ISTITUTO NAZIONALE DI FISICA NUCLEARE
Laboratori Nazionali di Frascati

LNF-80/66

G. Parisi:

FIELD-THEORETIC APPROACH TO SECOND-ORDER PHASE TRANSITIONS IN TWO- AND THREE-DIMENSIONAL SYSTEMS

Estratto da :

J. of Statistical Phys. 23, 49 (1980)

Servizio Documentazione
dei Laboratori Nazionali di Frascati
Cas. Postale 13 - Frascati (Roma)

Journal of Statistical Physics, Vol. 23, No. 1, 1980

Field-Theoretic Approach to Second-Order Phase Transitions in Two- and Three-Dimensional Systems

Giorgio Parisi[1]

Received November 15, 1979

We review the physical principles which are at the basis of recent field-theoretic computations of the critical exponents in two- and three-dimensional systems. We concentrate on those points that do not show up explicitly in the more standard ϵ-expansion: they must be discussed with care if one uses a perturbative approach at fixed space dimensions (the loop expansion). We present in detail simple computations of the critical exponents, while we summarize the results of longer and more accurate computations.

KEY WORDS: Field theory; second-order transitions; critical exponents; ϵ-expansion; loop expansion; renormalization group; Callan–Symanzik equation.

1. INTRODUCTION

There have been many recent approaches toward a deeper understanding of second-order phase transitions. The phenomenological Kadanoff–Widom scaling laws[1,2] are in quite good agreement with the experimental data and the high-temperature expansions (for a review see Ref. 3). Also, the universality hypothesis[1] (independence of the critical exponents from the detailed structure of the interaction) seems to be satisfied. The intensive use of the renormalization group[4–7] has produced a neat derivation of the scaling laws for static and dynamic phenomena and has clarified the deep reasons for the validity of the universality hypothesis. Very simple approximate computations have been done for many systems, ranging from spin-glasses[8] to Reggeon field theory.[9]

In the framework of a field-theoretic approach high-precision estimates of the critical exponents of the three-dimensional Ising model have been done.[10,11] These estimates involve an explicit evaluation of all diagrams up

[1] Laboratori Nazionali INFN, Frascati, Italy.

to six loops and approximate predictions for the size of the neglected higher loop diagrams.[12]

The aim of this paper is to describe in detail the foundations of the method used in these high-precision estimates; indeed, this paper is a shortened, revised version of unpublished 1973 Cargese Lecture Notes,[13] which were the basis of the computations of Refs. 10 and 11.

To present a microscopic derivation of the scaling laws is rather difficult. Many subtle questions must be settled: thermodynamic quantities must be computed just at the point where their dependence on the temperature is not analytic; any sort of high- or low-temperature expansion is divergent. The theory of second-order phase transitions is dominated by the quest for functions which are regular near the transition.

In this paper we concentrate on the derivation of the static scaling laws for a magnetic system above the transition at zero external field. Our treatment has many points in common with the standard approach[14–18]; the main difference is that we always work in the massive (finite correlation length) theory also at the critical temperature. This approach bypasses the problems connected with the infrared divergences present in a perturbative approach; the theory admits a closed formulation in a system of arbitrary dimensions.

This paper is divided into nine sections. In Section 2 we present the model which we will consider as the prototype of a system undergoing a second-order phase transition and which we will study in the other sections. In Section 3 we explain why a straightforward approach does not work near the critical temperature, and how infrared divergences arise. In analogy with quantum electrodynamics, we conjecture that the introduction of renormalized quantities will avoid the problems. All the results derived in the rest of the paper are based on this conjecture: a rigorous proof is lacking, although there are results, based on the Lebowitz inequality, which go in this direction.[19] In Section 4 we derive the Kadanoff–Widom scaling laws and prove that the critical exponents are connected to renormalized correlation functions computed at some finite value of the external momenta. In the next section we study the behavior of the correlation functions at the critical temperature. An exact expression for these correlation functions has been found. If it is expanded in powers of the bare coupling constant, infrared divergences appear; however, if a different expansion is used, finite results are obtained also at the critical temperature. Infrared divergences show up in a non-analytical dependence of the correlations functions on the bare coupling constant. In Section 6 we present simple applications of the formalism; critical exponents are computed using very simple approximations; the results of much more lengthy and accurate computations are reported. In the next section we show that the range of allowed values for the renormalized coupling constant is a closed interval. In Section 8 we show how the concept of a universal critical behavior arises naturally in this framework; we derive a universal equation for the correlation functions at the critical temperature. In the last section we present our final conclusions and comments.

In the appendix we explain some notations used in this paper.

2. THE MODEL

In this paper we concentrate on a special kind of continuous Ising model in the limit of zero lattice spacing. In presence of an external magnetic field H the partition function is

$$Z(H, \beta) \propto \int d[\phi] \exp\left[- \int d^D x \, \mathscr{L}(x) \right] \tag{2.1}$$

where

$$\begin{aligned} \mathscr{L}(x) &= \tfrac{1}{2}\partial_u \phi \, \partial_u \phi + \tfrac{1}{2} M^2(\beta)\phi^2 + (u/4!)\phi^4 \\ M^2(\beta) &= M^2(0) + \beta; \qquad \beta = 1/KT \end{aligned} \tag{2.2}$$

D is the dimension of the space and $\int d[\phi]$ stands for functional integration. In the two- and three-dimensional cases, there are rigorous proofs of the existence of the model and a precise mathematical meaning can be given to the words "functional integration." The only ultraviolet divergences present can be absorbed by assuming that $M^2(0)$ is a polynomial in g with infinite coefficient; i.e., in the language of field theory, the interaction is super-renormalizable for $D < 4$.[19]

In the limit $u = 0$ we obtain the Gaussian model and the perturbative expansion in powers of u can be easily obtained. It is identical to the Feynman graph expansion for a relativistic ϕ^4 theory in $D - 1$ space, one time dimension. It has been rigorously proved that the correlation functions of the two theories are connected through a Wick rotation.[20,21] The correlation functions of statistical mechanics are the Schwinger functions[22] (Wightman functions[23] at imaginary time in relativistic theory). In Fourier space they are the analytic continuation of the time-ordered functions in the Euclidean region.

This model can be generalized by introducing a multiplet of N fields which form a representation of the $O(N)$ group. The partition function is[6,24]

$$Z(H_i, \beta) \propto \int \prod_{i=1}^{N} d[\phi_i] \exp\left[- \int d^D \mathscr{L}(x) \right] \tag{2.3}$$

where

$$\mathscr{L}(x) = \tfrac{1}{2}\partial_u \phi_i \, \partial_u \phi_i + \tfrac{1}{2} M^2(\beta)\phi_i\phi_i + (1/4!)(\phi_i\phi_i)^2 \tag{2.4}$$

If we take $N = 1$, this model reduces to the first one. For $N = 2$ we recover both the Ginzburg–Landau[25] partition function of a superconductor and the Landau[26,27] partition function of superfluid helium near the λ transition. A special kind of continuous Heisenberg model is obtained for $N = 3$. In the limit N goes to infinity we get the spherical model.[28]

At each order in perturbation theory the N dependence of the correlation functions comes from multiplicity factors which multiply each Feynman diagram. These factors are polynomial in N and they can be analytically continued to noninteger N. It has been suggested that the $N = 0$ correlation functions are connected to some properties of the self-avoiding walk problem.[29] It may be of interest to note that for $N = -2$ the two-point correlation function above the transition coincides with that of the Gaussian model.[30]

The partition function of a "realistic" spin-1/2 model can be represented in a similar way: even higher powers of ϕ are present and the Lagrangian is no longer a polynomial.[31] According to the conventional wisdom, higher order couplings are irrelevant as far as the critical behavior is concerned; the model (2.2) and the Ising model on a real lattice belong to the same universality class. The arguments which lead to this belief are briefly discussed in Section 8.

The following simple remark will be quite useful in the rest of the paper: the argument of the exponential is a pure number and all the quantities have the dimension of a length to some power. Defining the length dimension to be -1, we find

$$\begin{gathered}
[x] = -1, \quad [d/dx] = 1, \quad [\phi] = (D-2)/2, \quad [M] = 1, \\
[u] = 4 - D, \quad [\phi^2] = D - 2, \quad [G_2] = -2, \quad [\Gamma_2] = 2 \\
[G_N] = D - N(\tfrac{1}{2}D + 1), \quad [\Gamma_N] = D - N[-1 + \tfrac{1}{2}D], \\
[G_{N\phi^2}] = [G_N] - 2, \quad [\Gamma_{N\phi^2}] = [\Gamma_N] - 2, \quad [D_{\phi^2\phi^2}] = D - 4
\end{gathered} \tag{2.5}$$

where the square brackets stand for "dimension of," and G_N and Γ_N are respectively the Fourier transforms of the connected N-point correlation function and of the amputated, one-particle, irreducible N-point correlation function. The notation is defined in the appendix.

Dimensional analysis may be used extensively; e.g.,

$$G_2(P, u, M) = \frac{1}{M^2} A\left[\frac{u}{M^{4-D}}, \frac{P}{M}\right] = \frac{1}{P^2} A^1\left[\frac{u}{P^{4-D}}, \frac{P}{M}\right] = \cdots \tag{2.6}$$

3. THE ASSUMPTIONS

The simplest approximation we can make is to take $u = 0$, or neglect terms proportional to high powers of the field. Naive arguments suggest that this approximation may be valid only for dimensions greater than four. However, in lower dimensions also it gives a first semiqualitative description of a second-order phase transition.[1]

In this situation, the model can be easily solved.[32] The two-point correlation function is

$$G_2(K, T) = 1/(K^2 + M^2) \tag{3.1}$$

All other connected correlation functions are zero. The transition temperature is at $M^2 = 0$. The transition is characterized by the fact that the correlation functions become singular at zero external momenta. These singularities are produced from the nonexponential decrease of correlation functions in the configuration space at large distances.

Introducing the reduced temperature $\tau \propto (T_c - T)/T_c$, we can write $G_2(K, \tau)$ for small τ as

$$G_2(K, \tau) = \tau^{-\gamma} f(K/\tau^{\nu}) \tag{3.2}$$

where

$$\gamma = 1, \qquad \nu = \tfrac{1}{2}, \qquad f(x) = 1/(1 + x^2) \tag{3.3}$$

The scaling laws[1] state that a formula similar to (3.2) is valid near the transition for not too large K and also for u different from zero. The critical exponents γ and ν and the function h may, however, be different from (3.3). It is also assumed that (3.2) has a finite, nonzero limit when $\tau \to 0$ at K different from zero:

$$G(K, 0) = 1/K^{2-\eta}, \qquad 2 - \eta = \gamma/\nu \tag{3.4}$$

The main problem of the theory of second-order phase transitions is to prove this hypothesis and to compute ν, γ, and h.

