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Critical Behavior of Branched Polymers and the Lee-Yang Edge Singularity

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The exponents for large branched dilute polymers (which are also connected with the exponents of the lattice animals) are related in D dimensions to the exponents of the Lee-Yang edge singularity of the Ising model in $D-2$ dimensions. From the exact solution of the Ising model in zero and one dimension, one gets the polymer exponents in two and three dimensions, $\theta(D=2)=1$ and $\theta(D=3)=\frac{3}{2}$.

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In this Letter we study the statistics of the dilute limit of branched polymers in a good solvent. This problem belongs to the same universality class as the statistics of lattice animals.[1] The two systems have the same critical exponents which have been computed at first order in $\epsilon = 8 - D$ (D is the space dimension) by Lubensky and Isaacson[1] using a field-theory formalism.

It is well known that in order to study the statistics of polymers, one can represent them as a self-avoiding chain on a regular lattice. Occupied bonds on the lattice represent monomers and connected clusters of occupied bonds represent polymers. In the limit of very dilute branched polymers in a good solvent we can neglect the influence which different polymers exercise on each other and study the statistics of a single cluster.

The statistics of clusters is also studied in the context of the percolation problem. A single connected cluster is usually called a lattice animal in the percolation terminology. The particular properties of the polymers we will study are the number $\mathfrak{N}$ of possible configurations of a branched polymer made from N monomers and the mean end-to-end distance R. The behavior for large N is expected to be the following[1]:

$$\mathfrak{N} \sim N^{-\theta}\lambda^N, \quad R \sim N^{\nu}. \tag{1}$$

The critical exponents θ and ν are universal but the constant λ is not.

We show that the critical exponents θ and ν in D dimensions are related to the critical exponent σ which controls the behavior of the magnetization near the Lee-Yang edge singularity[2] in the Ising model in the presence of an imaginary external field in $d=D-2$ dimensions

$$\theta(D) = \sigma(d) + 2. \tag{2}$$

The Josephson scaling law (or modern field-theory arguments) connects the singularity of the free energy F with the singularity of the correlation length ξ ($F \sim \xi^{-d}$). We get

$$\nu(D) = [\sigma(d) + 1]/d. \tag{3}$$

In the case of the Ising model it is well known from the Lee-Yang theorem that in the presence of an external magnetic field $H=H_1+iH_2$ the correlation functions may be singular only for $H_1=\mathrm{Re}(H)=0$. Above the critical temperature T_c there are singularities only for $H_2=\mathrm{Im}(H)\neq 0$. Let H_0 be the distance from the real axis of the closest singularity, $H=iH_0$. Fisher[2] has studied this singularity (which he calls the Lee-Yang edge singularity) and has shown that in its neighborhood the magnetization behaves like

$$m\equiv M-M_0\sim(H-iH_0)^\sigma,$$

where M_0 is the magnetization at iH_0.

While H_0 is not universal (it depends on the particular lattice, on the temperature, etc.), the exponent σ is universal like the ordinary critical exponents and has been computed by renormalization-group methods[2,3] in the $\epsilon=d-6$ expansion.

Because of the shift of the dimensionality appearing in Eqs. (2) and (3) from the exact solution of the Ising model (in the presence of an external field) in $d=0$ and $d=1$ dimensions, we get the exact values of θ and ν in the interesting case of three dimensions: $\theta(3)=\frac{3}{2}$, $\nu(3)=\frac{1}{2}$, and $\theta(2)=1$. These values can be compared with the results of "high-temperature expansions"[4] (enumeration of the bond animals up to $N=9$) $\theta(2)=1.00\pm 0.01$ and $\theta(3)=1.55\pm 0.05$ and the result of Monte Carlo simulations $\nu(3)=0.45\pm 0.06$,[5] or $\nu(3)=0.53\pm 0.02$.[6] From the $\epsilon'=6-d$ expansion for σ,[3] we also obtain the first three terms in the $\epsilon=8-D$ expansion for θ:

$$\theta=\frac{5}{2}-\frac{\epsilon}{12}-\frac{79}{3888}\epsilon^2+\left(\frac{\zeta(3)}{81}-\frac{10445}{1\,259\,712}\right)\epsilon^3. \tag{4}$$

In order to establish this connection between θ and σ we show that the effective Hamiltonian written by Lubensky and Isaacson for the description of the critical properties of the dilute branched polymers is equivalent to the effective Hamiltonian for the Ising model in a quenched random external field. Then we use the equivalence —in the critical region—of a D-dimensional magnetic system in a quenched random external field to a $d=D-2$ magnetic system without random external field.

