

To be submitted to  
Nuclear Physics

ISTITUTO NAZIONALE DI FISICA NUCLEARE  
Laboratori Nazionali di Frascati

LNF-80/55(P)  
10 Ottobre 1980  
(revised version  
19 Novembre 1980)

F. Palumbo: NONRELATIVISTIC GAUGE THEORIES.

F. Palumbo: NONRELATIVISTIC GAUGE THEORIES.

ABSTRACT.

Abelian and non-abelian gauge theories are formulated, which are invariant under Galilei transformations. It is shown that in the abelian case the infrared behaviour is the same of relativistic QED.

1. - INTRODUCTION.

The bound state problem is of fundamental importance in non-abelian gauge theories, being related to quark confinement. The investigation of this problem however, requires the knowledge of the quark-quark interaction, and this cannot be studied by perturbation theory which is not applicable at low energy.

The nonrelativistic version of non-abelian gauge theories could allow us to get insight into this difficult problem. Nonrelativistic approximations of the so far well understood relativistic theories provide in fact an accurate description of low energy phenomena, and require simple calculations especially as far as bound state are concerned. The properties of two- and three-body systems can indeed be determined by numerical computations to any desired accuracy.

Well known examples are the Coulomb and the  $\pi$ -N interactions in quantum mechanics.

One could argue that not all of the low energy properties of relativistic theories can be found in their nonrelativistic approximations. For instance a distinctive feature of QED are infrared divergencies and radiative corrections due to emission of soft photons. These corrections have a finite limit for  $c \rightarrow \infty$ , but cannot obviously appear in a nonrelativistic theory with Coulomb interaction. Such a theory however, is not invariant under local gauge transformations, so that cannot be considered the true nonrelativistic version of QED.

In this paper we consider the problem of constructing abelian and non-abelian gauge theories which are Galilei invariant. It is not at all obvious a priori that such theories exist, because nonrelativistic real fields cannot be quantized, while a characteristic feature of relativistic non-abelian gauge fields is their self-interaction. Nevertheless it turns out that nonrelativistic gauge theories can be formulated, and in the abelian case they have the same infrared behaviour of relativistic QED.

In Sect. 2 we will derive the field equations and in Sect. 3 we will show how these equations can be obtained from the relativistic field equations in the limit  $c \rightarrow \infty$ . In Sect. 4 we will perform the quantization for the abelian case and will study its infrared behaviour.

## 2. - THE FIELD EQUATIONS.

We will put  $\hbar = 1$ , while obviously  $c$  will never appear with the exception of Sect. 3.

In order to keep as close as possible to the relativistic formulation we start from equations of motion for the spinor field which are linear in space-time derivatives<sup>1</sup>

$$-i\partial_t \psi + i\partial_k \sigma_k \chi = 0 , \quad -i\partial_k \sigma_k \psi + 2m\chi = 0 . \quad (1)$$

In the above equations  $\sigma_k$  are Pauli matrices,  $m$  the fermion mass, and  $\chi$  an auxiliary fermion field. Its elimination yields the Schrödinger equation for  $\psi$ . Eqs. (1) are invariant<sup>1</sup> under Galilei transformations

$$\begin{aligned} \psi'(x', t') &= e^{if(x, t)} \psi(x, t) , \\ \chi'(x', t') &= e^{if(x, t)} \left[ \chi(x, t) - \frac{1}{2} v_k \sigma_k \psi(x, t) \right] , \end{aligned} \quad (2)$$

where

$$f(x, t) = \frac{1}{2} m v^2 t + m v_k x_k . \quad (3)$$

We now introduce gauge fields  $V^a$ ,  $A_k^a$  belonging to the regular representation of a compact Lie group  $G$ , whose generators are represented by Hermitian matrices  $t^a$  with commutation relations

$$[t^a, t^b] = if^{abc} t^c . \quad (4)$$

The gauge fields have gauge transformations

$$\begin{aligned} V^a &\rightarrow V^a - \partial_t \omega^a - gf^{abc} \omega^b V^c , \\ A_k^a &\rightarrow A_k^a + \partial_k \omega^a - gf^{abc} \omega^b A_k^c , \end{aligned} \quad (5)$$

