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A SYSTEMATICAL IMPROVEMENT OF THE MIGDAL RECURSION FORMULA

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We construct a new perturbative expansion whose zeroth-order approximation gives the Migdal–Kadanoff recursion equations. By this expansion it is therefore possible to improve systematically the Migdal–Kadanoff results. The first-order corrections are computed for the Ising and bond percolation models. The second-order corrections are computed only for the two-dimensional Ising model. Our method can be easily extended to other systems like the non-linear σ model or gauge theories.

1. Introduction

The real space normalization group is a very powerful technique which allows one to compute the critical properties of a system, in particular the transition temperature and the critical exponents. Unfortunately, in the exact approach one should consider the renormalization equations for an infinite number of independent interactions. In practice, when we make a numerical computation, we are forced to project the infinite-dimensional space of the hamiltonians onto a finite-dimensional subspace. This projecting operation is the most delicate point of the approximate approach: it is not clear which principles should guide us.

The Migdal recursion method [1] revised by Kadanoff [2] has a privileged place among other approaches; it works magnificently for systems whose transition temperature is very near or equal to zero (i.e., near the lower critical dimension, when the continuum field theory associated to the low-temperature expansion is renormalizable). This happens, for example, in dimension $D = 2$ for the non-linear σ model and in dimension $D = 4$ for non-abelian gauge theories. The Migdal approximation is the only real space renormalization group approach which reproduces automatically the "asymptotic freedom" behaviour for systems at the lower critical dimension [1, 3].

It was stressed to one of us (G.P.) by Migdal that it should be possible to improve this approach by considering the Migdal recursion formulas at the zeroth-order approximation of a perturbation theory and by finding higher-order corrections in a systematic way.

Stimulated by this suggestion, we have accomplished this program by computing the first-order corrections of a few simple cases and the second-order corrections in a particular case. The corrections we find are not small and they improve the Migdal–Kadanoff results in a substantial manner. Our approach is applicable to any model.

In sect. 2 we recall the principles of the Migdal–Kadanoff recursion formulas and we describe the principle of the method we shall use to construct this new perturbative expansion. In sect. 3 we apply our method to the two-dimensional Ising model while in sect. 4 we study the D-dimensional Ising model. In sect. 5 we extend our results to the bound percolation problem. In sect. 6 we present a general discussion of the results and the possible developments.

2. The method

In this section we reconsider critically the Migdal–Kadanoff approach to the renormalization group in order to introduce corrections to their results. For definiteness we limit ourselves to the two-dimensional Ising model with the nearest neighbour spin interactions (NNI).

First let us recall the general principles of the renormalization group.

To obtain the renormalization group equations, one proceeds in the following way: given the initial partition function Z, corresponding to a hamiltonian $H^{(1)}(\sigma)$, one eliminates a part of the spins $\tilde{\sigma}$ (this procedure is called decimation). After this, one obtains the result that Z depends on a new effective hamiltonian $H^{(2)}$, where $H^{(2)}$ is a function of the remaining spins μ.

More precisely,

$$Z = \sum_{\{\mu\}} \exp\left[-H^{(2)}(\mu)\right] = \sum_{\{\mu\}} \left(\sum_{\{\tilde{\sigma}\}} \exp\left[-H^{(1)}(\tilde{\sigma}, \mu)\right] \right). \tag{2.1}$$

We have absorbed the inverse temperature β in the definition of H. Eq. (2.1) defines the renormalization group transformation R which gives $H^{(2)}(\mu)$ as a function of the initial hamiltonian $H^{(1)}(\sigma)$

$$R: H^{(2)} = R[H^{(1)}]. \tag{2.2}$$

One can also define the hamiltonian $H^{(n)}$ as:

$$H^{(n)} = R[H^{(n-1)}]. \tag{2.3}$$

According to the conventional wisdom, the stable fixed points of the transformation R (i.e., $H_f = R[H_f]$) are connected with the phases of the spin system [4].

The domains of attraction of these fixed points are separated by a critical surface and the unstable fixed point on the critical surface controls the critical properties of the system (like, for example, the critical exponents). The critical temperature is fixed by the intersect of the critical surface with the surface of the hamiltonian one is

considering (e.g., for the NNI, the subspace is one-dimensional and is characterized only by the parameter β).

Unfortunately, a very simple hamiltonian under the renormalization group transformation will generate a very complex one. For example, if $H^{(1)}$ is the NNI Ising model hamiltonian, $H^{(2)}$ contains next-to-neighbouring interactions, four-spin interactions and so on. Then the fixed point will be characterized by an infinite number of parameters. In practical cases, one tries to avoid these difficulties by defining an operator P which projects the hamiltonian, after decimation, onto a new hamiltonian depending on a finite number of parameters. The renormalization group transformation becomes [4]

$$H^{(n)} = R_P[H^{(n-1)}] = P\{R[H^{(n-1)}\} . \tag{2.4}$$

R_P is obviously defined as a finite-dimensional space. The choice of P is crucial but it is not evident *a priori* how to make a good choice. Another method to simplify the renormalization group transformation is given by the Kadanoff version of the Migdal recursion equations [2]. One defines the transformation as

$$H^{(2)} = R[H^{(1)} + \tilde{H}] , \tag{2.5}$$

where $\tilde{H}$ is in general a certain function of $H^{(1)}$ chosen in such a way as to simplify the transformation. The choice of $\tilde{H}$ is equivalent in the previous case to the choice of P.

Let us consider in detail the two-dimensional Ising system. In this case the Kadanoff operator $\tilde{H}$ is a bond moving operator whose action is pictorially shown in fig. 1. For the moment we limit the discussion to the simple case where only half of the bonds have been moved. Then half of the spins are connected to the rest of the system by only two bonds. The sum over the configurations of these spins can easily be done using the formulas

$$\exp(\beta\sigma_1\sigma_2) = \cosh(\beta) \cdot \{1 + [\mathrm{tgh}(\beta)]\sigma_1\sigma_2\} , \tag{2.6a}$$

$$\sum_{\{\sigma=\pm1\}} (1 + k\mu_1\sigma)(1 + k\sigma\mu_2) = 1 + k^2\mu_1\mu_2 , \tag{2.6b}$$

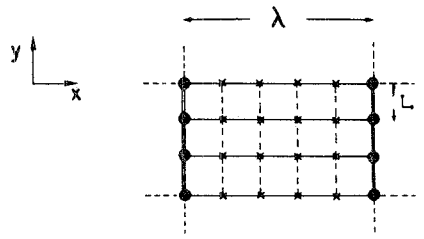

Fig. 1. A pictorial representation of the lattice after the shifting operation. The simple lines are the unshifted bonds. The double lines are the strong bonds, the dashed lines the weak bonds. The corresponding interaction strength is $\lambda\beta$ and 0 in the Migdal–Kadanoff limit, $\beta_y + (\lambda - 1)(1 - \varepsilon)\beta_y$ and $\varepsilon\beta_y$ in our approach. The x are the decimated spins ($\tilde{\sigma}$), the ● surviving spins (μ).

from which follows

$$\sum_{\{\sigma=\pm 1\}} \exp(\mu_1\sigma+\sigma\mu_2) = \text{const} \cdot \exp\{\text{tgh}^{-1}[\text{tgh}^2(\beta)]\cdot\mu_1\mu_2\}. \tag{2.7}$$

(If we are not interested in explicitly computing the free energy but only the renormalization group equations of the coupling constants, the overall multiplicative factors can be neglected.)