If u is different from zero, we can develop the partition function and the correlation function in powers of u, expanding the exponential in (2.10) (see the appendix)

$$Z \propto \sum_{k=0}^{\infty} \frac{u^k}{k!} \int d[\phi] \int \prod_{i=1}^{k} d^D y_i \, \phi^4(y_i) \times \exp \int d^D x \left[-\tfrac{1}{2}\partial_\mu \phi(x)\, \partial^\mu \phi(x) - M^2 \phi^2(x) \right] \tag{3.5}$$

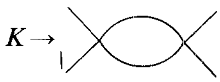
Unfortunately, this expansion is not convergent and it may be regarded only as an asymptotic expansion. It has been proved[33] that (3.5) is not convergent, but its Borel sum exists and is equal to the functional integral (2.10). These problems are connected with the nonexistence of the theory for negative u.

Near the phase transition new pathologies arise. The first effect of introducing a nonzero coupling constant is a shift in the transition temperature. This difficulty may be bypassed by making a perturbation in u not at fixed M^2, but at fixed $M^2 - M_c^2 = \bar{M}^2$. Then $M_c^2(u)$ is the point at which the phase transition occurs. Near the transition, the most singular part of the dependence of the correlation function on the temperature comes from $\bar{M}^2$ and not from u. We can consistently assume that $\bar{M}^2$ is proportional to τ and neglect the temperature dependence of u. The error involved in this approximation affects only terms that are not singular near the critical temperature.

Although the introduction of $\bar{M}$ improves the situation, it remains hopeless. The dimensionless coupling constant in which we are making our expansion is $u/\bar{M}^{4-D}$; it goes to infinity when $\bar{M}$ goes to zero in any dimension less than four. The perturbative expansion is useless if $\bar{M}$ is small, i.e., near the transition. It is well known that it is very hard to reconstruct the behavior of a function $f(x)$ when x goes to infinity from the knowledge of the first few terms of its Taylor expansion around $x = 0$.

The situation is still worse if we start directly from $M = 0$ and limit ourselves to the study of correlation functions at nonzero external momenta. In this case the perturbative expansion does not exist, because of infrared divergences.

Let us study a simple example: the diagram



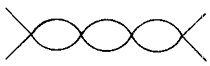
(3.6)

in the massless case is proportional to K^{-4+D} if $2 < D < 4$. Its contribution to Γ_4 is proportional to the integral

$$\int d^D p \frac{1}{p^2(p+K)^2} \tag{3.7}$$

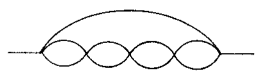
This integral is infrared-convergent at K different from zero if $D > 2$ and ultraviolet-convergent if $D < 4$; in this interval of dimension, power counting implies a simple power behavior in K.

The chain of N bubbles



(3.8)

factorizes into a product of N integrals and is proportional to $K^{(D-4)N}$. If $(D - 4)N < -D$, the Fourier transform with respect to K cannot be performed, and the vertex is no longer an L^1 function. Any diagram having this chain as subgraph is divergent, e.g.,

 (3.9)

We will see later that these divergences are connected to the fact that correlation functions are not C^{∞} functions of u around $u = 0$ in the zero-mass case. This effect is a subtle one; it is present only at an order $N > D/(4 - D)$. If we work in a dimension near enough to four, it appears only at a very high order in the coupling constant and it disappears completely in dimensions infinitesimally near to four. We stress that any computation of the critical exponents done in $4 - \epsilon$ dimensions[5] using the renormalization group in the zero-mass theory does not contain any information on the critical behavior of a system in 3.99 dimensions without an additional hypothesis on the resummation of these infrared singularities.

The conclusion is that the limit $\bar{M} \to 0$ at fixed u is strictly connected with the limit $u \to \infty$ at fixed $\bar{M}$ and cannot be studied by treating u as a small perturbation.

Similar problems are also present in renormalizable theories such as quantum electrodynamics (QED) in four dimensions: at each finite order in perturbation theory, the presence of the ultraviolet divergence destroys the possibility of performing an expansion in powers of the bare coupling constant.[34] After more than fifteen years of theoretical work, this difficulty has been overcome by introducing renormalized parameters such as the physical charge and mass of the electron and by performing an expansion of correlation functions in powers of the renormalized parameters.[35,36] The perturbative expansion for correlation functions at low external momenta is perfectly well defined and regular, although the bare parameters are infinite at each order in perturbation theory. The drawback is that in the large-momentum region the effective coupling constant is of order $\alpha \lg K^2$, and clearly blows up when the momenta go to infinity.[36] Renormalized perturbation expansion is useless in this region because the large-momentum region is controlled by the bare coupling constant and not by the renormalized one. The same phenomenon happens also in our case if $D = 4$. If $D < 4$, problems in the critical region arise from the fact that the dimensionless bare coupling constant $u/\bar{M}^{(4-D)/2}$ goes to infinity, as in QED. We hope that the introduction of the renormalized integral equation will be of some help. The perturbative solution of these equations produces the renormalized perturbation expansion (RPE). The analogy with QED suggests that we can trust RPE only for the results concerning correlation functions computed at low external momenta, but we should not believe in any direct computation of the bare quantities or of the behavior of correlation functions in the large-momentum region, although all these quantities have a well-defined perturbative expansion in powers of the renormalized coupling constant.

We are led to the following conjecture: All renormalized correlation functions (to be defined later) at fixed external momenta have a finite limit when the bare coupling constant goes to infinity at fixed m. The whole theory of second-order phase transitions is explicitly or implicitly founded on this hypothesis, whose intuitive justification comes from the idea that a too strong repulsive interaction shields itself, producing a finite result. An explicit realization of this phenomenon may be found in the large-N limit[37–39] at all orders in $1/N$ or in the $D = 1$ case (anharmonic quantum oscillator). We note that the bare coupling constant is the limit of the four-point functions when the external momenta become very large. Our conjecture states that we can reach a situation where the four-point function goes to infinity in the large-momentum region, but no infinity is present in the correlation functions in the finite-momentum region. In a very rough sense, the low- and the high-momentum correlation functions are decoupled in the integral equations. The large-momentum behavior of correlation functions comes from the internal region of integration where all the momenta are large, and the main contribution to the correlation functions in the low-momentum region comes from the region of integration where all the momenta are low.[40] We suppose that we can find solutions to these equations whose high-energy behavior is singular with a perfectly regular low-energy behavior. This may be possible if the large-momentum behavior of the four-point function is such that no new ultraviolet divergences are created when the bare coupling constant goes to infinity. If this condition is not violated, the low-momentum behavior is quite insensitive to the high-momentum behavior. The construction of RPE is straightforward and it is discussed in detail in many books (see, e.g., Refs. 36). We present here only a sketch of the fundamental steps. We introduce renormalized ϕ and ϕ^2 fields which are proportional to the bare ones:

$$\phi_R = Z_1^{-1/2}\phi, \qquad \phi_R{}^2 = Z_2^{-1}\phi^2 \tag{3.10}$$

The constants of proportionality are fixed from the conditions

$$\frac{d}{dK^2}[\Gamma_2{}^R(K^2)]|_{K^2=0} = 1, \qquad \Gamma^R_{2,\phi^2}(0, 0) = 1 \tag{3.11}$$

The renormalized mass and coupling constant are defined as

$$m^2 = \Gamma_2{}^R(0) = [G_2{}^R(0)]^{-1}, \qquad g = \Gamma_4{}^R(0, 0, 0)m^{D-4} \tag{3.12}$$

The Z_1, Z_2, and g are clearly dimensionless. Note that in general $\phi_R{}^2$ is only proportional but not equal to $(\phi_R)^2$. It is easy to check that

$$\begin{aligned} m^2 &= M^2[1 + O(u/M^{4-D})] \\ Z_1 &= 1 + O[u^2/M^{2(4-D)}] \\ Z_2 &= 1 + O(u/M^{4-D}) \\ g &= u/m^{4-D} + O[u^2/M^{2(4-D)}] \end{aligned} \tag{3.13}$$

The physical meaning of these parameters follows from the relation

$$G_2(K) = Z_1/[m^2 + K^2 + O(K^4)] \tag{3.14}$$

m^{-1} is the correlation length ξ; $\bar{m}^{-2} = \bar{m}^{-2} \cdot Z_1$ is the magnetic susceptibility χ; Z_2 is $Z_1(d/d\tau)\bar{m}^2 = Z_1(d/dM^2)\bar{m}^2$; and $D_{\phi^2,\phi^2}(0)$ is the specific heat, i.e., the second derivative with respect to the temperature of the logarithm of the partition function. All the correlation functions have an expansion in powers of g at fixed m, where no divergences appear in four dimensions also. However, in this framework we cannot compute the bare coupling constant without studying the large-momentum behavior of the theory, going outside of the range of validity of RPE. The problem that we have to solve is to compute the bare coupling constant using as input only correlation functions in the low-momentum region, or to compute the large-momentum behavior from the low-momentum behavior. This problem seems very hard because we know that the two momentum regions are decoupled in the integral equations. The answer to such problems is contained in the following sections and is the main result of this work.

We recall that the following has been rigorously proved, using the Lebowitz inequality[(19)]:

$$0 \leqslant g \leqslant A \tag{3.15}$$

where A is a computable constant of order 1. If $g(u)$ is a monotonically increasing function (or has only a finite number of oscillations), as is reasonable, then $\lim_{u\to\infty} g(u)$ exists and is finite. Although a crucial inequality is missing to extend this argument to all Green's functions, the rigorous result (3.15) strongly supports the hypothesis of finiteness of the renormalized Green's functions in the infinite coupling limit.

The reader may observe that the whole procedure seems terribly complicated. Why does one not use the integral equation directly in the large-momentum region, which decouples from the low momenta, and solve the equations in the large-momentum region? The answer is that this alternative approach has been tried in the past.[(40–44)] Solving integral equations is not an easy job and it is hard to make good approximations.

An explicit computation of the state equations seems to be nearly impossible. As far as the critical exponents are concerned, safe results are obtained in the $1/N$ expansion[(44)] and they coincide with those obtained in the conventional approach. Rather good results for the critical exponents have been obtained also in the case $N = 1$ by truncating in an appropriate way the nonlinear integral equations for the massless theory and by making full use of the constraints dictated by conformal invariance.[(45)]

4. THE SCALING LAWS

The problem we will study in this section is how to compute the bare coupling constant and the renormalization constants as functions of the renormalized coupling constant.

It is possible to prove by direct inspection of the diagrams in perturbation theory that

$$
\begin{aligned}
Z_1^{-1} &= \lim_{\lambda\to\infty} \lambda^2 K^2 G_2{}^R(\lambda K), \qquad |K| \neq 0 \\
(Z_2 Z_1)^{-1} &= \lim_{\lambda\to\infty} \lambda^2 K^2 G^R_{2\phi^2}(\lambda K, \lambda p), |K| \neq 0, \quad |p| \neq 0, \quad |K+p| \neq 0 \\
Z_1{}^2 u &= \lim_{\lambda\to\infty} \Gamma_4{}^R(\lambda p_1, \lambda p_2, \lambda p_3), \qquad |p_1| \neq 0, \quad |p_2| \neq 0, \quad |p_3| \neq 0 \\
&\qquad |p_1 + p_2| \neq 0, \quad |p_1 + p_3| \neq 0 \\
&\qquad |p_3 + p_2| \neq 0, \quad |p_1 + p_2 + p_3| \neq 0
\end{aligned} \tag{4.1}
$$

The interaction does not change the large-momentum behavior of the correlation functions.