and Galilei transformations<sup>1</sup>

$$V^a \rightarrow V^a + v_k A_k^a , \quad A_k^a \rightarrow A_k^a . \quad (6)$$

Assuming the fermion fields to belong to any representation of  $G$ , we couple fermions to gauge fields by the standard replacement of ordinary derivatives by covariant derivatives

$$-iD_t \psi + iD_k \sigma_k \chi = 0 , \quad -iD_k \sigma_k \psi + 2m\chi = 0 , \quad (7)$$

where

$$D_t = \partial_t + igV^a t^a , \quad D_k = \partial_k - igA_k^a t^a . \quad (8)$$

Eqs.(7) are gauge-invariant and Galilei-invariant. Using standard procedures we can show that the fermion current

$$\begin{aligned} \rho &= \psi^\dagger \psi \quad , \\ J_k &= \frac{1}{2mi} (\psi^\dagger \partial_k \psi - \partial_k \psi^\dagger \psi) - \frac{g}{m} A_k^a \psi^\dagger t^a \psi + \\ &\quad + \frac{1}{2m} \epsilon_{kh\ell} \partial_k (\psi^\dagger \sigma_\ell \psi) \end{aligned} \quad (9)$$

is conserved

$$\partial_t \rho + \partial_k J_k = 0 . \quad (10)$$

Moreover, using the matrix elements of the covariant derivative in the adjoint representation

$$D_t^{ab} = \partial_t \delta^{ab} + gf^{abc} V^c, \quad D_k^{ab} = \partial_k \delta^{ab} - gf^{abc} A_k^c . \quad (11)$$

we can define the color current

$$\begin{aligned} \rho^a &= \psi^\dagger t^a \psi \quad , \\ J_k^a &= \frac{1}{2mi} (\psi^\dagger t^a D_k \psi - D_k^\dagger \psi^\dagger t^a \psi) + \\ &\quad + \frac{1}{2m} \epsilon_{kh\ell} D_h^{ab} \psi^\dagger t^b \sigma_\ell \psi \quad , \end{aligned} \quad (12)$$

which satisfies the equation

$$D_t^{ab} \rho^b + D_k^{ab} J_k^b = 0 . \quad (13)$$

We see that fermions have a magnetic moment given by

$$\mu_k = \frac{g}{2m} \sigma_k$$

as predicted for a relativistic fermion (in absence of radiative corrections)

This is a consequence of the use of eqs. (1) for the spinor field and of minimal coupling leading to eqs. (7).

We can introduce antifermion fields  $\lambda$  and  $\zeta$  satisfying eqs.(7) with  $g$  replaced by  $-g$ . They obey separate conservation laws.

It remains to determine the equations for the gauge fields. As a first step we define the tensors

$$\begin{aligned} F_{ok}^a &= \partial_t A_k^a + \partial_k V^a + gf^{abc} A_k^b V^c, \\ F_{ij}^a &= \partial_i A_j^a - \partial_j A_i^a + gf^{abc} A_i^b A_j^c, \end{aligned} \quad (14)$$

which have the following Galilei transformations

$$F_{ok}^a \rightarrow F_{ok}^a + v_i F_{ik}^a, \quad F_{ij}^a \rightarrow F_{ij}^a. \quad (15)$$

The unique gauge-invariant equation we can write for  $F_{ok}^a$  is identical to the relativistic one

$$D_k^{ab} F_{ok}^b = g(\psi^* \gamma_t^a \psi - \lambda^* \gamma_t^a \lambda). \quad (16)$$

This equation is Galilei-invariant provided

$$D_j^{ab} F_{ij}^b = 0. \quad (17)_1$$

This last equation is also a possible Galilei-invariant equation for  $F_{ij}^a$ . The other possible equation is

$$F_{ij}^a = 0. \quad (17)_2$$

As we see Galilei-invariance plus gauge invariance determine uniquely the theory but for the constraints. Note that  $F_{ij}^a$  cannot be coupled to the current  $J_k^a$  because under Galilei transformations

$$J_k^a \rightarrow J_k^a + v_k \rho^a, \quad (18)$$

and with the tensors (14) and the covariant derivatives (11) we cannot construct anything transforming like that.