After the decimation in the x direction (see fig. 1) we obtain an anisotropic hamiltonian whose next neighbour coupling constants are given by

$$\begin{aligned}\beta'_x &= \text{tgh}^{-1}[\text{tgh}^2(\beta)],\\ \beta'_y &= 2\beta.\end{aligned} \tag{2.8}$$

We proceed to a new decimation by changing the rôle of the x and y axes.

After the two transformations we obtain a lattice whose lattice spacing is twice the original one ($L\to 2L$) and the renormalization group equations are

$$\begin{aligned}\beta''_x &= 2\,\text{tgh}^{-1}[\text{tgh}^2(\beta)],\\ \beta''_y &= \text{tgh}^{-1}[\text{tgh}^2(2\beta)].\end{aligned} \tag{2.9}$$

The two transformations do not commute and we do not restore the isotropy of the original system. This effect is an artefact of the transformation we used and can be eliminated by projecting the final hamiltonian on the space of the isotropic hamiltonians by defining

$$\beta'' = \tfrac{1}{2}(\beta''_x+\beta''_y). \tag{2.10}$$

The generalization of eqs. (2.8)–(2.10) to the case $L\to\lambda L$ gives the following formulas:

$$\begin{aligned}\beta^\lambda_x &= \lambda\ \text{tgh}^{-1}[\text{tgh}^\lambda(\beta)],\\ \beta^\lambda_y &= \text{tgh}^{-1}[\text{tgh}^\lambda(\lambda\beta)],\\ \beta^\lambda &= \tfrac{1}{2}(\beta^\lambda_x+\beta^\lambda_y) = f_\lambda(\beta).\end{aligned} \tag{2.11}$$

The dilatation of the lattice spacing is defined for integer λ. Eqs. (2.11) can be formally continued to non-integer λ without difficulties. If we set $\lambda = 1+\Delta$, with Δ infinitesimal, we get

$$\begin{aligned}\beta^\Delta_x &= \beta + f(\beta)\Delta + O(\Delta^2),\\ \beta^\Delta_y &= \beta + f(\beta)\Delta + O(\Delta^2),\end{aligned} \tag{2.12}$$

with

$$f(\beta) = \beta + \sinh(\beta)\cosh(\beta)\ln[\text{tgh}(\beta)].$$

We see that for Δ infinitesimal the symmetry between x and y is restored. After N

spin decimations with a scale factor $\lambda = 1+\Delta$, the spacing is increased by a factor $(1+\Delta)^N \sim \exp(N\Delta)$. Calling $t = N\Delta$, in the limit $\Delta \to 0$, t becomes the continuum variable and $\beta(t)$ satisfies the differential equation:

$$\frac{\mathrm{d}\beta(t)}{\mathrm{d}t} = f(\beta)\,. \tag{2.13}$$

The function $f(\beta)$ is positive for large β, negative for small β and it has only one zero at $\beta = \beta_c$. The asymptotic behaviour for large t crucially depends on the value of β^0 at $t = 0$, i.e., the value of the NNI when the decimation process starts. One finds that for $t \to \infty$

$$\begin{aligned} &\beta(t) \to 0\,, && \text{if } \beta^0 < \beta_c\,,\\ &\beta(t) \to \infty\,, && \text{if } \beta^0 > \beta_c\,,\\ &\beta(t) \to \beta_c\,, && \text{if } \beta^0 = \beta_c\,. \end{aligned} \tag{2.14}$$

There are three fixed points: $\beta = 0$ and $\beta = \infty$ are stable fixed points, $\beta = \beta_c$ is unstable. Putting $\beta^0 = \beta_c + \delta\beta$ we have

$$\begin{aligned} &\beta(t) \simeq \beta_c + \delta\beta \exp(\omega t)\,,\\ &\omega = \frac{\mathrm{d}}{\mathrm{d}\beta} f(\beta)|_{\beta=\beta_c}\,, \qquad \text{for } |\delta\beta \exp(\omega t)| \ll 1\,. \end{aligned} \tag{2.15}$$

Recalling that the scale has increased by a factor e^t we find $\beta(l) - \beta_c \simeq l^{\omega}(\beta_0 - \beta_c)$. This defines the critical exponent of the correlation length $\nu = 1/\omega$.

For general λ we have qualitatively the same situation:

$$\begin{aligned} &\beta_c = f_\lambda(\beta_c)\\ &\omega = \ln\left[\left(\frac{\mathrm{d}f_\lambda}{\mathrm{d}\beta}\right)_{\beta=\beta_c}\right] \Big/ \ln(\lambda)\,. \end{aligned} \tag{2.16}$$

The dependence of β_c and ω on λ is not very strong in the region $\lambda \sim 1$. We note that for reasons which are not clear one gets the exact result for the critical temperature in the limit $\lambda \to 1$ while the critical exponent ν is overestimated. The approach can be generalized to any dimension D. For infinitesimal Δ one gets

$$f(\beta) = (D-1)\beta + \sinh(\beta)\cosh(\beta)\ln(\tanh(\beta))\,. \tag{2.17}$$

β_c and ν as functions of D are shown in figs. 2, 3. The results are rather satisfactory for low dimensions and their quality deteriorates with increasing D. As expected, if $D = 1$, there is no transition; if $D \geqslant 1$, $\beta_c \sim 1/(D-1)$ and $\nu \sim 1/(D-1)$ [5]. When $D \to \infty$, however, in this approximation $\beta_c \sim \mathrm{e}^{-(D-1)}$ and $\nu \to 1$ while the "exact" results [6] are $\beta_c = 1/2D + \mathrm{O}(1/D^2)$ and $\nu = \frac{1}{2}$ for $D > 4$.

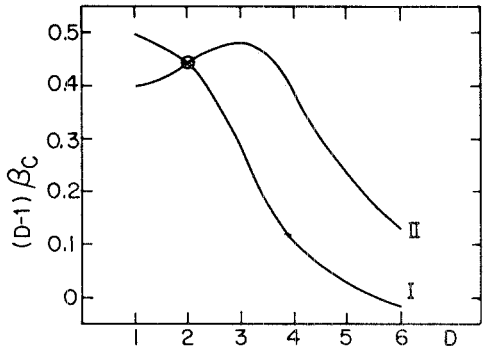

Fig. 2. $(D-1)\beta_c$ as a function of the dimension at zeroth order (line I) and at first order (line II) in ε (Ising model).