However, (4.1) is of no practical interest insofar as it involves correlation functions computed in the very large-momentum region, where we cannot trust RPE. Our goal is to find formulas which are equivalent to (4.1), but involve only correlation functions computed at fixed external momenta.

We introduce the differential operator $\bar{\Delta}$ defined by[46,47]

$$
\bar{\Delta} = \bar{m} \left.\frac{\partial}{\partial \bar{m}^2}\right|_u = \left.\frac{\partial}{\partial \lg \bar{m}^2}\right|_u \tag{4.2}
$$

Using definitions (3.10) and (3.11), we prove the following chain of identities:

$$
\begin{aligned}
\bar{m}^2 \left.\frac{\partial G_N}{\partial \bar{m}^2}\right|_u &= \bar{m}^2 \left.\frac{\partial M^2}{\partial \bar{m}^2}\right|_u \left.\frac{\partial G_N}{\partial M^2}\right|_u = \bar{m}^2 \left.\frac{\partial M^2}{\partial \bar{m}^2}\right|_u G_{N\phi^2} \\
&= \bar{m}^2 \left.\frac{\partial M^2}{\partial \bar{m}^2}\right|_u Z_2 G_{N\phi_R{}^2} = \bar{m}^2 G_{N\phi_R{}^2}
\end{aligned} \tag{4.3}
$$

The action of the $\bar{\Delta}$ operator on correlation functions can be computed in RPE; finite results are obtained also if the theory is renormalizable, like QED, in four dimensions. Instead of $\bar{\Delta}$ we will use the operator Δ, which is proportional to $\bar{\Delta}$; the proportionality constant is fixed from the condition $\Delta m^2 = m^2$. We introduce Δ only to be closer to the standard notation.[46] The following functions can be defined.

$$Z_1^{-2}\Delta\Gamma_4(0, 0, 0) = m^{4-D}h(g)\Delta \frac{\partial}{\partial K^2}\Gamma_2{}^R(K^2) = -c_1(g);$$
$$\Delta\Gamma^R_{2,\phi_R{}^2}(0, 0) = c_2(g) + c_1(g) \tag{4.4}$$

Using definitions (3.10)–(3.12), we find

$$\Delta g = -\tfrac{1}{2}(4 - D)g + h(g) - 2gc_1(g) \equiv b(g) = -\tfrac{1}{2}(4 - D)g + O(g^2)$$
$$\Delta Z_1 = c_1(g)Z_1, \qquad \Delta Z_2 = c_2(g)Z_2 \tag{4.5}$$

We introduce the notation

$$u = m^{4-D}N(g) \tag{4.6}$$

We apply the Δ operator to both sides of (4.6):

$$\Delta u = 0 = m^{4-D}[\tfrac{1}{2}(4 - D) + b(g)\,\delta/\delta g]N[g] \tag{4.7}$$

The solution of the differential equation (4.2) is uniquely fixed by the condition (3.13)

$$N(g) = g \exp \int_0^g \left(\frac{D-4}{2b(g')} - \frac{1}{g'}\right) dg' \tag{4.8}$$

$b(g)$ is negative for small g, $N(g)$ is monotonically increasing in the region between 0 and the first zero of $b(g)$. In this region we can define an inverse function ρ such that

$$g = \rho[u/m^{4-D}] \tag{4.9}$$

The same technique can be used to obtain

$$Z_1(g) = \exp \int_0^g \frac{c_1(g')}{b(g')}\, dg', \qquad Z_2(g) = \exp \int_0^g \frac{c_2(g')}{b(g')}\, dg' \tag{4.10}$$

Let us now try to use these formulas to study the critical behavior of the partition function.

In the limit $u/m^{4-D} \to \infty$ the integral defining $N(g)$ must diverge, and this is possible only if the $b(g)$ function has a zero at a positive point g_c. If we suppose that the function has a simple zero with slope b', then

$$b' = \frac{d}{dg}b(g)\bigg|_{g=g_c} > 0 \tag{4.11}$$

Equations (4.8)–(4.10) simplify in the limit $u \to \infty$

$$\frac{u}{m^{4-D}} = H\left(\frac{g_c - g}{g_c}\right)^{-(4-D)/2b'}, \qquad g = g_c\left[1 - \left(\frac{u}{Hm^{4-D}}\right)^{-2b'/(4-D)}\right] \tag{4.12}$$

where

$$H = g_c \exp \int_0^{g_c} \left[-\frac{4-D}{2b(g')} - \frac{1}{g} - \frac{4-D}{2b'(g_c - g)} \right] dg' \tag{4.13}$$

If $b' = 0$, but $b'' \neq 0$,

$$g = g_c - \frac{1}{\lg(u/m^{4-D})} + \cdots \tag{4.14}$$

In any case g goes to g_c when $u^{1/4-D}/m$ goes to infinity. Note that the bare coupling constant has a negative power in (4.12).

Putting (4.12) into (4.10), we obtain

$$Z_1 = m^{2c_1^c}\left\{1 + O\left[\left(\frac{m}{u^{1/4-D}}\right)^{2b'}\right]\right\}, \qquad Z_2 = m^{2c_2^c}\left\{1 + O\left[\left(\frac{m}{u^{1/4-D}}\right)^{2b'}\right]\right\} \tag{4.15}$$

where $c_1{}^c = c_1(g_c)$ and $c_2{}^c = c_2(g_c)$. Using the definitions of Z_1 and Z_2, we find

$$dm^2/dM^2 = dm^2/dT = m^{2(1/2\nu - 1)} \tag{4.16}$$

Relation (4.16) implies that, if $\nu > 0$,

$$m = \tau^\nu, \qquad Z_1 = m^\eta \tag{4.17}$$

where

$$\eta = 2c_1{}^c, \qquad \nu = 1/(2 - 2c_2{}^c) \tag{4.18}$$

The relations (2.13) and (3.14) imply the scaling law for the two-point correlation function:

$$G_2(K, \tau) = (1/\tau^\gamma)\{f_0[K/\tau^\nu] + \tau^{\nu\omega} f_1[K/\tau^\nu] + \cdots\} \tag{4.19}$$

where

$$\gamma = \nu[2 - \eta], \qquad \omega = 2b' \tag{4.20}$$

The limit $\tau \to 0$ is done at K/τ^ν constant; the study of the behavior of correlation functions in the limit $\tau \Rightarrow 0$ at fixed K requires a different and more complicated analysis, which will be the object of the next section.

The corrections to scaling laws have a simple power behavior, which is connected to b'. Similar scaling laws can be derived for high-order correlation functions. The same technique can be used to study the behavior of the specific heat $C = D_{\phi^2\phi^2}(0)$. We introduce the function

$$\Delta[D_{\phi_R{}^2\phi_R{}^2}(0)] = (1/m^{4-D})l(g) \tag{4.21}$$

Using the definition of the renormalized field, we find

$$\frac{\partial}{\partial \lg m^2} D_{\phi^2\phi^2}(0) = \Delta D_{\phi^2\phi^2}(0) = (Z_2)^2 \Delta D_{\phi_R{}^2\phi_R{}^2}(0) + 2c_2(g) D_{\phi^2\phi^2}(0) \tag{4.22}$$

The solution of this equation is

$$C(g) \equiv D_{\phi^2\phi^2}(0) = \frac{1}{u}\int_0^g \frac{dg'}{b(g')} N(g')[Z_2(g')]^2 l(g') \tag{4.23}$$

Near the transition it simplifies to

$$C \simeq C_1 m^{\alpha/\nu}/\alpha + C_0 = \tilde{C}_1 \tau^\alpha/\alpha + C_0 \tag{4.24}$$

where

$$\alpha = 2 - D\nu \tag{4.25}$$

and C_0 and C_1 are computable constants. For $\alpha = 0$ the singularity of the specific heat is logarithmic, and for α negative we find a constant term plus an irregular term. The presence of the regular term C_0 allows a positive specific heat also for negative α.

We stress that all our results rely on the assumption that correlation functions have a limit when the bare coupling constant goes to infinity at fixed mass, i.e., they have a limit when g goes to g_c. If we relinquish this hypothesis, no conclusion can be drawn, e.g., if $c_2(g) = \sin[1/(g_c - g)]$ we find an oscillatory behavior. If $l(g)$ is singular at $g = g_c$, we obtain a violation of the scaling law for the specific heat.

5. THE CORRELATION FUNCTION AT THE CRITICAL TEMPERATURE

In this section we study the behavior of correlation functions at the critical temperature, i.e., in the zero-mass theory. These correlation functions are connected by an infinite scale transformation with the correlation function of the massive theory at the infinite bare coupling constant. Our aim is to express the correlation functions in both situations as an integral over correlation functions of the massive theory. Such an integral should be dominated by the region of integration where the external momenta are comparable with the mass.

For the sake of simplicity we study only the correlation function of the ϕ field. A similar analysis can be performed on the correlation function of ϕ^2. From (3.10) and (4.5) we derive the identity[46–48,14]

$$\begin{aligned}\Delta G_N^R(P|m^2, g) &= \left[\Delta m^2 \frac{\delta}{\delta m^2} + \Delta g \frac{\delta}{\delta g} + \Delta Z \frac{\delta}{\delta Z}\right] G_N^R(P|m^2, g) \\ &= \left[m^2 \frac{\delta}{\delta m^2} + b(g)\frac{\delta}{\delta g} + \frac{N}{2} c_1(g)\right] G_N^R(P|m^2, g)\end{aligned} \tag{5.1}$$