The above equations must be supplemented by the gauge conditions

$$\partial_k A_k^a = 0 . \quad (19)$$

It is perhaps worth while noticing that although eqs. (17) are time independent we are not in a static limit because  $\partial_t A_k^a$  enter eqs. (16).

### 3. - THE $c \rightarrow \infty$ LIMIT.

We will write relativistic quantities in italic. Let us start by the tensors

$$\mathcal{F}_{oi}^a = \frac{1}{c} \partial_t \mathcal{A}_i^a - \partial_i \mathcal{A}_o^a + \frac{1}{c} g^{abc} \mathcal{A}_o^b \mathcal{A}_i^c , \quad (20)$$

$$\mathcal{F}_{ik}^a = \partial_i \mathcal{A}_k^a - \partial_k \mathcal{A}_i^a + \frac{1}{c} g^{abc} \mathcal{A}_i^b \mathcal{A}_k^c .$$

If we take the limit  $c \rightarrow \infty$  other things being finite, we loose the gauge invariance. This is mantained if performing the limit we rescale the potentials according to

$$\mathcal{A}_o^a = - V^a , \quad \mathcal{A}_i^a = c A_i^a . \quad (21)$$

With the above definitions

$$\mathcal{F}_{oi}^a = F_{oi}^a , \quad \mathcal{F}_{ij}^a = c F_{ij}^a . \quad (22)$$

Now the relativistic field equations

$$\mathcal{D}_k^{ab} \mathcal{F}_{ko}^b = g \mathcal{Y}_o^a , \quad \mathcal{D}_k^{ab} \mathcal{F}_{ki}^b - \mathcal{D}_o^{ab} \mathcal{F}_{oi}^b = \frac{1}{c} g \mathcal{Y}_i^a , \quad (23)$$

can be rewritten

$$D_k^{ab} F_{ko}^b = g \varrho^a, \quad c D_k^{ab} F_{ki}^b - \frac{1}{c} D_t^{ab} F_{oi}^b = \frac{1}{c} g j_i. \quad (24)$$

Taking the limit  $c \rightarrow \infty$  we obtain eqs. (16) and the constraints  $(17)_1$ .

The constraints  $(17)_2$  can be obtained by performing the limit on the Lagrangian

$$\mathcal{L} = \frac{1}{2} \mathcal{F}_{oi}^a \mathcal{F}_{oi}^a - \frac{1}{2} \mathcal{F}_{ij}^a \mathcal{F}_{ij}^a + \mathcal{L}_F(-\mathcal{A}_0, \frac{\vec{\mathcal{A}}}{c}).$$

The limit on the fermion part  $\mathcal{L}_F$  can be performed in the standard way with the result

$$\begin{aligned} \mathcal{L}_F^{(0)} = & \frac{1}{2} \bar{\psi} i D_t \psi - \frac{1}{2} i D_t \bar{\psi} \psi + 2m \bar{\chi} \chi - \bar{\psi} i D_k \sigma_k \chi \\ & - \bar{\chi} i D_k \sigma_k \psi + (\psi \rightarrow \lambda, \chi \rightarrow \xi), \end{aligned}$$

so that for  $c \rightarrow \infty$

$$\mathcal{L} \longrightarrow \frac{1}{2} F_{oi}^a F_{oi}^a + \mathcal{L}_F^{(0)}(V, \vec{A})$$

with the constraints  $(17)_2$ .

One can ask whether a Lagrangian can be constructed which embodies the constraints by using Lagrange multipliers. For the constraints  $(17)_2$  the answer is positive. Introducing the auxiliary fields  $C_{ik}^a$  we have

$$\mathcal{L}^{(0)} = \frac{1}{2} F_{oi}^a F_{oi}^a - C_{ik}^a F_{ik}^a + \mathcal{L}_F^{(0)}(V, \vec{A}). \quad (25)$$

Variation of  $\mathcal{L}$  w.r. to  $\bar{\psi}$ ,  $\bar{\chi}$ ,  $V^a$  and  $C_{ik}^a$  gives eqs. (7), (16) and  $(17)_2$ . It is a peculiarity of the theory that these equations do not involve the auxiliary fields  $C_{ik}^a$ , and therefore separate from the equations obtained by



variation w.r. to  $A_k^a$ . These latter equations provide only the definition of  $C_{ik}^a$  in terms of the other fields.