Let us generalize the Migdal–Kadanoff approach as expressed by eq. (2.5). We define the transformation R_ε depending on the parameter ε as

$$H^{(2)} = R_\varepsilon[H^{(1)}] = R[H^{(1)} + (1-\varepsilon)\tilde{H}], \tag{2.18}$$

where $\tilde{H}$ is still a bond shifting operator. Obviously, for $\varepsilon = 0$ we recover the Migdal–Kadanoff transformation while for $\varepsilon = 1$ we get the exact renormalization group transformation of eq. (2.2). Using the renormalization group transformation [eq. (2.18)] we get a critical β, $\beta_c = \beta_c(\varepsilon)$, and a critical exponent $\nu = \nu(\varepsilon)$ which depends on ε. It was observed by Kadanoff that if we choose $\tilde{H}$ in an appropriate way,

$$Z(H) = Z(H + (1-\varepsilon)\tilde{H})(1 + O(1-\varepsilon)^2), \tag{2.19}$$

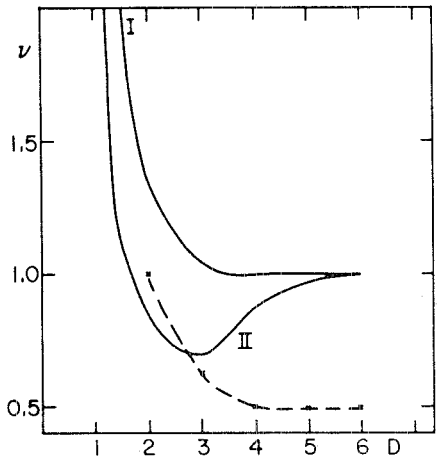

Fig. 3. ν as function of the dimension at zeroth order (line I) and first order (line II) in ε. The dashed line represents the correct result (Ising model) [8].

the difference between $\beta_c(\varepsilon = 1)$ (which is the exact one) and $\beta_c(\varepsilon)$ should be of order $(1-\varepsilon)^2$.

For $\varepsilon \neq 0$ it is not possible to compute $\beta_c(\varepsilon)$, $\nu(\varepsilon)$ and $H_f(\varepsilon)$ [the fixed point hamiltonian satisfying the condition $H_f = R_\varepsilon(H_f(\varepsilon))$] in a closed form because of the infinite number of interactions generated by the decimation. However, we shall prove that it is possible to decompose the space S of the hamiltonians generated by the renormalization group transformation in the direct sum of spaces S_m ($S = \bigoplus_{m=0}^{\infty} S_m$) in such a way that the S_m are *finite*-dimensional spaces and the component of the fixed point hamiltonian $H_f(\varepsilon)$ in the space S_m is of order ε^m (S_0 is the space of the nearest-neighbour interaction). Indeed if the starting hamiltonian belongs to S_0 after n transformation, the renormalized hamiltonian $H^{(n)}(\varepsilon)$ will have its component in S_m proportional to ε^m. $H_f(\varepsilon)$, $\beta_c(\varepsilon)$ and $\nu(\varepsilon)$ can be expanded in power series of ε and the coefficients of the Taylor expansion can be computed in a closed form. Calling P_M the projection operator on the space $S^M = \bigoplus_{m=0}^{M} S_m$ (S^M being finite dimensional) and defining $R_\varepsilon^M = P_M R_\varepsilon$, the results obtained using R_ε^M will differ from those obtained using R_ε only by $O(\varepsilon^{M+1})$. Evidently the coefficients of the Taylor expansion in ε can be computed working with only a finite number of interactions (for the two-dimensional Ising model, S^0, S^1, S^2 have dimensions 1, 2, 7, respectively).

Our goal is to compute the critical exponents at $\varepsilon = 1$, i.e., the exact transformation, using as input the coefficients of the expansion in powers of ε. The Migdal–Kadanoff recursion formulas are the zeroth-order approximation of this new perturbative expansion.

In this approach the analytic properties of physical quantities in the complex ε plane are crucial. We do not see any reason for having a singularity around $\varepsilon = 0$ or on the 0–1 interval. The most delicate point is $\varepsilon = 1$. It is not clear if the fixed point hamiltonian $H_f(\varepsilon)$ has a "good" limit when $\varepsilon \to 1$ (it may be argued that in this limit it should become an infinite range hamiltonian); we hope that the behaviour of physical quantities like $\nu(\varepsilon)$ and $\beta(\varepsilon)$ will be smooth enough so that it would be possible to extrapolate their values around $\varepsilon = 1$ from the region $\varepsilon \sim 0$. In sect. 3 we shall show how this method works for the two-dimensional Ising model.

3. The two-dimensional Ising model

As explained in sect. 2, we study the renormalization group equation for the hamiltonian $H_\varepsilon = H + (1-\varepsilon)\tilde{H}$, where $\tilde{H}$ is the bond shifting operator whose action is shown in fig. 1. For the moment let us discuss the case $\lambda = 2$. Then half of the vertical bonds have strength $(2-\varepsilon)\beta$, half being weak ($= \varepsilon\beta$). For infinitesimal ε, we construct the perturbative expansion in the weak bonds. Let us consider the elementary cell of fig. 4:

$$H^{(1)}_{\text{cell}}(\mu,\sigma) = -\beta_x(\mu_1+\mu_2)\sigma_5 - \beta_x(\mu_3+\mu_4)\sigma_6 - (2-\varepsilon)\beta_y(\mu_1\mu_3+\mu_2\mu_4) - \varepsilon\beta_y\sigma_5\sigma_6 . \tag{3.1}$$

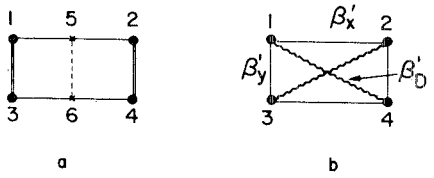

Fig. 4. The elementary cell for $\lambda = 2$ (a) before (2a) and (b) after (2b) decimation.

Here we have called σ_i the spins that we want to decimate and μ_i the spins left after decimation.

Taking care of the effects due to the nearby cells, after decimation we will find

$$\exp(-H^{(2)}_{\text{cell}}(\mu)) = \exp[\beta'_x(\mu_1\mu_2+\mu_3\mu_4) + \beta'_y(\mu_1\mu_3+\mu_2\mu_4)+\beta'_D(\mu_1\mu_4+\mu_2\mu_3)]+O(\varepsilon^2) \tag{3.2}$$

(all inessential multiplicative constants have been omitted) where, defining

$$\begin{aligned} k_x &= \text{tgh}(\beta_x), \\ \tilde{k}_x &= k_x^2, \\ k_y &= \text{tgh}[(2-\varepsilon)\beta_y], \\ \tilde{k}_y &= k_y + (1-k_y^2)\Delta k_y, \\ \Delta k_y &= 2\varepsilon\beta_y k_x^2/(1+k_x^2)^2. \end{aligned}$$

To this order we have

$$\begin{aligned} \beta'_x &= \text{tgh}^{-1}(\tilde{k}_x), \\ \beta'_y &= \text{tgh}^{-1}(\tilde{k}_y), \\ \beta'_D &= \tfrac{1}{2}\Delta k_y. \end{aligned} \tag{3.3}$$

Here we have used the identities (2.6) and (3.4):

$$\begin{aligned} \frac{1}{(1+k^2\mu_1\mu_2)} &= \frac{1}{1-k^4}(1-k^2\mu_1\mu_2), \\ \frac{\mu_1+\mu_2}{(1+k^2\mu_1\mu_2)} &= \frac{\mu_1+\mu_2}{(1+k^2)}. \end{aligned} \tag{3.4}$$

β'_D is clearly of order ε. Other interactions which are generated by decimation are at least of order ε^2.