Using dimensional analysis, it can be written as

$$\Delta G_N{}^R(\lambda P|m^2, g) = \left[-\frac{1}{2}\lambda\frac{\partial}{\partial\lambda} + b(g)\frac{\delta}{\delta g} + \frac{1}{2}d_N + \frac{N}{2}c_1(g)\right] G_N(\lambda P|m^2, g)$$

$$d_N = [G_N] \tag{5.2}$$

A similar equation can be derived for the Γ_N functions

$$\Delta\Gamma_N{}^R(\lambda P|m^2, g) = \left[-\frac{1}{2}\lambda\frac{\partial}{\partial\lambda} + b(g)\frac{\delta}{\delta g} + \frac{1}{2}\delta_N - \frac{N}{2}c_1(g)\right] \Gamma_N{}^R(\lambda P|m^2, g)$$

$$\delta_N = [\Gamma_N] \tag{5.3}$$

Equations (5.3) are identities which are valid at each order in perturbation theory; however, we can also consider them as partial differential equations for the function G_N (Γ_N), assuming ΔG_N ($\Delta\Gamma_N$) is known. This linear, inhomogeneous differential equation has an infinite number of solutions. However, as shown in Ref. 49, there is only one solution which satisfies the physical requirement of being regular in λ and g around the lines $\lambda = 0, g = 0$. This solution is

$$G_N{}^R(\lambda P|m^2, g)$$

$$= \int_0^g \frac{dg'}{b(g')}\left[\frac{R(g)}{R(g')}\right]^{-d_N}\left[\frac{Z_1(g)}{Z_1(g')}\right]^{-N/2} \Delta G_N{}^R\left(\lambda\frac{R(g)}{R(g')}P|m^2, g'\right)$$

$$= -2\lambda^{d_N}\int_0^\lambda \frac{dx}{x}x^{-d_N}\left[\frac{Z_1(g)}{Z_1[\bar{g}(g, \lambda/x)]}\right]^{-N/2} \Delta G_N{}^R\left(xP|m^2, \bar{g}\left(g, \frac{\lambda}{x}\right)\right) \tag{5.4}$$

where we have introduced the new functions

$$R(g) = [N(g)]^{-1/(4-D)}, \qquad R^{-1}(x) = \rho[x^{-(4-D)}],$$
$$\bar{g}(g, x) = R^{-1}[xR(g)] \tag{5.5}$$

The functions N, Z, and ρ are defined in Eqs. (4.8)–(4.10). These functions have the following limits:

$$g \to 0: \quad R(g) \to g^{-1/(4-D)}, \qquad g \to g_c: \quad R(g) \to A(g_c - g)^{1/2b'}$$
$$A = g_c^{-1/2b'}H^{-1/(4-D)}$$
$$x \to \infty: \quad R^{-1}(x) \to x^{-(4-D)}, \qquad x \to 0: \quad R^{-1}(x) \to g_c - (x/A)^{2b'}$$
$$\bar{g}(g, 1) = g$$
$$x \to \infty: \quad \bar{g}(g, x) \to N(g)x^{-(4-D)} \tag{5.6}$$

Note that the function ΔG can be computed from the same loop integral as G; the only difference is that one propagator $1/(K^2 + m^2)$ is changed to

$-1/(K^2 + m^2)^2$. This substitution suppresses the high-momentum region in the loop integration. At each finite order in perturbation theory, we find for generical momenta P,[48-50]

$$\Delta G_N{}^R(\lambda P)/G_N{}^R(\lambda P) \to 0 \tag{5.7}$$

However, perturbation theory is useless in the large-momentum region for very large coupling constant. Nevertheless, we conjecture that (5.7) is true. The whole analysis of the behavior of correlation functions at the critical temperature depends heavily on this assumption. It is interesting to note that (5.7) is satisfied at all orders in perturbation theory both for renormalizable and superrenormalizable theories. Nonperturbative arguments may only suggest the consistency of (5.7). For example, one can try to compute $G_{N\phi^2}(P, 0)$ from $\Delta G_{N\phi^2}(P, 0)$. If one assumes that $\Delta G_{N\phi^2}/G_{N\phi^2} \to 0$, one finds that $\Delta G/G \to 0$. The analysis is not so simple, because of the exceptional momentum configuration. However, the problem can be studied in great detail.[50]

The main consequence of (5.7) is that the relevant region of integration in x remains bounded when λ goes to infinity also. Three different limits can be studied: (I) λ goes to infinity at fixed $g \neq g_c$; (II) λ goes to infinity at $g = g_c$; (III) $\bar{M}^2$ goes to zero at fixed P and u, or equivalently $\lambda \to \infty$ and $g \to g_c$ together. Let us study, in order, these three limits.

When $\lambda \to \infty$, $x/\lambda \to 0$. In this limit we can freely substitute in $\bar{g}(\lambda/x, g)$ its asymptotic limit $(x/\lambda)^{4-D}N(g)$ and extend the integral to infinity. The leading term comes from the region where g is very small. If the first nonzero contribution to G is of order k

$$\begin{aligned} G_N{}^R(P|m^2, g) &= g^k G_N^{R(k)}(P|m^2) + O(g^{k+1}) \\ \Delta G_N{}^R(P|m^2, g) &= g^K\, \Delta G_N^{R(k)}(P|m^2) + O(g^{k+1}) \end{aligned} \tag{5.8}$$

we find

$$\begin{aligned} G_N{}^R(\lambda P|m^2, g) \underset{\lambda \to \infty}{\longrightarrow} & Z_1^{-N/2}(g)\left[\frac{u}{m^{4-D}}\right]^k \lambda^{-[k(4-D)-d_N]} \\ & \times \int_0^\infty \frac{dx}{x}\, x^{-d_N + k(4-D)}\, \Delta G_N^{R(k)}(xP|m^2) \\ & \sim Z_1^{N/2}(g)\left[\frac{u}{m^{4-D}}\right]^k G_N^{R(k)}(\lambda P|m^2) \end{aligned} \tag{5.9}$$

We have just recovered the expansion in powers of the bare coupling constant, which is known to give the dominant term in the large-momentum region. It is easy to see that (4.1) is identically satisfied.

If we start directly from $g = g_c$, the situation changes dramatically. The bare parameters are infinite; however,

$$\bar{g}(g_c, x) \equiv g_c, \qquad \lim_{g \to g_c} Z_1[\bar{g}(g, x)]/Z_1(g) = x^{2c_1{}^c} \tag{5.10}$$

Equation (5.5) simplies to

$$G_N{}^R(\lambda P|m^2, g_c) = -2\lambda^{d_N + Nc_1{}^c} \int_0^\lambda \frac{dx}{x} x^{-d_N - Nc_1{}^c} \, \Delta G_N{}^R(xP|m^2, g_c) \tag{5.11}$$

Our assumption implies that the integral is convergent when $\lambda \to \infty$. The upper limit of integration can be taken equal to infinity; neglecting terms that vanish in this limit, we obtain

$$G_N{}^R(\lambda P|m^2, g) \underset{\lambda \to \infty}{\longrightarrow} -2\lambda^{d_N - Nc_1{}^c} \int_0^\infty \frac{dx}{x} x^{-d_N - Nc_1{}^c} \, \Delta G_N{}^R(xP|m^2, g_c) \tag{5.12}$$

The integral no longer has any λ dependence. The correlation function in the large-momentum region satisfies a simple scaling law. The functional form of the correlation function is given by an integral on the correlation function computed in the low-momentum region. We have solved the problem of computing functions in the large-momentum region, where renormalized perturbation expansion is not convergent, using as input only correlation functions computed in the low-momentum region, where the renormalized perturbation expansion is more reliable.

We are now ready to study the limit $m \to 0$ at fixed u of the correlation function of the unrenormalized field.

Using dimensional arguments and the definition of renormalized correlation functions, we can derive the following chain of identities:

$$\begin{aligned} G_N{}^R(\mu P|m^2/\lambda^2, u) &= \lambda^{-d_N} G_N{}^B(\lambda\mu P|m^2, u\lambda^{4-D}) \\ &= Z_1^{N/2}(g)\lambda^{-d_N} G_N{}^R(\lambda\mu P|m^2, g) \end{aligned} \tag{5.13}$$

where g is a function of λ, u, and m

$$g = \rho[\lambda^{4-D}u/m^{4-D}] = R^{-1}[m/\lambda u^{1/(4-D)}] \tag{5.14}$$

Using (5.4) and (5.5), we obtain

$$\begin{aligned} &G_N{}^B\left(\mu P|\frac{m^2}{\lambda^2}, u\right) \\ &\quad = -2\mu^{d_N} \int_0^{\lambda\mu} \frac{dx}{x} x^{-d_N} Z_1^{N/2}\left[\bar{g}\left(g, \frac{\mu\lambda}{x}\right)\right] \Delta G_N{}^R\left[xP|m^2, \bar{g}\left(g, \frac{\mu\lambda}{x}\right)\right] \end{aligned} \tag{5.15}$$

From the definition of $\bar{g}$, Eq. (5.5), it follows that

$$\bar{g}\left(g, \frac{\mu\lambda}{x}\right) = R^{-1}\left[\frac{\mu\lambda}{x} R(g)\right] = R^{-1}\left(\frac{\mu}{x} \frac{m}{u^{1/(4-D)}}\right) \tag{5.16}$$

Without loss of generality we can take $m^2 = u^{2/(4-D)}$; any other value of m^2 can be reached by changing λ

$$G_N{}^B\left(\mu P| \frac{u^{2/(4-D)}}{\lambda^2}, u\right)$$
$$= -2\mu^{d_N} \int_0^{\lambda\mu} \frac{dx}{x} x^{-d_N} Z_1^{N/2}\left[R^{-1}\left(\frac{\mu}{x}\right)\right] \Delta G_N{}^R\left(xP|u^{2/(4-D)}, R^{-1}\left[\frac{\mu}{x}\right]\right) \tag{5.17}$$

We stress that (5.17) is an identity which can be derived without assumptions. If we send λ to infinity, the integral may diverge or converge. The possible region of divergence comes from x very large. In this situation $R^{-1}[\mu/x] \to g_c$. If the high-momentum behavior of ΔG_N at $g = g_c$ is good [see Eq. (5.7)], the integral is convergent. However, the limit $\lambda \to \infty$ does not exist in the perturbation expansion. $R^{-1}(1/x)$ is a finite-order polynomial of $x^{(4-D)}$ and clearly diverges when $x \to \infty$. If we include enough higher orders in the bare coupling constant, the decrease of $\Delta G_N(xP)$ when $x \to \infty$ can no longer compensate for the faster and faster increase of $R^{-1}(1/x)$, and infrared divergences appear. However, if we use a function R which satisfies (5.6) and ΔG_N computed at any order in perturbation theory, infrared divergences disappear.

If the zero-mass theory holds, or our two assumptions on the finiteness of the renormalized correlation functions in the limit $u \to \infty$ at fixed m and on the large-x behavior of $\Delta G_N{}^R(xP)$ are valid, the upper limit of integration can be shifted to infinity

$$G_N{}^B(\mu P|0, u) = -2\mu^{d_N} \int_0^{\infty} \frac{dx}{x} x^{-d_N} Z^{N/2}\left[R^{-1}\left(\frac{\mu}{x}\right)\right]$$
$$\times \Delta G_N{}^R\left[xP|u^{2/(4-D)}, R^{-1}\left(\frac{\mu}{x}\right)\right] \tag{5.18}$$

Let us study Eq. (5.18) in the two different limits. When $\mu \to \infty$, the situation is very similar to the first case we have studied. An expansion in powers of u/μ^{4-D} can be obtained. This expansion breaks down at the same order in u at which infrared divergences appear in perturbation theory. The coefficient of the next power in μ is no longer analytic in u. When $\mu \to 0$,

x/μ goes to infinity, and at the leading order in μ, barring corrections of order $\mu^{2b'}$, we find

$$G_N{}^B(\mu P|0, u) \xrightarrow[\mu\to 0]{} -2\mu^{d_N + Nc_1{}^c} \int_0^\infty \frac{dx}{x} x^{-d_N - N/2c_1{}^c} \, \Delta G_N(xP|u^{2/(4-D)}, g_c) \tag{5.19}$$

The coefficient of $\mu^{d_N + \mathrm{N}c_1{}^c}$ is strongly nonanalytic in the coupling constant: e.g., if we consider the two-point correlation function, we obtain

$$G_2{}^B(P) \xrightarrow[P\to 0]{} au^{-2c_1{}^c/(4-D)} P^{2-2c_1{}^c} \tag{5.20}$$

where a is a pure number:

$$a = -2 \int_0^\infty dx \, x^{1-x} m^{c_1{}^c} \, \Delta G_2{}^R(x|m^2, g_c) \tag{5.21}$$

Correlation functions are no longer C^0 in the coupling constant in the small-momentum region. It is not a surprise that the attempt to compute the Taylor expansion of a discontinuous function produces infrared divergences.