Our method therefore relies entirely on renormalization-group ideas and field theory. We show that the two problems, which look very different, belong to the same universality class. The excellent agreement we find for $\theta(2)$ where the numerical analysis is more accurate may be viewed as another verification of renormalization-group ideas.

For the polymer problem the effective interaction has been shown to be[1]

$$\mathcal{H}=\tfrac{1}{2}\sum_i[(\partial_r\varphi_i)^2+r\varphi_i^2]-w\sum_i\varphi_i^3+u\Big(\sum_i\varphi_i^2\Big)^2+H\sum_i\varphi_i, \tag{5}$$

where $i=1,\ldots,n$ and the limit $n\to 0$ has to be taken at the end. The field φ_i appearing in Eq. (5) is to be identified with the field ψ_{1i} of Ref. 1. The other components ψ_{ji}, $j\neq 1$, do not become critical at the phase transition and their presence can be neglected for the computation of the critical exponents, as was already noticed by the authors of Ref. 1. It was also shown in Ref. 1 that for the Hamiltonian (5) the first infrared singularities in the one-loop diagram level appear for $D=8$, contrary to naive expectations ($D=6$). This result holds only for $n=0$.

As usual, we first shift the field φ_i ($\varphi_i\to\varphi_i+Q$) in order to eliminate the linear term in (5) and we get the effective Hamiltonian

$$\mathcal{H}=\tfrac{1}{2}\sum_i[(\partial_r\varphi_i)^2+m^2\varphi_i^2-\lambda\varphi_i^3]+4uQ^2\sum_{i,j}\varphi_i\varphi_j+4uQ\sum_{i,j}\varphi_i^2\varphi_j. \tag{6}$$

For the computation of the critical exponents one can take account of the leading infrared divergences only, neglecting power corrections. [This is why we omitted the quartic term from Eq. (6).]

The propagator P_{ij} given by Eq. (6) is not diagonal in the fields but has the form

$$P_{ij}=P_{ij}^{(1)}+P_{ij}^{(2)},\quad P_{ij}^{(1)}=\delta_{ij}/(p^2+m^2),$$
$$P_{ij}^{(2)}=\Pi_{ij}/[(p^2+m^2)(p^2+m^2+nuQ^2)], \tag{7}$$

where $\Pi_{ij}=1$, $\Pi^2=n\Pi$.

There are thus two propagators $P^{(1)}$ and $P^{(2)}$ and two cubic vertices. Figure 1 represents three Feynman diagrams contributing to the self-ener-

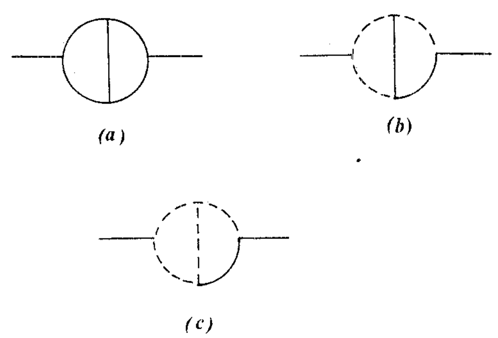

FIG. 1. Feynman diagrams contributing to the self-energy.