We then proceed to a new decimation. We have to decide now which are the bonds to be shifted (having one more coupling β_D). We find it convenient to leave the diagonal bonds in place and to move only the horizontal (or the vertical bonds). If we remember that $\beta_D \sim O(\varepsilon)$, we find all the interactions that can be generated,

besides β_x, β_y and β_D, are at least of order ε^2. To this order we have only the new coupling β_D.

The complete renormalization group equations to this order will be as before:

$$\begin{aligned} \beta'_x &= \mathrm{tgh}^{-1}(\tilde{k}_x)\,, \\ \beta'_y &= \mathrm{tgh}^{-1}(\tilde{k}_y)\,, \\ \beta'_D &= \tfrac{1}{2}\Delta k_y\,, \end{aligned} \tag{3.5}$$

but now

$$\Delta k_y = \frac{2\varepsilon\beta_y k_x^2}{(1+k_x^2)^2} + \frac{4k_x\beta_D}{(1+k_x^2)}\,.$$

This defines the transformation R_x (decimation along the x axis). The transformation R_y is defined by interchanging the rôle of the x and y axes while S is defined as

$$\beta''_x = \beta''_y = \tfrac{1}{2}(\beta'_x + \beta'_y)\,, \qquad \beta''_D = \beta'_D\,. \tag{3.6}$$

The total transformation will be

$$R^{[1]}(H) = S\{R_y[R_x(H)]\}\,. \tag{3.7}$$

Using the transformation (3.7) we find the values for $\beta_c(\varepsilon)$ and $\omega(\varepsilon) = 1/\nu(\varepsilon)$ at the first order in ε. [$\omega(\varepsilon)$ can be found by linearizing the renormalization group transformations near the fixed point.] We obtain the following numerical results:

$$\begin{aligned} \beta_c(\varepsilon) &= \beta_0 + \beta_1\varepsilon\,, & \beta_0 &\simeq 0.4359\,, \\ & & \beta_1 &\simeq 0.024\,, \\ \omega(\varepsilon) &= \omega_0 + \omega_1\varepsilon\,, & \omega_0 &\simeq 0.687\,, \\ & & \omega_1 &\simeq 1.14\,. \end{aligned} \tag{3.8}$$

For simplicity the values of β_1 and ω_1 have been estimated by computing $\beta_c(\varepsilon)$ and $\omega(\varepsilon)$ using the transformation (3.7) for finite small values of ε and then by numerically differentiating the results (shown in figs. 5 and 6).

We now face the problem of extrapolating eqs. (3.8) to $\varepsilon = 1$; it is convenient to impose the condition

$$\left.\frac{\mathrm{d}\beta_c(\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=1} = \left.\frac{\mathrm{d}\omega(\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=1} = 0\,. \tag{3.9}$$

This can be done by defining the functions

$$\begin{aligned} \tilde{\beta}_c(\varepsilon) &= \beta_0 + \beta_1\varepsilon(1-\tfrac{1}{2}\varepsilon)\,, \\ \tilde{\omega}(\varepsilon) &= \omega_0 + \omega_1\varepsilon(1-\tfrac{1}{2}\varepsilon)\,. \end{aligned} \tag{3.10}$$

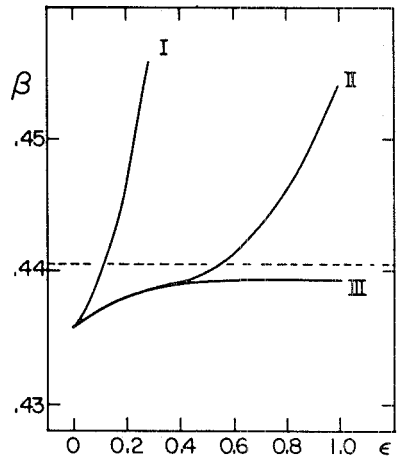

Fig. 5. Lines I, II and III are, respectively, the values of β_c for the two-dimensional Ising model as function of ε at first order in ε [*eqs.* (3.5)], at second order in ε [eqs. (3.12), (3.13), (3.14)] and for the modified Padé approximant [eq. (3.17)]. The dashed lines denote the exact result.

One then obtains $\beta_c(1) = 0.4486$, $\omega(1) = 1.26 \rightarrow \nu = 0.796$. We see that the corrections to the Migdal results go in the correct direction, although there is a tendency to overshoot.

Let us consider what happens at the next order in ε. Four-spin couplings and next-to-neighbour couplings are needed at order ε^2. In total we have eight couplings after the symmetrization along the two axes by the transformation S (12 for the asymmetric hamiltonian). Some of the new couplings are pictorially shown in fig. 7 for a 3×3 spin cell.

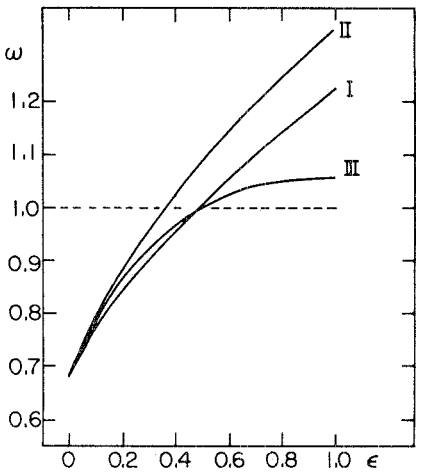

Fig. 6. Lines I, II and III refer to the values of $\omega(\varepsilon) = 1/\nu(\varepsilon)$ for the two-dimensional Ising model, respectively, at first order in ε, second order in ε, and using the modified Padé approximant.

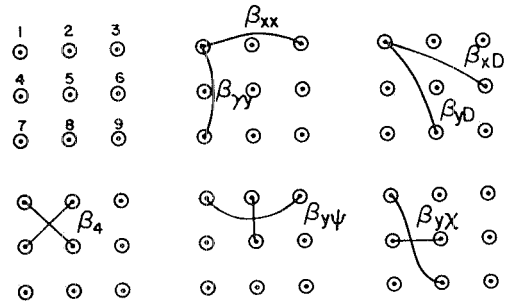

Fig. 7. The spins used to define the coupling constants in eq. (3.11) and some of the couplings of order ε^2.

