.043
		4.0	0.667	0.639	0.800	1.342	1.006
		4.5	0.688	0.639	0.933	1.410	0.883
Υ_1	10.016	3.5	0.655	0.644	0.639	1.168	0.608
		4.0	0.665	0.644	0.745	1.287	0.596
		4.5	0.680	0.644	0.864	1.373	0.557
Υ_2	10.350	3.5	0.655	0.647	0.617	1.132	0.433
		4.0	0.665	0.647	0.717	1.254	0.427
		4.5	0.678	0.647	0.828	1.347	0.407
Υ_3	10.570	3.5	0.656	0.649	0.603	1.108	0.337
		4.0	0.665	0.649	0.699	1.232	0.334
		4.5	0.677	0.649	0.806	1.328	0.321

these widths agree rather well with the data. Gounaris[22] has an argument that since the $\psi'(3.685)$ and $\psi''(3.768)$ are almost degenerate duality should not be applied to them separately but to some weighted average of the two. He gets better agreement with experiment for a linear combination of the two states. It is not necessary for our purposes to enter into these details. The point we wish to make is that the correspondence between the mass spectrum and the leptonic widths established by duality is experimentally valid. We have used this correspondence actually in three ways in this paper:

a) to compute leptonic widths for a given mass spectrum as just discussed

b) to estimate the relativistic and radiative corrections to the non-relativistic Van Royen-Weiskopf formula

c) to provide an alternative and simpler derivation of the Karplus-Klein formula in positronium decay.

Does it follow from these applications of duality that short range forces, and not the long range confining ones, are dominant in determining the structure of quarkonium states? Poggio and Schnitzer[8] have argued that this is what the validity of the analogy with QED in applying the radiative corrections would imply. Actually duality leads to an even more puzzling paradox namely that the effect of the quark binding fores can be neglected. Following the work of Shifman, Vainstein and Zakharov (SVZ)[17], Bell and Bertlmann[18] have reformulated the problem thus: Under what conditions will the confining potential act as a small perturbation? The problem is still not well understood. However in the framework of potential models the idea of duality is limited but clear: it is no more than an alternative derivation of the WKB approximation for the wave function at the origin. This latter in no way implies that the binding potential is not effective. In the relativistic theory we conclude therefore that duality does not imply that only the short range part or no binding forces at all are effective. The fact that this principle applies to positronium is of course consistent with the usual short time arguments[10,18]. These arguments are seemingly not sufficient for understanding the validity of duality in the case of confined systems.

We think that all this adds up not to a mystery of duality but of confinement. Regarding the question of Bell and Bertlmann then we would speculate that a large part of the confining interaction goes into renormalising the quark mass down to the mass parameter of that name appearing in all our equations. A choice of this parameter can thus be made such that what remains of the confining interaction can be treated as a small perturbation.

Finally we have compared our result Eq. (15), with the definition (17) of the wave function at the origin with the parametrisation of Quigg and Rosner[3] of the overall correction to the Van Royen-Weiskopf formula. Their parameter ρ is related to our r_n and C_n by the formula

$$(m_n/2M)^2 \, r_n \, C_n \, \rho_n = 1 \tag{22}$$

The values of ρ_n calculated from Eq. (22) are shown also in the Tables. They are in the range found by Quigg and Rosner by taking ρ constant, that is independent of the state n.

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