Comparing (5.11) with (5.18), we verify that the low-momentum behavior of the zero-mass theory and the large-momentum behavior of the theory computed at $g = g_c$ are the same.

It is clear that in this approach infrared divergences are completely washed out; the only welcome trace of their presence comes from the nonanalyticity in the coupling constant. The problem of computing the zero-mass behavior of correlation functions is reduced to the computation of correlation functions at finite values of the external momenta. The scaling law (4.14) is satisfied also at $K \neq 0$, $\tau \to 0$. The corrections to the scaling law are ruled by the same exponent ω

$$G_2(0, \tau) = \tau^{-\gamma}[1 + O(\tau^{\nu\omega})], \qquad G_2(K, 0) = (1/K^{2-\eta})[1 + O(K^\omega)] \tag{5.22}$$

6. SIMPLE EXAMPLES

The general analysis of the correlation function near the transition for the partition function (2.1) is now completed. The critical exponent and correlation function of the zero-mass theory can be computed as integrals over the correlation functions of the massive theory. Before looking for the possible generalizations of the model, we want to study a few concrete examples.

We compute the b, c_1, c_2, and l functions in the one-loop approximation. The relevant Feynman diagrams are shown in Fig. 1. Diagrams (a), (b), and (c) contribute respectively to the functions b, c_2, and l. No diagram contributes to c_1.

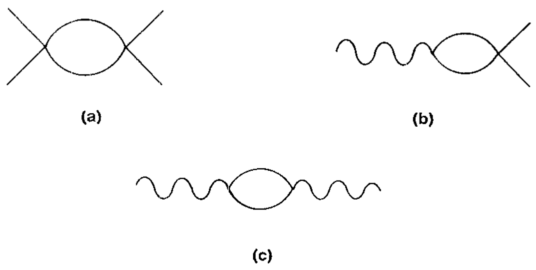


Fig. 1. Diagrams involved in the computation of the b, c_2, and l functions in the one-loop approximation.

The final result is[51]

$$\begin{aligned}
\Gamma_4(0,0,0) &= u + u^2 \frac{N+8}{6} I(m^2) + O(u^3) \\
\Delta\Gamma_4 &= \frac{N+8}{6} u^2 m^2 \frac{\partial}{\partial m^2} I(m^2) + O(u^3) \\
&= \frac{N+8}{6} m^{2(4-D)} g^2 m^2 \frac{\partial}{\partial m^2} I(m^2) \\
I(m^2) &= \frac{1}{(4\pi)^{D/2}} \Gamma\left(2 - \frac{D}{2}\right) m^{2(2-D/2)} \\
m^2 \frac{d}{dm^2} I(m^2) &= \frac{1}{(4\pi)^{D/2}} \Gamma\left(3 - \frac{D}{2}\right) m^{2(2-D/2)}
\end{aligned} \tag{6.1}$$

Note that when $D \to 4$, Γ_4 diverges (a renormalization of the coupling constant is needed) but $\Delta\Gamma_4$ remains finite. It is very important to realize that $\Delta\Gamma_4$ is nonzero in four dimensions only because Γ_4 is divergent.

In the same way we find

$$\Delta Z_2 = \frac{1}{2} \frac{g}{(4\pi)^2} \frac{N+2}{6} \Gamma\left(3 - \frac{D}{2}\right), \qquad \Delta G = \frac{1}{2} \frac{N(m)^{D-4}}{(4\pi)^2} \Gamma\left(3 - \frac{D}{2}\right) \tag{6.2}$$

These formulas imply

$$\begin{aligned}
b(g) &= -\frac{4-D}{2} g \left[1 - \frac{g}{g_c}\right] + O(g^3) \\
c_1(g) &= O + O(g^2), \qquad c_2(g) = c_2{}^c g/g_c + O(g^2) \\
l(g) &= \frac{N}{2(4\pi)^2} \Gamma\left(3 - \frac{D}{2}\right) + O(g) \\
c_2{}^c &= \frac{4-D}{2} \frac{N+2}{N+8}, \qquad g_c = \frac{(4\pi)^{D/2} 3(4-D)}{(N+8)\Gamma(3-D/2)}
\end{aligned} \tag{6.3}$$

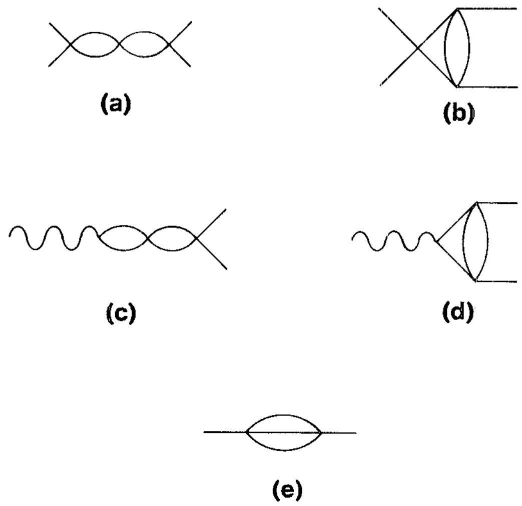


Fig. 2. Diagrams involved in the computation of the b, c_1, and c_2 functions in the two-loop approximation.

Putting (6.3) into (4.8) and (4.10), we obtain

$$
\begin{aligned}
\frac{u}{m^{4-D/2}} &= \frac{g}{1-g/g_c} \;. \\
\frac{dm^2}{dT} &= \left[1+\frac{u}{g_c m^{4-D}}\right]^{2c_2{}^c/4-D} \\
C &= \frac{N}{2(4\pi)^2}\,\Gamma\!\left(3-\frac{D}{2}\right)\int_{-\infty}^{m^2}\frac{d\tilde{m}^2}{\tilde{m}^{(2+4-D)}}\left[1+\frac{u}{g_c\tilde{m}^{4-D}}\right]^{-4C_2{}^c/(4-D)} \\
G_2(K, m^2) &= \frac{1}{K^2+m^2}
\end{aligned}
\tag{6.4}
$$

Near the transition, these formulas simplify to

$$
\begin{aligned}
g &= g_c - \frac{m^{4-D}g_c{}^2}{u} \\
\frac{\partial m^2}{\partial T} &= \left[\frac{u}{g_c}\right]^{2c_2{}^c/(4-D)} m^{-2c_2{}^c} \\
C &= \frac{N}{2(4\pi)^2}\,\Gamma\!\left(3-\frac{D}{2}\right)\left[\frac{g_c}{u}\right]^{+4c_2{}^c/(4-D)} m^{D-4+4c_2}
\end{aligned}
\tag{6.5}
$$

The critical exponents are

$$\nu = \frac{1}{2 - [(N+2)/(N+8)](D-4)} \qquad (6.6)$$
$$\gamma = 2\nu, \qquad \eta = 0, \qquad \alpha = 2 - D\nu,$$

For the three-dimensional Ising model we obtain

$$\gamma = 1.2, \qquad \nu = 0.6, \qquad \eta = 0, \qquad \lambda_c \equiv g_c \Gamma(3 - D/2)3/(4\pi)^{D/2} = 1 \quad (6.7)$$

For D sufficiently small we find $\nu < 0$. A negative value of ν implies that the system has no transition at finite temperature. The reason is simple. The integrated form of the scaling law $dM^2/dm = m^{(1/\nu)-1}$ is

$$M^2 - M_0{}^2 \propto (1/\nu)m^{1/\nu} \qquad (6.8)$$

where $M_0{}^2$ is an integration constant. If $1/\nu$ is positive, $M_c{}^2$ (the critical value of the temperature) is simply $M_0{}^2$; however, if $1/\nu$ is negative or zero, $M_c{}^2$ is located at $-\infty$ and cannot reach finite temperature. It depends on the details of the model whether $M_c{}^2 \to -\infty$ corresponds to zero temperature.

A transition is also present in the one-dimensional Ising model ($1/\nu = 1 > 0$), while no transition is correctly found in the spherical model if $D \leqslant 2$ ($\nu < 0$). It is of interest to note that at this simple order the exponents for the spherical model are the exact ones. In the case of the Ising model the corrections to the classical exponents seem to be in the right direction. How can the approximation be improved?

A first possibility, advocated by Wilson,[5] is based on the observation that g_c is of the order $4 - D$ when D is near to 4. This property is not destroyed in high orders in perturbation theory: we can compute both g_c and the critical exponents as a formal power of $\epsilon = 4 - D$. This is possible because the linear term in $b(g)$ is proportional to $-\frac{1}{2}\epsilon g$ and in the limit $\epsilon \to 0$, $b(g)$ is a finite, nontrivial function of g. The finiteness of $b(g)$ in dimension 4 is far from being a trivial statement: it is connected to the existence of the renormalized four-dimensional ϕ^4 theory in perturbation theory.[36]

In this situation we need to compute only the first k-loop diagrams to get the critical exponents up to the order ϵ^k. In the case of the N-component model, the following results have been found:

$$\gamma = 1 + \frac{N+2}{2(N+8)}\epsilon + \frac{(N+2)(N^2+22N+52)}{4(N+8)^3}\epsilon^2 + O(\epsilon^3)$$
$$\eta = \frac{\epsilon^2(N+2)}{2(N+8)^2}\left\{1 + \left[\frac{6(3N+14)}{(N+8)^2} - \frac{1}{4}\right]\epsilon\right\} + O(\epsilon^4) \qquad (6.9)$$

If we use Eq. (6.9) in the case $N = 1$, $D = 3$, we get

$$\gamma = 1.244, \qquad \eta = 0.037, \qquad \nu = 0.628 \qquad (6.10)$$

Higher order corrections up to ϵ^4 have been computed.(12,51) The expansion is not convergent; the generical terms increase like $\epsilon^k k!$.(52) However, there are good arguments which suggest that the correct results are obtained by using the Borel summation technique. Indeed one finds, for $N = 1$, $D = 3$,(51)

$$\gamma = 1.235 \pm 0.004, \qquad \eta = 0.0333 \pm 0.0001, \qquad \nu = 0.628 \pm 0.002 \tag{6.11}$$