We write a typical term of the hamiltonian (the others can be obtained by translational invariance) making reference to the spins shown in fig. 7:

$$\begin{aligned} H = &-\beta_x\mu_4\mu_5 - \beta_y\mu_2\mu_5 - \beta_D\mu_1\mu_5 - \beta_{yy}\mu_2\mu_8 - \beta_{xx}\mu_4\mu_6 \\ &- \beta_{xD}\mu_1\mu_6 - \beta_{yD}\mu_1\mu_8 - \beta_4\mu_1\mu_2\mu_4\mu_5 - \beta_{x\psi}\mu_3\mu_5\mu_6\mu_9 \\ &- \beta_{y\psi}\mu_1\mu_2\mu_3\mu_5 - \beta_{y\chi}\mu_1\mu_4\mu_5\mu_8 - \beta_{x\chi}\mu_1\mu_2\mu_5\mu_6 + \cdots . \end{aligned} \tag{3.11}$$

We define the renormalization group transformation as before by shifting only the vertical and horizontal nearest neighbour bonds. After a long computation we find the following transformation equations for the decimation along the x axis:

$$\begin{aligned} \beta'_x &= \mathrm{tgh}^{-1}(\tilde{k}_x) , \\ \beta'_y &= \mathrm{tgh}^{-1}(\tilde{k}_y) , \\ \tilde{k}_x &= k_x^2 + (1-k_x^4)\Delta k_x , \\ \tilde{k}_y &= (k_y + \Delta k_y)/(1 + k_y \Delta k_y) , \end{aligned} \tag{3.12}$$

where k_x and k_y are defined as before and

$$\begin{aligned} \Delta k_x = &\, 2(\beta_D^2 + 2\varepsilon\beta_y\beta_D A_x + 4\beta_D^2 A_x^2) - 4(\varepsilon\beta_y A_x^2 + 2\beta_D A_x)^2 \\ &+ 8\beta_{x\chi}A_x^2 + 4\beta_4 A_x^2 + \beta_{xx}(1+2A_x^2) , \\ \Delta k_y = &\, 2(\varepsilon\beta_y A_x^2 + 2\beta_D A_x) + 6\beta_{y\psi}A_x^2 + 2\beta_{xD}A_x^2 . \end{aligned} \tag{3.13}$$

The various couplings β are defined by eq. (3.11) and fig. 7:

$$A_x = k_x/(1+k_x^2) .$$

For the other couplings we have the following transformation equations:

$$
\begin{aligned}
\beta'_{\mathrm{D}} &= \varepsilon\beta_y A_x^2 + 2\beta_{\mathrm{D}} A_x + 4\beta_{y\psi} A_x^2 + \beta_{x\mathrm{D}}(1+2A_x^2)\,,\\
\beta'_{xx} &= \beta_{xx} A_x^2\,,\\
\beta'_{yy} &= 2(\beta_{\mathrm{D}} + \varepsilon\beta_y A_x)^2(1-2A_x^2) + 2\beta_{x\psi}(A_x^2 + 2A_x^3)\,,\\
\beta'_{x\mathrm{D}} &= \beta_{x\mathrm{D}} A_x^2\,,\\
\beta'_{y\mathrm{D}} &= (\beta_{\mathrm{D}} + \varepsilon\beta_y A_x)^2(1-2A_x^2) + 2(\beta_{x\psi} A_x^3 + \beta_{y\chi} A_x^2) + \beta_{yy} A_x^2 + 2\beta_{y\mathrm{D}} A_x\,,\\
\beta'_4 &= 4A_x^2\beta_{\mathrm{D}}^2 - 2(\varepsilon\beta_y A_x^2 + 2\beta_{\mathrm{D}} A_x)^2 + 2\beta_{x\chi} A_x^2 + 2\beta_4 A_x^2\,,\\
\beta'_{x\psi} &= -2(\varepsilon\beta_y A_x^2 + \beta_{\mathrm{D}} A_x)^2 + \beta_{x\psi}(A_x + 2A_x^3) + 2\beta_{y\chi} A_x^2\,,\\
\beta'_{y\psi} &= \beta_{y\psi} A_x^2\,,\\
\beta'_{x\chi} &= \beta_{x\chi} A_x^2\,,\\
\beta'_{y\chi} &= -2(\varepsilon\beta_y A_x^2 + \beta_{\mathrm{D}} A_x)^2 + 2\beta_{x\psi} A_x^3 + 2\beta_{y\chi} A_x^2\,.
\end{aligned}
\tag{3.14}
$$

The transformation $R^{[2]}$ is defined as

$$
R^{[2]}(H) = S\{R_y^{[2]}[R_x^{[2]}(H)]\}\,. \tag{3.15}
$$

We show in figs. 5 and 6 the results obtained by using the group transformation $R^{[2]}$. These results are corrected only at order ε^2. As before, by numerical differentiation we get

$$
\begin{aligned}
\beta(\varepsilon) &= \beta_0 + \beta_1\varepsilon + \beta_2\varepsilon^2\,, \qquad & \beta_2 &= -0.109\,,\\
\omega(\varepsilon) &= \omega_0 + \omega_1\varepsilon + \omega_2\varepsilon^2\,, \qquad & \omega_2 &= -1.21\,.
\end{aligned}
\tag{3.16}
$$

The second-order corrections are large and have the tendency to correct the overshooting due to the first-order approximation.

The reader is free to use the method he prefers to extract the values of $\beta(1)$ and $\omega(1)$ from the numbers of eqs. (3.8), (3.16). Of course, our series are very short and the extrapolation is not safe. We found it reasonable to use Padé approximants (the series has alternative signs). At this stage we can concentrate our attention only on the [1, 1] Padé approximant:

$$
\beta^{[1,1]}(\varepsilon) = \left.\frac{\beta_0[1 + (\beta_1/\beta_0 - \beta_2/\beta_1)\varepsilon + C\varepsilon^2]}{1 - (\beta_2/\beta_1)\varepsilon + C\varepsilon^2}\right|_{C=0}. \tag{3.17}
$$

The Padé approximant cannot satisfy the condition (3.9). We can impose this condition by taking a value different from zero for C ($C = 1$ is appropriate). This is not the only way to modify the Padé approximation scheme to impose the validity of eqs. (3.9). We could also consider, at fixed ε, the sequence

$$
\beta_0\,, \qquad \beta_0 + \beta_1\varepsilon(1 - \tfrac{1}{2}\varepsilon)\,, \qquad \beta_0 + \beta_1\varepsilon(1-\tfrac{1}{2}\varepsilon) + (\beta_2 + \tfrac{1}{2}\beta_1)\varepsilon^2(1 - \tfrac{2}{3}\varepsilon), \ldots, \tag{3.18}
$$

and to extrapolate the sequence (3.18) we use Padé approximants. Using the two different methods, we find, respectively,

$$\begin{aligned} &\beta_c(1) = 0.4396\,, \qquad \nu(1) = 0.94\,, \\ &\beta_c(1) = 0.4395\,, \qquad \nu(1) = 0.92\,. \end{aligned} \tag{3.19}$$

The results obtained with the two different methods are practically equivalent. Many other numerical extrapolations could be used, but it is difficult to judge the efficacy of the method without knowing the structure of the singularities in ε of the series.

The results of the second-order approximation are very satisfactory, also if we do not have computed enough terms of the series to deduce a sensible estimate of the error of our final predictions.