gy. Continuous lines represent the $P^{(1)}$ propagator and dashed lines represent the $P^{(2)}$ propagator. In the limit $n\to 0$, $P^{(2)}$ becomes more singular as $m^2\to 0$ and has to be taken a maximum number of times in order to get the most singular diagram. [The diagram 1(b) is more singular than 1(a).] The diagram 1(c) is proportional to n (because $\Pi^2 = n\Pi$) and does not contribute to the limit $n\to 0$ although more singular (one first takes $n\to 0$ and then $m^2\to 0$). This limits the number of $P^{(2)}$ propagators. An analysis similar to that of Houghton, Reeve, and Wallace[7] can be carried out. One easily sees that only the $\sum_i \varphi_i^3$ cubic vertex contributes to the most singular diagrams; the vertex $\sum_{i,j}\varphi_i^2\varphi_j$ combined with $P^{(2)}$ giving additional n factors is suppressed by the $n\to 0$ limit. If we omit this term, the Hamiltonian (6) also describes the φ^3 theory with a quenched random imaginary external field h (with a Gaussian distribution of width uQ^2) in the replica formalism[8] (h must be imaginary because of the interaction of polymers being repulsive, $u>0$).

For real magnetic field it was proven in a perturbative framework[9] that, at the leading divergence level, the correlation functions of the quenched system in D dimensions are the same as those of a magnetic system without the random external field in dimension $d=D-2$. As we have previously noted,[10] the leading infrared diagrams correspond to the solution of the stochastic equation

$$-\Delta\varphi + m^2\varphi + w\varphi^2 = ih, \quad \langle h(x)h(y)\rangle = \delta(x-y), \tag{8}$$

where $h(x)$ is a random magnetic field. By redefining φ as $i\varphi$, we get

$$-\Delta\varphi + m^2\varphi + iw\varphi^2 = h. \tag{9}$$

If we write this equation in a field-theory formalism, a hidden supersymmetry appears which allows us to prove the shift of dimensionality in a compact nonperturbative way.[10] Therefore the problem has been reduced to the computation of the critical exponents of a φ^3 interaction with an imaginary coupling constant. Fisher[2] has proved that the same interaction appears in the study of the Lee-Yang edge singularity in the presence of an imaginary magnetic field. In this case, one finds that the magnetization has a branch point of the form[2]

$$m \sim (h-h_0)^\sigma, \quad \sigma = (d-2+\eta)/(d+2+\eta). \tag{10}$$

Consequently, the free energy has a singularity of the form $(h-h_0)^{\sigma+1}$. The singularity of the free energy of the d-dimensional Ising model is therefore the same as for the D-dimensional polymer problem ($D=d+2$). From Eq. (1) it follows that the polymer generating function $Z(K)=\sum_n \mathfrak{N}(N)K^N$ has a singularity of the form $(K-1/\lambda)^{\theta-1}$ and this proves Eq. (2).

The exponent σ has been computed up to the third order in the ϵ expansion[3] and from this we obtain Eq. (4). The Ising model with an external field is soluble for $d=0$ (one site) and $d=1$. One finds that $\sigma(0)=-1$ and $\sigma(1)=-\frac{1}{2}$, from which the exact values for θ and ν in two and three dimensions follow as listed in the introduction. Using the $\epsilon=8-D$ expansion and the exact value of θ for $D=3$, we find the following parametrization for θ:

$$\theta = 2.5 - 0.0833\epsilon \times \frac{1+0.5661\epsilon}{1+0.3223\epsilon - 0.0406\epsilon^2}, \tag{11}$$

from which we find, using Eq. (3), that $\nu(2)=[d\theta/dD]_{D=2}=0.61$. If we add the numerical results of series expansion for the Ising model in two and three dimensions,[2] we get the estimates for θ in four and five dimensions shown in Table I. From ν, the Hausdorf dimension of the animals $d_H=1/\nu$ follows. Going from eight to three dimensions, the Hausdorf dimension is reduced from 4 to 2.

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TABLE I. Values of the exponent θ at different dimensions. θ_1 is the value obtained in Ref. 3 from bond animal enumeration. θ_2 is obtained from the high-temperature series evaluation of σ given in Ref. 2 through Eq. (2); and θ_3 is given by Eq. (11).

D	2	3	4	5	6	7
θ_1	1.00 ± 0.01	1.55 ± 0.05	1.90 ± 0.07	2.2 ± 0.1	2.3 ± 0.2	2.4 ± 0.2
θ_2	1	1.5	1.85	2.10	···	···
θ_3	1.006	1.5	1.84	2.08	2.26	2.40

tality.

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