A different approach, which avoids the use of noninteger dimensions at intermediate steps, consists in computing directly the k-loop contributions at fixed dimensions and in a first approximation neglecting higher orders in g.(13,53) The two methods produce the same results up to the order ϵ^k but are different at finite ϵ. The final answer is

$$\begin{aligned}
2b(\lambda) &= -(4-D)\lambda + \frac{N+8}{9}\lambda^2 - \left[\frac{10N+44}{81} f(D) - \frac{N+2}{81} h(D)\right]\lambda^3 + O(\lambda^4) \\
C_2(\lambda) &= \frac{N+2}{18}\lambda - [6f(D) - h(D)]\frac{N+2}{324}\lambda^2 + O(\lambda^3) \\
C_1(\lambda) &= \frac{N+2}{324} h(D)\lambda^2 + O(\lambda^3) \\
\lambda &= \frac{3\Gamma(3-D/2)}{(4\pi)^{D/2}} g
\end{aligned} \tag{6.12}$$

where

$$\begin{array}{lll} f(4) = 1.0, & f(3) = 0.2/3, & f(2) \simeq 0.56 \\ h(4) = 1.0, & h(3) \simeq 0.59, & h(2) \simeq 0.46 \end{array} \tag{6.13}$$

Unfortunately, at this approximation the function $b(\lambda)$ in Eq. (6.12) has no zero in dimensions less than 3.5 if $N = 1$. We know a priori that the loop expansion is not convergent and that resummation techniques should be used; although the Borel technique is the best suited to cope with this divergent series, we use for simplicity a Padé approximant. The presence of a zero is restored and the critical exponents can be evaluated.

In Fig. 3 we show the functions $\nu(D)$ computed to the first and the second order in ϵ, computed in the one- and two-loop approximations, using the Taylor expansion, and that in the two-loop approximation using the Padé approximant. The upper part of IIP is the analytic continuation of the lower part and it has no physical meaning as far as phase transitions are concerned. No transition is found in the one-dimensional Ising model, i.e., $1/\nu$ slightly negative.

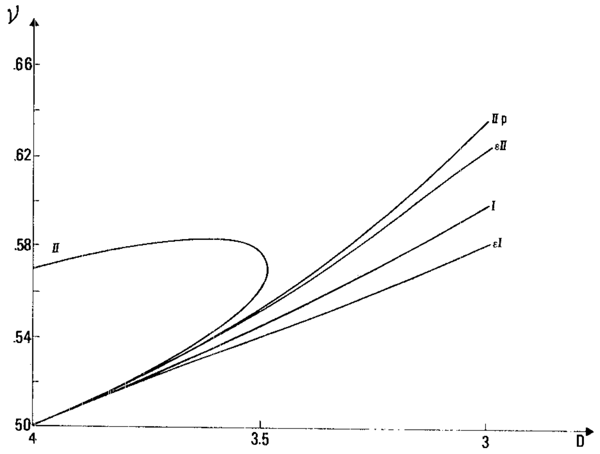


Fig. 3. Plot of the critical exponent ν of the Ising model in different approximations: ϵI and ϵII are, respectively, the first and the second order in ϵ; I and II are the critical exponents produced respectively by the one-loop and the two-loop approximations using the g_c given by the Taylor expansion, while in IIP we use the g_c computed from the [2, 1] Padé approximant.

In the three-dimensional Ising case we find

$$\gamma = 1.256, \qquad \eta = 0.033, \qquad \nu = 0.638, \qquad \lambda_c = 1.73 \tag{6.14}$$

Always at the two-loop level, but using the Padé–Borel resummation technique, we get[10]

$$\gamma = 1.247, \qquad \eta = 0.028, \qquad \nu = 0.633, \qquad \lambda_c = 1.60 \tag{6.15}$$

The success of this computation has led people to use more industrious techniques: all the diagrams up to six loops have been computed (1042 diagrams),[54] and asymptotic estimates for higher loop diagrams have been found.[12]

$$\begin{gathered} b(\lambda) = \sum_k b_k \lambda^k, \qquad b_k \underset{k \to \infty}{\longrightarrow} C(-A)^k \Gamma(k + 9/2) \\ C = 0.03996, \qquad A = 0.147742 \end{gathered} \tag{6.16}$$

The final result is[10,11]

$$\begin{gathered} \gamma = 1.241 \pm 0.002, \qquad \eta = 0.031 \pm 0.004 \\ \nu = 0.630 \pm 0.0015, \qquad \lambda_c = 1.416 \pm 0.005 \end{gathered} \tag{6.17}$$

For the two-dimensional Ising model a similar procedure, starting from the knowledge of only four-loop diagrams, gives[11]

$$\begin{aligned} \gamma &= 1.79 \pm 0.09, & \eta &= 0.13 \pm 0.07 \\ \nu &= 0.97 \pm 0.08, & \lambda_c &= 1.85 \pm 0.10 \end{aligned} \tag{6.18}$$

The results in the two-dimensional case are worse than those in the three-dimensional case. There are two reasons for this: the effective value of g is larger and the precision is decreased; the result of any computation of this kind is automatically an analytic function of N; however, we know that the critical exponents are discontinuous around $N = 2$: the critical exponent jumps from 1/4 to 0 for N greater than 2.[55,56] If we do not consider the contribution of very high orders in g, poor results should be obtained for N too near to 2.

7. THE RANGE OF THE RENORMALIZED COUPLING CONSTANT

An interesting problem is the determination of the possible values that the renormalized coupling constant may assume. It is clear that when the bare coupling constant u is in the 0 to ∞ range, the renormalized coupling ranges from 0 to g_c. A theory with g greater than g_c cannot be obtained as limit of theories where the bare coupling constant is finite, but it could be obtained as an analytic continuation in the renormalized coupling constant from the good region $g \leqslant g_c$. We shall see below that there is a cut, starting from g_c, which forbids the analytic continuation to greater values of the renormalized coupling constant, and that g_c is the greatest allowed value. This result implies that, if ultraviolet divergences are eliminated using dimensional regularization, the four-dimensional theory is always free.[6]

Let us assume that all the correlation functions are C^∞ functions of the coupling constant and that $b(g)$ has a simple zero at $g = g_c$. This hypothesis is not consistent with the differential equation (5.23) unless an infinite number of independent sum rules is satisfied; this last possibility is rather unrealistic; moreover, in the framework of the $1/N$ expansion it is possible to check that the sum rules are not satisfied and a cut is present at g_c. To be specific, let us consider the case of the Γ^6 function computed at zero external momentum. Then Eq. (5.4) reads

$$\begin{aligned} \Gamma_6(g) &= \int_0^g \frac{dg'}{b(g')} \left[\frac{R(g)}{R(g')}\right]^{2D-6} \left[\frac{Z_1(g)}{Z_1(g')}\right]^3 \Delta\Gamma_6(g) \\ &= \int_0^g dg' \frac{g_c - g'}{b(g')} \frac{f(g)}{f(g')} \frac{(g_c - g)^{B_6}}{(g_c - g')^{B_6+1}} \Delta\Gamma_6(g') \end{aligned} \tag{7.1}$$

where $B_6 = (D - 3 + 3c_1{}^c)/b'$ and the function $f(g)$ is C^∞ around g_c. If B_6 were negative, Eq. (7.1) could be written as

$$\Gamma_6(g) = f(g)(g_c - g)^{B_6} \int_0^{g_c} dg' \frac{(g_c - g')^{-B_6}}{b(g')f(g')} \Delta\Gamma_6(g')$$
$$- \int_g^{g_c} dg' \frac{(g_c - g)^{B_6}}{b(g')(g_c - g')^{B_6}} \frac{f(g)}{f(g')} \Delta\Gamma_6(g') \qquad (7.2)$$

If B_6 is positive, but is not an integer, Eq. (7.2) is still true provided that the integrals are defined as analytic continuations in B_6. In both cases, the second integral is C^∞ around g_c; the first term, if it is not identically zero, has a cut of the type $(g_c - g)^{B_6}$.[49]

Equation (7.2) implies that the renormalized Green's functions must be singular at g_c; this equation, however, cannot be correct as it stands: if $D = 3$ the value of B_6 is only slightly positive and becomes negative for D slightly less than 3. Indeed there is a loophole in the argument; if $\Gamma_6(g)$ has a cut at $g = g_c$, then $\Delta\Gamma(g)$ also must have a cut of the same strength as $\Gamma_6(g)$.

It has been shown in Ref. 57 that under a technically simplifying hypothesis, we can write the following equation (we have neglected operator mixing):

$$[b(g)\, \partial/\partial g + 3 - D - C_6(g)]\Gamma^6(g) = \tilde{\Delta}\Gamma_6(g) \qquad (7.3)$$

where the singularity of $\tilde{\Delta}\Gamma_6(g)$ at $g = g_c$ is less strong than that of $\Gamma_6(g)$, the function $C_6(g)$ being the "anomalous dimension" of the operator ϕ^6:

$$C_6(g) \sim \frac{12 + 3N}{8 + N} \lambda + O(\lambda^2) \qquad (7.4)$$

The quantity $A_6 = 3(D - 2) + C_6(g_c)$ is the effective dimension of the renormalized ϕ^6 operator at the critical point:

$$\langle \phi^6(x)\phi^6(0)\rangle \xrightarrow[x \to 0]{} \frac{1}{x^{2A_6}}, \qquad A_6 = 6 + 2\frac{2 + N}{8 + N}\epsilon + O(\epsilon^2) \qquad (7.5)$$

An equation similar to (7.2) holds, where now $B_6 = (A_6 - D)/2b'$. The power of the singularity is controlled by the dimension of the ϕ^6 operator, as it should be.[58,59] The disaster $B_6 < 0$ happens only if $A_6 < D$, which would be a definite signal of serious problems (see next section).