What happens at higher orders? It is clear that the first decimation, at order ε^N, will generate interactions connecting spins which are separated, at maximum, by N steps in the vertical or horizontal directions. It is easy to convince ourselves that if we stop at order ε^N only a finite number of coupling constants will be generated (so that one can work always with a finite number of coupling constants).

The computation of higher orders is straightforward but rather long: it could be simplified by using a computer to do algebraic manipulations. The use of a triangular lattice, instead of a square one, would strongly shorten the computation and decrease the number of couplings. We have preferred the square lattice in order to simplify the generalization to higher dimensions which will be the subject of sect. 4.

4. D-dimensional Ising model

In sect. 3 we have studied the two-dimensional Ising model with a scale factor $\lambda = 2$. It is rather simple to extend the renormalization group transformation (3.5) to generic λ. By repeated use of eqs. (2.6) and (3.5) and of the identity

$$\sum_{m=0}^{\lambda} k^m = (1-k^{\lambda+1})/(1-k)\,,$$

we find, at first order in ε, the following transformation $R_x^{(\lambda)}$:

$$\begin{aligned} &\beta'_x = \mathrm{tgh}^{-1}(k_x^\lambda)\,, \qquad && k_x = \mathrm{tgh}\,(\beta_x)\,, \\ &\beta'_y = \mathrm{tgh}^{-1}(\tilde{k}_y)\,, && \tilde{k}_y = k_y + 2(1-k_y^2)\Delta k_y\,, \\ & && k_y = \mathrm{tgh}\,[\beta_y + (\lambda-1)(1-\varepsilon)\beta_y]\,, \end{aligned} \tag{4.1}$$

where β'_D and Δk_y are solutions of the following equations:

$$\begin{aligned} &\Delta k_y = (1+k_x^{2\lambda}) + \beta'_D k_x^\lambda = \frac{[\varepsilon\beta_y(k_x^2 - k_x^{2\lambda}) + 2\beta_D k_x(1-k_x^{2\lambda})]}{1-k_x^2}\,, \\ &2\Delta k_y k_x^\lambda + \beta'_D(1+k_x^{2\lambda}) = k_x^\lambda\left[(\lambda-1)\varepsilon\beta_y + \frac{\lambda\beta_D}{k_x}(1+k_x^2)\right]. \end{aligned} \tag{4.2}$$

The details of the computation can be found in the appendix: for $\lambda = 2$ we recover the results of sect. 3.

For an infinitesimal $\lambda - 1$, R_x^λ and R_y^λ commute [up to $O((\lambda-1)^2)$] so that the application of the renormalization group transformation $R[H] = R_y[R_x(H)]$ does not introduce an anisotropy in the initially isotropic hamiltonian. In terms of the two couplings (β and β_D), eqs. (4.1) and (4.2) become

$$\begin{aligned} \dot{\beta} &= \frac{k \ln (k)}{1-k^2} + F_\beta(\beta, \beta_D) , \\ \dot{\beta}_D &= F_{\beta_D}(\beta, \beta_D) . \end{aligned} \tag{4.3}$$

F_β and F_{β_D} can easily be obtained by differentiating eqs. (4.1) and (4.2) with respect to λ at $\lambda = 1$ [notice that eqs. (4.1) and (4.2) become the identity for $\lambda = 1$].

Again, by computing the fixed points of eqs. (4.3), we obtain

$$\begin{aligned} &\beta_c(\varepsilon) = \beta_0 + \beta_1 \varepsilon + O(\varepsilon^2) , && \beta_0 = 0.4407 = \tfrac{1}{2} \ln (1+\sqrt{2}) , \\ &&& \beta_1 = 0 , \\ &\omega(\varepsilon) = \omega_0 + \omega_1 \varepsilon + O(\varepsilon^2) , && \omega_0 = 0.754 , \\ &&& \omega_1 = 0.82 . \end{aligned} \tag{4.4}$$

We found rather surprising the result that the exact value for β_c, given already by the zeroth-order Migdal approximation for $\varepsilon = 0$, is preserved at first order in ε. The deep reasons for this success are mysterious, although duality of an Ising model in two dimensions may play an important rôle.

By extrapolating $\beta_c(1)$ and $\omega(1)$ as done in sect. 3 [eqs. (3.9), (3.10)], we find

$$\beta_c = \tfrac{1}{2} \ln (+\sqrt{2}) , \qquad \nu = 0.86 . \tag{4.5}$$

The result obtained for ν is of the same quality as the one obtained in the case $\lambda = 2$.

The equations found for λ infinitesimal can be easily generalized to three or more dimensions. However, in this case, we have to face a new problem.

In two dimensions we decided to move only horizontal and vertical bonds, leaving the diagonal bonds fixed in place. This choice was arbitrary: all we need is that there is no bond moving for $\varepsilon = 1$. In dimension three, following the same procedure as in sect. 3, we would generate new couplings between spins at opposite vertices of a cube (fig. 8). There is no difficulty in working with more than one diagonal coupling; however, it would be very difficult to generalize this procedure for non-integer dimensions (in general we must add $D-2$ new diagonal couplings).

To avoid this complication we have decided to move a part of the diagonal couplings also. For example, in the three-dimensional case, we move the diagonal couplings of the y–z plane when we change the scale in the x direction. Taking the same for a generic dimension D, we must move all diagonal couplings in the planes

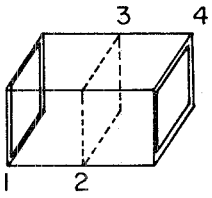

Fig. 8. The elementary cell in dimension $D = 3$. When we decimate spins 2 and 3, the diagonal coupling between them generates a new coupling of the same order between spins 1 and 4.

which do not contain the x direction. For infinitesimal λ we obtain

$$\begin{aligned} \dot{\beta} &= \frac{k \ln (k)}{1-k^2} + (D-1) F_\beta(\beta, \beta_D) , \\ \dot{\beta}_D &= (D-2)\beta_D + F_{\beta_D}(\beta, \beta_D) . \end{aligned} \tag{4.6}$$

Using eqs. (4.6) we find for the critical temperature and for ν the results shown in figs. 2, 3.

The behaviour of β_c versus D and the values of ν definitely improve the Migdal approximation, although, in our case, the results also deteriorate for higher dimensions. In the limit $D \to 1$ we find numerically:

$$\nu(D) \simeq \frac{1 - 0.15\varepsilon + O(\varepsilon^2)}{D-1} \tag{4.7}$$

while the expected result [5] is:

$$\nu(D) \sim \frac{1}{D-1} , \tag{4.8}$$

i.e., the $\varepsilon = 0$ result.

It is worth noticing, however, that our procedure to move $D-2$ diagonal bounds ($D-2 \to -1$ for $D \sim 1$) seriously violates the geometry of the lattice near $D = 1$. It is possible that with a different bound moving one could obtain the correct results in the limit $D \to 1$. At this stage of art we have not been worried by the discrepancy between (4.7) and (4.8).

5. The bond percolation problem

Percolation is a particular case of the Potts model: the N-component Potts model corresponds to the Ising model, to bond percolation and to the random resistor model for $N = 2$, 1 and 0, respectively.