The physical origin of these singularities is quite clear; an a priori argument can be given for their existence; the only difficulty is to see their appearance in the present formalism. In the limit $\omega \to \infty$ the following expansion for the renormalized Green's functions is expected to hold:

$$\Gamma^R(\omega) \to \Gamma^R(\infty) + \sum_n R_n u^{(D-A_n)/\epsilon}, \qquad D = 4 - \epsilon \tag{7.6}$$

where the A_n are the effective dimensions of all the possible local operators. Let us consider for simplicity only the ϕ^n operators with even n ($n \neq 2$). Equation (.) clearly shows why we need the condition $A_n > D$, and comparing it with Eq. (4.12), we get the identification

$$2b' = A_4 - D \tag{7.7}$$

It is easy to see that Eq. (7.6) implies the presence of a cut in the renormalized constant complex plane of the form

$$(g_c - g)^{(A_6 - D)/(A_4 - D)} \tag{7.8}$$

As a last remark, we notice that in $4 - \epsilon$ dimensions we have a relation between the singularities of the Green's functions at $g = g_c = O(\epsilon)$ and the singularities due to ultraviolet divergences of the Borel transform of the Green's functions.[57] We only quote the result:

$$\Gamma_6(\lambda) \sim (\lambda_c - \lambda)^{B_6}[\pi/\sin(\pi B_6)]\, \tilde{\Delta} C_6(2) \tag{7.9}$$

which can be written also as

$$\operatorname{Im} \Gamma_6(\lambda) \sim (\lambda - \lambda_c)^{B_6} \theta(\lambda - \lambda_c) \pi \tilde{\Delta} C_6(2) \tag{7.10}$$

where $\Delta\tilde{C}_6(t)$ is the Borel transform of $\Gamma_6(\lambda)$ with respect to λ [see Eq. (6.12)]:

$$\tilde{\Delta}\Gamma_6(\lambda) = \sum_n^{\infty} t_n \lambda^n, \qquad \tilde{\Delta} C_6(Z) = \sum_1^{\infty}{}_n \frac{t_n Z^n}{(n-1)!} \tag{7.11}$$

8. ON UNIVERSALITY

Until now we have discussed only a $\lambda\phi^4$ interaction. It is clear that the whole approach is of little interest if the results cannot be extended to much more general interactions.

We consider a more complicated model

$$Z \propto \int d[\phi] \exp \int d^D x\, \mathscr{L}(x) \tag{8.1}$$

where

$$-\mathscr{L}(x) = \frac{1}{2}\,\partial_\mu\phi(x)\,\partial^\mu\phi(x) + \frac{1}{2}\,M^2\phi^2 + \frac{u_0}{4!}\,\phi(x^4) + u_1\phi^6(x) + u_2[\Delta\phi(x)]^2 + \cdots \tag{8.2}$$

The field ϕ is still defined on a continuous space M, and the u's are arbitrary functions of the temperature. The number of parameters may be arbitrary.

For simplicity we assume that all interaction constants u, M_c^2, and $\bar{M}^2 = M^2 - M_c^2(u)$ are regular functions of the temperature [$M_c^2(u)$ is the value of the bare mass at the critical temperature]. If in the limit $\bar{M}^2 \to 0$ thermodynamic quantities are regular functions of the other parameters, we can neglect their temperature dependence near the transition. This is the general case, although more complicated situations can occur near a tricritical point.[50] In conclusion, without any loss of generality we assume that near a simple critical point the partition function depends on the temperature only through the $\bar{M}^2$ term: $\bar{M}^2 \propto T - T_c$. The techniques used in the other sections can be easily extended. Renormalized fields, mass, and coupling constants are defined in such a way that the only dimensional parameter is the mass. For example, in a theory described by the Lagrangian (8.2), a possible choice of coupling constants could be

$$\begin{gathered} g_0 = \Gamma^4(0,0,0)m^{D-4}, \qquad g_1 = \Gamma^6(0,0,0,0,0)m^{2D-6} \\ g_2 = (d/dK^2)^2\Gamma^2(K^2)|_{K^2=0}m^2 \end{gathered} \tag{8.3}$$

It may be convenient to define the coupling constants in such a way that if the interaction contains only a ϕ^4 term, all coupling constants except g_0 are zero.

We introduce the Δ operator

$$\Delta = m^2 \left.\frac{\partial}{\partial m^2}\right|_{u_i} = \left.\frac{\partial}{\partial \lg m^2}\right|_{u_i} \tag{8.4}$$

and we define a set of b functions

$$\Delta g_i = b_i(g_0, \ldots, g_j) \tag{8.5}$$

In compact notation

$$\left.\frac{\partial}{\partial \lg m^2}\right|_{\mathbf{u}} \mathbf{g} = \mathbf{b}(\mathbf{g}) \tag{8.6}$$

The integration of the system (8.6) yields the renormalized coupling constants as functions of the bare ones and the renormalized mass

$$\mathbf{g} = \rho(m, \mathbf{u}) \tag{8.7}$$

The bare coupling constants play the role of integration constants or boundary conditions to the solutions of the differential equations in the limit $m \to \infty$.

Approaching the transition, m goes to zero and $\log m$ goes to $-\infty$. If the limit

$$\mathbf{g}_c = \lim_{m \to 0} \rho(m, \mathbf{u}) \tag{8.8}$$

exists and is finite, then

$$\mathbf{b}(\mathbf{g}_c) = 0 \tag{8.9}$$

We say that g_c is a fixed point if (8.9) is satisfied. However, if $g_c^{(1)}$ is a particular fixed point, the actual limit in (8.8) may be different from $g_c^{(1)}$ unless this fixed point is attractive and the set of bare coupling constants is inside its domain of attraction.[5] The first condition implies that the real part of all the eigenvalues of the matrix

$$B_{ik} = \left.\frac{\partial b_i}{\partial g_k}\right|_{g=g_c} \tag{8.10}$$

are positive. Its eigenvalues control the deviation from scaling near the transition. The exponent ω defined in (4.20) is $1/2B_{\min}$, where $B_{\min}$ is the minimum eigenvalue of B. If we recall Eq. (7.6), we find that the eigenvalues B_n of the matrix B are connected to the effective dimensions of the renormalized operator by

$$B_n = A_n - D \tag{8.11}$$

The condition $A_n > D$ (i.e., absence of relevant operators) imposed in the previous section is equivalent to the stability of the fixed point.

Both the critical exponent and the detailed form of the correlation functions in the critical region depend only on the fixed point. They are the same in all the theories which belong to the domain of abstraction of the same fixed point. The dependence of the critical exponent on the detailed form of the interaction is absent; only discontinuous behavior can be found. [We have assumed that the system of equations $b(g) = 0$ has only a discrete set of solutions. If lines of zeros are present, the critical exponent can have a continuous dependence on the parameters of the bare interaction.]

We stress that by increasing the complexity of the interaction, new fixed points can be created, but the old ones cannot disappear. However, they can become unstable. The results of the $\lambda\phi^4$ model can be applied to other models only if the matrix A does not develop negative eigenvalues when new interactions are introduced, and the bare coupling constants of the new interactions are not too large.

In dimensions near enough to 4 all local interactions which involve more derivatives or higher powers of the fields have positive eigenvalues, and the smallest one is b'. If $N \neq 1$, the only possible relevant operators must be quadrilinear or bilinear with different dependence on the internal degree of symmetry; their presence accounts for the different behavior of the isotropic and anisotropic Heisenberg models.

We have just arrived at the conclusion that a wide class of interactions have the same renormalized correlation functions at the critical point. It would be nice to compute these correlation functions without committing ourselves to any particular interaction. This can be done by deriving integral nonlinear equations for the renormalized correlation functions which are valid only at the critical point.

Comparing (5.4) with (5.11), we find that the fixed point is characterized by the fact that the correlation functions $G_N(\Gamma_N)$ can be computed as integrals over the correlation functions $\Delta G_N(\Delta\Gamma_N)$ evaluated at the same value of the coupling constant. If we are not at the fixed point, (5.4) also involves $\Delta G_N(\Delta\Gamma_N)$ functions computed at all possible values of the coupling constant. Moreover, the $\Delta G_N(\Delta\Gamma_N)$ can be easily computed from the knowledge of the first $N + 2$ of the $G(\Gamma)$ functions. The correlation functions of ${\phi_R}^2$ with the other N of the ϕ_R fields can be written as an integral over some nonlinear combination of the correlation function of the $N + 2$ field, using the relation ${\phi_R}^2 \propto (\phi_R)^2$.

The generating function of the connected (amputated one-line irreducible) correlation functions satisfies a nonlinear functional equation

$$G = F(G) \qquad [\Gamma = \tilde{F}(\Gamma)] \tag{8.12}$$

This equation has the same structure as the Schwinger equation[31] for the generating function of a polynomial interaction. The main difference is that no possible small parameter exists in (8.12): it is not clear how to construct a simple algorithm which would allow the iterative solution of this equation.

9. CONCLUSIONS AND OUTLOOK

In this paper we have formulated a field-theoretic approach to the theory of second-order phase transitions, based on the renormalization group; in this approach, spaces of noninteger dimensions play no role and everything can be done without leaving the three (two) dimensional space.

Simple computations of the critical exponents of the three-dimensional Ising model have been shown in detail; reasonable results are obtained. If longer computations are done, better results are obtained: the critical exponents are estimated with an error less than 10^{-3}.

We cannot claim to have proved the validity of the scaling laws: we have only derived them, starting "reasonable" hypotheses. However, it is gratifying that these hypotheses can be explicitly checked in the $1/N$ expansion.

The extension of this method to other systems undergoing a second-order phase transition is straightforward: indeed, it has been used to compute the critical exponents of the Reggeon field theory with good accuracy.[60]

APPENDIX

In this appendix we define the notations used in the text.

We consider the partition function (2.10) in the presence of a point-dependent magnetic field H

$$Z[H] \propto \int d[\phi] \exp\left[- \int d^D x \, \mathscr{L}(x, H) \right] \tag{A.1}$$

where

$$\mathscr{L}(x, H) = -\frac{1}{2} \partial_\mu \phi(x) \, \partial^\mu \phi(x) + \frac{1}{2} M^2 \phi^2(x) + \frac{u}{4!} \phi^4(x) + H(x)\phi(x) \tag{A.2}$$

is a functional of the magnetic field H. We define the free energy functional

$$G[H] = \ln\{Z[H]\} \tag{A.3}$$

Z is the generating functional of the correlation function of the field ϕ, while G is the generating functional of the connected (truncated) correlation functions

$$\left.\frac{\delta^N Z}{\partial H(x_1) \cdots \delta H(x_N)}\right|_{H=0} \equiv \langle \phi(x_1) \cdots \phi(x_N) \rangle$$

$$= \int d[\phi] \, \phi(x_1) \cdots \phi(x_N) \exp\left[- \int d^D x \, \mathscr{L}(x, 0) \right]$$

$$\frac{\delta^N G}{H(x_1) \cdots H(x_N)} = \langle \phi(x_1) \cdots \phi(x_N) \rangle_c = G_N(x_1 \cdots x_N) \tag{A.4}$$

or

$$G[H] = \sum_{D}^{\infty}{}_{n} \frac{1}{n!} \int \prod_{1}^{n}{}_{i} H(x_i)\, d^D x_i \prod_{1}^{n}{}_{i} \phi(x_i) \tag{A.5}$$

The magnetization of the system is

$$\mu(x, H] = \partial G/\delta H(x) \tag{A.6}$$

$\mu(x, H]$ is a function of x and a functional of H.

The functional (A.6) can be inverted, producing the magnetic field as a functional of the magnetization $H(x, \mu]$.

The free energy at constant magnetization can be introduced by performing a Legendre transformation

$$\Gamma[\mu] = \int d^D x\, \mu(x) H(x, \mu] - G[H[\mu]] \tag{A.7}$$

$\Gamma[\mu]$ is the generating functional of the connected one-line (one-particle) irreducible correlation functions

$$\left.\frac{\delta^N \Gamma}{\delta\mu(x_1) \cdots \delta\mu(x_N)}\right|_{\mu=0} = \Gamma_N(x_1 \cdots x_N) \tag{A.8}$$

If $u = 0$, the partition function Z can be explicitly computed

$$Z_0[H] \propto \exp - \int dx\, dy\, \tfrac{1}{2} D(x-y, M) H(x) H(y) \tag{A.9}$$

where $D(x - y, M)$ satisfies the differential equation

$$(\Delta_x + M^2) D(x-y, M) = (\Delta_y + M^2) D(x-y, M) = \delta^D(x-y) \tag{A.10}$$

If u is different from zero, (A.1) can be formally written in compact notation as

$$Z[H] \propto \exp\left\{-\frac{u}{4!} \int \left[\frac{\delta}{\delta H(x)}\right]^4 d^D x\right\} Z_0[H] \tag{A.11}$$

The expansion of the first exponential in (A.10) in powers of u reproduces the standard perturbation expansion, Eq. (3.5).

The functional Z satisfies the Schwinger functional differential equation:

$$\left[(\Delta_x + M^2) \frac{\delta}{H(x)} + \frac{u}{3!}\left(\frac{\delta}{\delta H(x)}\right)^3 + H(x)\right] Z[H] = 0 \tag{A.12}$$

Equation (A.12) is very similar to a Kirkwood–Salisbury equation; it yields the N-point correlation functions as an integral over the $N + 2$ correlation functions

$$\frac{\delta Z}{\delta H(x)} = \int d^D y\, D(x-y, M)\left[H(y) + \frac{u^3}{3!}\left(\frac{\delta}{\delta H(y)}\right)^3\right] Z[H] \quad \text{(A.13)}$$

The ϕ field is defined as the square of ϕ^2. Its correlation functions are defined by

$$Z_{N,\phi^2}(x_1 \cdots x_N, y) = \langle \phi(x_1) \cdots \phi(x_N)\phi(y)\phi(y)\rangle \quad \text{(A.14)}$$

From (A.14) one can obtain the connected and the amputated correlation functions of ϕ^2.

In this paper one normally considers the Fourier transforms of the correlation functions G_N and Γ_N. They are defined as follows:

$$G_N(P_1 \cdots P_{N-1}) = \int d^D x_1 \cdots d^D x_{N-1} \exp\{i[P_1 x_1 + \cdots + P_{N-1}x_{N-1}]\}\, G_N(x_1 \cdots x_{N-1}, 0)$$

$$G_{N\phi^2}(P_1 \cdots P_{N-1}, K) = \int d^D x_1 \cdots d^D x_{N-1}\, d^D y \quad \text{(A.15)}$$

$$\times \exp\{i[P_1 x_1 + \cdots + P_{N-1}x_{N-1} + Ky]\}\, G_{N+2}(x_1 \cdots x_{N-1}, 0, y, y)$$

$$D_{\phi^2\phi^2}(P) = \int d^D x\, e^{iPx}\langle \phi^2(x)\phi^2(0)\rangle_c$$

Similar definitions are valid for the Γ_N functions. Sometimes the vector K is omitted from the argument of $G_{N\phi^2}$; in such a case it is supposed to be zero, e.g.,

$$G_{2\phi^2}(P) = \int d^D x\, d^D y\, e^{iPx} G_{2\phi^2}(x, 0, y) \quad \text{(A.16)}$$

The correlation functions of the renormalized theory field are denoted by $G_N^R(\Gamma_N^R)$. To simplify the notation the subscript R is omitted in Section 8. In Section 5 the correlation function of the bare field acquires a subscript B to prevent any confusion.

REFERENCES

1. L. P. Kadanoff *et al.*, *Rev. Mod. Phys.* **39**:395 (1967).
2. B. Widom, *J. Chem. Phys.* **43**:3893 (1965).
3. C. Domb and H. S. Green, eds., *Phase Transitions and Critical Phenomena* (Academic Press, 1975), Vol. III.

4. C. Di Castro and G. Jona-Lasinio, *Phys. Lett.* **29A**:332 (1969).
5. K. G. Wilson and M. E. Fisher, *Phys. Rev. Lett.* **28**:234 (1972); K. G. Wilson, *Phys. Rev. Lett.* **28**:548 (1972); K. G. Wilson and J. Kogut, *Phys. Rep.* **12C**:77 (1974).
6. E. Brezin, J. C. Le Guillou, B. G. Nikel, and J. Zinn Justin, *Phys. Lett.* **44A**:227 (1973).
7. C. De Dominicis and L. Peliti, *Phys. Rev. B* **18**:353 (1978), and references therein.
8. A. B. Harris, T. C. Lubensky, and J. H. Chen, *Phys. Rev. Lett.* **36**:415 (1976).
9. M. Moshe, *Phys. Rep.* **43C**:197 (1977).
10. G. A. Baker, B. G. Nikel, M. S. Green, and D. I. Meiron, *Phys. Rev. Lett.* **36**:1351 (1976); *Phys. Rev. B* **17**:1365 (1978).
11. J. C. Le Guillou and J. Zinn Justin, *Phys. Rev. Lett.* **39**:95 (1977); Saclay Preprint DPh-T/79/94.
12. E. Brézin and G. Parisi, *J. Stat. Phys.* **19**:269 (1978), and references therein.
13. G. Parisi, Cargese Lectures Notes, Columbia University preprint (1973), unpublished.
14. C. Di Castro, *Nuovo Cimento Lett.* **5**:69 (1972).
15. K. Symanzik, unpublished, quoted in G. Mack, *Lecture Notes in Physics*, Vol. 17 (Springer Verlag, 1972).
16. W. F. Jegerlehrer and B. Schoer, *Acta Physica Austriaca Suppl.* (1973); D. K. Mitter, *Phys. Rev. D* **7**:2927 (1973); E. Brézin, J. Le Guillou, and J. Zinn Justin, *Phys. Rev. D* **8**:2418 (1973); F. De Pasquale and P. Tombesi, *Nuovo Cimento* **12B**:43 (1972).
17. E. Brézin, J. C. Le Guillou, and J. Zinn Justin, in *Phase Transitions and Critical Phenomena*, C. Domb and M. S. Green, eds. (Academic Press, 1976), Vol. VI.
18. D. J. Amit, *Field Theory, the Renormalization Group and Critical Phenomena* (McGraw-Hill, 1978).
19. J. Glimm and A. Jaffe, Cargese Lectures Notes (1976).
20. K. Symanzik, *J. Math. Phys.* **7**:510 (1966).
21. K. Osterwalder and J. Schrader, *Comm. Math. Phys.* **31**:83 (1973).
22. J. Schwinger, *Proc. Nat. Acad. Sci. (U.S.)* **44**:956 (1958).
23. R. Streater and A. S. Wightman, *PCT, Statistics and All That* (Benjamin, 1964).
24. A. A. Migdal, *Sov. Phys.—JEPT* **32**:552 (1971).
25. V. L. Ginzburg and L. D. Landau, *Sov. Phys.—JEPT* **20**:1060 (1950).
26. L. D. Landau and M. E. Lifshitz, *Statistical Physics* (Addison-Wesley, 1969).
27. A. A. Migdal, *Sov. Phys.—JEPT* **28**:1036 (1969).
28. A. Stanley, *Phys. Rev.* **176**:718 (1968).
29. P. G. De Gennes, *Phys. Lett.* **38A**:339 (1972).
30. R. Balian and G. Toulouse, *Phys. Rev. Lett.* **30**:544 (1973).
31. A. M. Poliakov, *Sov. Phys.—JEPT* **28**:533 (1969).
32. T. H. Berlin and M. Kac, *Phys. Rev.* **86**:821 (1952).
33. J. P. Eckman, J. Magnen, and R. Sénéor, *Comm. Math. Phys.* **39**:251 (1976).
34. P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford, The Clarendon Press, 1930).
35. J. S. Schwinger, *Selected Papers on Quantum Electrodynamics* (New York, Dover, 1958).
36. J. D. Bjorken and S. D. Drell, *Relativistic Quantum Field Theory* (New York, McGraw-Hill, 1965); C. Itzykson and J.-B. Zuber, *An Introduction to Field Theory* (McGraw-Hill, in press).
37. E. Brézin and D. J. Wallace, *Phys. Rev. B* **7**:1967 (1973).
38. K. Wilson, *Phys. Rev. D* **4**:2911 (1973).
39. G. Parisi, in Proc. Colloquium on Lagrangian Field Theory (Marseille 1974), *Nucl. Phys. B* **100**:368 (1975); *Cargese Lecture Notes* (1976).
40. A. M. Poliakov, *Sov. Phys.—JEPT* **30**:151 (1970).
41. A. M. Poliakov, *JEPT Lett.* **12**:381 (1970).
42. A. A. Migdal, *Phys. Lett.* **37B**:98, 386 (1971).
43. G. Parisi and L. Peliti, *Lett. Nuovo Cim.* **2**:627 (1971).

44. G. Parisi and L. Peliti, *Phys. Lett.* **41A**:331 (1972).
45. E. S. Fradkin, M. Y. Palchik, and V. N. Zaikin, Saclay preprint (1977).
46. K. Symanzik, *Comm. Math. Phys.* **18**:222 (1970).
47. J. Callan, Jr., *Phys. Rev. D* 1:1541 (1971).
48. K. Symanzik, *Comm. Math. Phys.* **23**:61 (1971).
49. G. Parisi, *Nuovo Cimento* **21A**:179 (1974).
50. S. Weinberg, *Phys. Rev.* **118**:838 (1960).
51. D. I. Kazakov, O. V. Tarasov, and A. A. Vladimirov, Dubna Preprint E2-12249 (1979).
52. E. Brézin, J. C. Le Guillou, and J. Zinn Justin, *Phys. Rev. D* **15**:1544 (1977).
53. G. Parisi, Rome University Preprint No. 426 (1973); and in *Proceedings of the Temple Conference on Field Theory and Critical Phenomena* (1973).
54. B. G. Nikel, D. I. Meiron, and G. A. Baker, Jr., University of Guelph Report (1977).
55. J. M. Kosterliz and D. J. Thouless, *J. Phys. C* **6**:1181 (1973); for a review see D. R. Nelson, *Phys. Rep.* **49C**:255 (1979).
56. A. M. Poliakov, *Phys. Lett.* **59B**:79 (1975); A. A. Migdal, *Sov. Phys.—JEPT* **42**:743 (1976); E. Brézin and J. Zinn Justin, *Phys. Rev. Lett.* **36**:961 (1976); *Phys. Rev. B* **14**:3110 (1976).
57. G. Parisi, *Phys. Lett.* **76B**:65 (1978); and *Phys. Rep.* **49C**:215 (1979).
58. K. Symanzik, Desy Preprint (1977).
59. Y. Araf'eva, *Theor. Math. Phys.* **31**:279 (1977).
60. J. L. Cardy, *Phys. Lett.* **67B**:97 (1977).