In the case of percolation, one assigns to each bond a probability p of its being occupied and a probability $1-p$ of its being free. It is then possible to extract randomly a bond distribution with this probability law: two points x and y are

connected if there is a chain of occupied bonds which links them; the cluster to which x belongs is the set of all points connected to x. We call $G(x, y, p)$ the probability that points x and y are connected: $G(x, y, p)$ is the percolation version of the two-spin correlation function for the Ising model. It is known that there is a critical value of p, p_c, at which the correlation length of $G(x, y, p)$ goes to infinity like $(p-p_c)^{-\nu}$; our aim is to apply our method to compute p_c and ν. Following Kirkpatrick [7], let us define the Migdal transformation starting from $\lambda = 2$ in two dimensions. The bond shifting operation consists in assigning two bonds to the double vertical lines of fig. 4 and zero bond to the dashed one. After decimation, for the points belonging to the remaining half of the lattice, we find that the probability of being horizontally connected is p^2 while the probability of being vertically connected is $2p-p^2$. Therefore the renormalization group transformation along the x direction will be

$$p'_x = p_x^2, \qquad p'_y = 2p_y - p_y^2. \tag{5.1}$$

For generical λ we find

$$p'_x = p_x^\lambda, \qquad p'_y = 1-(1-p_y)^\lambda. \tag{5.2}$$

For infinitesimal $\lambda - 1$ and generical dimension D using eqs. (5.2), the renormalization group transformation after rescaling in all the directions will be:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p \ln(p) - (D-1)(1-p)\ln(1-p). \tag{5.3}$$

(Remember that for infinitesimal $\lambda - 1$ the isotropy of the system after decimation is restored.) We want to set up the ε expansion for this model. First of all let us define a partial bond moving ($\varepsilon \neq 0$); having non-zero internal "weak" bonds we find for the probabilities along the y axis:

$$\bar{p}_y = 1-(1-p_y)^\varepsilon \simeq -\varepsilon \ln(1-p_y),$$

weak bonds (dashed lines in fig. 1);

$$p_{\mathrm{ext}} = 1-(1-p_y)^{\lambda-(\lambda-1)\varepsilon} \simeq 1-(1-p_y)^\lambda(1-(\lambda-1)\varepsilon \ln(1-p_y)], \tag{5.4}$$

strong bonds (double lines in fig. 1).

The bond moving is realized by moving the p's: we can visualize the p's by saying that we make connections on the lattice with certain probabilities p. To move the bonds simply means that we move these connections from site to site on the lattice, according to (5.2) or (5.4). Secondly we have to find the renormalization group transformation for the p's. We have had problems with the precise definition of the new couplings and of the bond-moving procedure. After many hesitations we have decided to use the following description. Let us introduce the probabilities q: q_x is the probability that two nearby points are directly connected along the x axis independently of whatever happens to the other points. In the same way we can

define the probabilities q_y and q_D for nearby points in the y direction and diagonal directions, respectively; we can also define the probabilities q_3 and q_4 for three or four nearby points simultaneously connected, independent of whatever happens to other points. The coupling constants p $(p_x, p_y, p_D, p_3, p_4)$* are the elementary probabilities of having bonds connecting some points on the lattice (with this definition the same points may be connected by many bounds). For elementary cells we can express the q probabilities as functions of the p's. If all the p's (except p_x) are infinitesimal, we have

$$\begin{aligned}
q_x &= p_x + 2p_3(1-p_x) + p_4(1-p_x)\,,\\
q_y &= p_y(1+p_x^2) + 2p_x p_D + 2p_3(1+p_x) + p_4\,,\\
q_D &= p_D(1+p_x^2) + 2p_y p_x + 2p_3(1+p_x) + p_4\,,\\
q_3 &= p_3(1+3p_x) + p_4 + p_x(1+p_x)(p_y+p_D)\,,\\
q_4 &= p_4 + 4p_3 p_x + 2(p_y+p_D)p_x^2\,.
\end{aligned} \tag{5.5}$$

At order ε the renormalization will generate only the five couplings $p(q)$ of eqs. (5.5). Let us now consider a $\lambda \times \lambda$ cell. We can compute the q probabilities between points at the vertices of the cell. After decimation these points will be directly connected due to the effects of the other points inside the cell (points that we want to eliminate). These q's are functions of the p's and λ:

$$\begin{aligned}
q_x(\lambda) &= p_x^\lambda + 2p_3 p_x^{\lambda-1}(1-p_x) + p_4 p_x^{\lambda-1}(1-p_x)\,,\\
q_y(\lambda) &= \bar{p}_y p_x^2\left[\frac{(1-p_x^{2(\lambda-1)})}{(1-p_x^2)}\right] + 2p_D p_x\left[\frac{(1-p_x^{2\lambda})}{(1-p_x^2)}\right]\\
&\quad + 2p_3(1+p_x)\left[\frac{(1-p_x^{2\lambda})}{(1-p_x^2)}\right] + p_4\left[\frac{(1-p_x^{2\lambda})}{(1-p_x^2)}\right],\\
q_D(\lambda) &= \lambda p_D p_x^{\lambda-1}(1+p_x^2) + (\lambda-1)\bar{p}_y p_x^\lambda + 2\lambda p_3 p_x^{\lambda-1}(1+p_x) + \lambda p_4 p_x^{\lambda-1}\,,\\
q_3(\lambda) &= p_3 p_x^{\lambda-1}(1+3p_x)\left[\frac{(1-p_x^\lambda)}{(1-p_x)}\right] + p_4 p_x^{\lambda-1}\left[\frac{(1-p_x^\lambda)}{(1-p_x)}\right]\\
&\quad + p_D p_x^\lambda(1+p_x)\left[\frac{(1-p_x^\lambda)}{(1-p_x)}\right] + \bar{p}_y p_x^{(\lambda+1)}\left[\frac{(1-p_x^{(\lambda-1)})}{(1-p_x)}\right],\\
q_4(\lambda) &= \lambda p_4 p_x^{2(\lambda-1)} + 4\lambda p_3 p_x^{(2\lambda-1)} + (\lambda-1)\bar{p}_y p_x^{2\lambda} + 2\lambda p_D p_x^{2\lambda}\,.
\end{aligned} \tag{5.6}$$

[Note that for $\lambda = 1$, eqs. (5.6) coincide with eqs. (5.5) when $p_y = 0$.]

By considering a new rescaled cell made only by the points situated at the vertices, the q's of eqs.6 (5.6) can be expressed, using eqs. (5.5), as functions of renormalized

* We limit ourselves to these five p's and q's because at order ε, only the probabilities can be generated by the renormalization group transformation.

$\tilde{p}(\lambda)$, the new "effective" elementary probabilities. In the case we are considering (all the couplings except p_x infinitesimal) $\tilde{p}_y$ of eqs. (5.6) is the internal "weak" bond and p_{ext} has not been included. Including p_{ext} and the effects of the nearby cells, the final transformation is

$$\begin{aligned} p'_x &= p_x^\lambda + 2(\tilde{p}_x(\lambda) - p_x^\lambda), \\ p'_y &= p_{\text{ext}} + 2(1-p_{\text{ext}})\tilde{p}_y(\lambda), \\ p'_D &= \tilde{p}_D(\lambda), \\ p'_3 &= \tilde{p}_3(\lambda), \\ p'_4 &= \tilde{p}_4(\lambda). \end{aligned} \tag{5.7}$$

$\tilde{p}(\lambda)$'s are the solutions of the equations obtained when equating the q's of eqs. (5.5)–(5.7).

The limit $\lambda \to 1$ and the generalization to a generical dimension are straightforward. One gets the results shown in figs. 9, 10. The behaviour of the correction is qualitatively similar to that obtained for the Ising model.

6. Conclusions

We have shown in this paper that the Migdal recursion approximation can be considered as the zeroth-order approximation of a new perturbative expansion. Explicit computations for the Ising model and for the percolation model show that higher-order corrections improve the Migdal results. It is possible to extend this approach to the non-linear σ model but we have been unable to find a satisfactory scheme for computing the corrections (the only one we have found seems to us unnecessarily complex); however we have checked, at first order in ε, that the lower

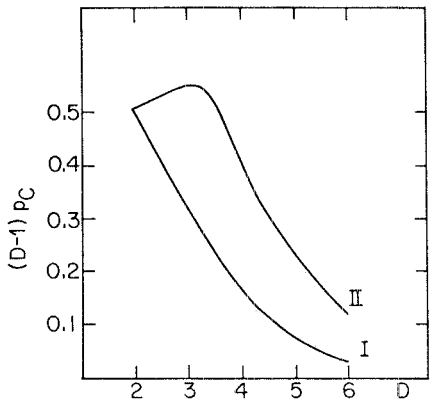

Fig. 9. $(D-1)p_c$ as a function of the dimension at zeroth order (line I) and at first order (line II) in ε for percolation.

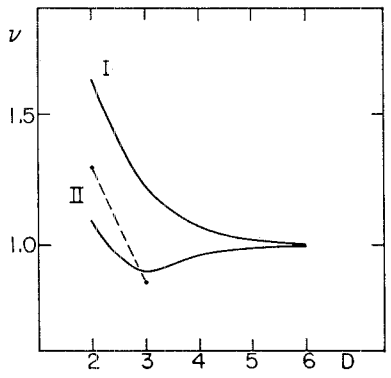

Fig. 10. ν as a function of the dimension at zeroth order (line I) and first order (line II) in ε for percolation.

critical dimension remains stable at $D = 2$, as it should do. The extension to gauge theories presents only technical problems. The whole approach appears to be promising although many questions remain unanswered, in particular the analytic properties of the functions $\beta(\varepsilon)$ and $\nu(\varepsilon)$. It would be quite interesting to use the bond-moving procedure together with other real space renormalization group techniques like the cumulant expansion or the Monte Carlo renormalization group. The idea is rather simple: before applying the renormalization group transformation we move the bonds; the amount of bond moving depends on a parameter ε and it is zero for $\varepsilon = 1$. The values of the critical exponents and of the critical temperature depend on ε, but according to eqs. (3.9) they must have zero derivative for $\varepsilon = 1$. We could use this constraint to check the quality of the results of the real space renormalization groups; one can also argue that the results for $0 < \varepsilon \neq 1$ are more reliable than those at $\varepsilon = 0$, so that one could compute the critical exponent (those for $\varepsilon = 1$) by extrapolating the results from $0 < \varepsilon \neq 1$.

It is a pleasure for us to thank A.A. Migdal for having stressed to us that the computations presented here were feasible: without this suggestion we would never have started work on this subject.

Appendix

In order to compute the renormalization transformation we start summing over all the internal spins of the cell shown in fig. 11:

$$\exp[-\tilde{H}(\mu)] = \sum_{\{\sigma,\sigma'\}} \exp\{\beta_x(\mu_1\sigma_1 + \sigma_1\sigma_2 + \cdots \sigma_{\lambda-1}\mu_2) + \varepsilon\beta_y(\sigma_1\sigma_1' + \sigma_2\sigma_2' + \cdots \sigma_{\lambda-1}\sigma_{\lambda-1}') + \beta_D(\mu_1\sigma_1' + \sigma_1\sigma_2' + \cdots \sigma_{\lambda-1}\mu_4 + \mu_3\sigma_1 + \sigma_1'\sigma_2 + \cdots \sigma_{\lambda-1}'\mu_2) + \cdots\}. \quad (A.1)$$

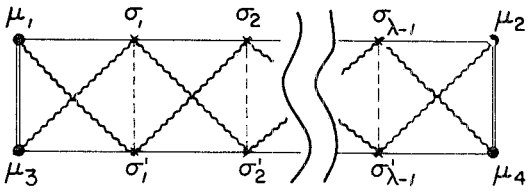

Fig. 11. Cell to be decimated for generical λ in two dimensions. All couplings are shown.

After some algebra, by taking into account only the terms up to first order in ε, we obtain

$$\exp[-\tilde{H}(\mu)] = (1+k_x^\lambda \mu_1\mu_2)(1+k_x^\lambda \mu_3\mu_4)$$
$$+\left[\varepsilon\beta_y(k_x^2-k_x^{2\lambda})+\frac{2\beta_D}{k_x}(k_x^2-k_x^{2(\lambda+1)})\right]\frac{(\mu_1\mu_3+\mu_2\mu_4)}{1-k_x^2}$$
$$+\left[(\lambda-1)\varepsilon\beta_y+\frac{\lambda\beta_D}{k_x}(1+k_x^2)\right]k_x^\lambda(\mu_1\mu_4+\mu_2\mu_3)\,, \tag{A.2}$$

where $k_x = \mathrm{tgh}\,(\beta_x)$.

If we have

$$\exp[-\tilde{H}(\mu)] = \exp\{\beta_x(\mu_1\mu_2+\mu_3\mu_4)+\Delta\beta_y(\mu_1\mu_3+\mu_2\mu_4)+\beta_D(\mu_1\mu_4+\mu_2\mu_3)\}\,, \tag{A.3}$$

with $\Delta\beta_y$ and $\beta_D \sim O(\varepsilon)$, this can be written as

$$\exp[-\tilde{H}(\mu)] = (1+k_x\mu_1\mu_2)(1+k_x\mu_3\mu_4)$$
$$+(\mu_1\mu_3+\mu_2\mu_4)[\Delta\beta_y(1+k_x^2)+2\beta_D k_x]+(\mu_1\mu_4+\mu_2\mu_3)$$
$$\times[2\Delta\beta_y k_x+\beta_D(1+k_x^2)]\,. \tag{A.4}$$

Equating (A.2) and (A.4) we find eqs. (4.2). By taking into account the non-infinitesimal bounds between $\mu_1-\mu_3$ and $\mu_2-\mu_4$ and the effects of nearby cells, we obtain the whole renormalization transformation of eqs. (4.1), (4.2).

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