The momenta conjugate to  $\chi$ ,  $\xi$ ,  $C_{ik}^a$  and  $V^a$  vanish. The dynamical variables are  $\psi$ ,  $\lambda$  and  $A_k^a$ . The momenta conjugate to  $A_k^a$  are

$$\frac{\partial \mathcal{L}}{\partial \partial_t A_i^a} = F_{oi}^a \quad (26)$$

as in the relativistic case. In the present case, however, in addition to the gauge constraints, the fields  $A_k^a$  must satisfy the constraints (17)<sub>2</sub>. Are these constraints which, enormously reducing the degrees of freedom make the nonrelativistic problem much easier.

Determining the independent variables may in general prove to be a difficult problem. Its solution becomes very simple in the abelian case.

#### 4. - THE ABELIAN CASE.

In the abelian case the constraints are

$$\partial_i A_k - \partial_k A_i = 0, \quad \partial_k A_k = 0, \quad (27)$$

In the presence of periodic boundary conditions they admit the solutions

$$A_i = \frac{1}{\sqrt{V}} q_i(t), \quad V = \text{quantization volume.} \quad (28)$$

The e m. field has only three degrees of freedom. The x-independence tells that it does not carry momentum.

According to eqs. (26) the conjugate momenta are

$$p_i = \partial_t q_i + \partial_i v. \quad (29)$$

Eliminating the auxiliary fields we obtain the Hamiltonian

$$\begin{aligned}
 H = H_R + \int d^3x \frac{1}{2m} (\partial_k \psi^* \partial_k \psi + \partial_k \lambda^* \partial_k \lambda) + \\
 + \frac{g^2}{8\pi} \int d^3x \int d^3x' [\psi^*(x)\psi(x) - \lambda^*(x)\lambda(x)] \cdot \\
 \cdot \frac{1}{|x-x'|} [\psi^*(x')\psi(x') - \lambda^*(x')\lambda(x')]
 \end{aligned} \quad (30)$$

where the radiation part is

$$H_R = \frac{1}{2} p_k^2 + \frac{1}{2} \omega^2 q_k^2 - \omega q_k I_k, \quad (31)$$

with

$$\omega^2 = \frac{g^2 N}{mV}, \quad N = \int d^3x [\psi^* \psi + \lambda^* \lambda], \quad (32)$$

$$I_k = \sqrt{\frac{m}{N}} \int d^3x \frac{1}{2mi} [\psi^* \partial_k \psi - \partial_k \psi^* \psi - \lambda^* \partial_k \lambda + \partial_k \lambda^* \lambda]. \quad (33)$$

It is convenient to introduce creation operators

$$a_k^+ = (2\omega)^{-1/2} (p_k + i\omega q_k) \quad (34)$$

in terms of which

$$H_R = \frac{3}{2} \omega + \omega a_k^+ a_k + (a_k^+ - a_k) i \sqrt{\frac{\omega}{2}} I_k. \quad (35)$$

$H_R$  describes photons of zero momentum and energy  $\sim \frac{1}{\sqrt{V}}$  interacting with the current  $I_k$ . It gives rise to divergences in perturbation theory due to vanishing energy denominators in the limit  $V \rightarrow \infty$ .

In the relativistic case the divergences associated with virtual photons and real photons cancel out leaving with a finite correction which exponentiates. The emission of real photons is the same as that of the classical current corresponding to incoming and outgoing particles. We will

show here that the radiation by a classical current is the same in the relativistic and nonrelativistic case.

Assuming  $I_k$  to be a c-number the S-matrix can be exactly determined<sup>2</sup>

$$S = \exp \left[ (a_k^+ + a_k) i \sqrt{\frac{\omega}{2}} \tilde{I}_k(\omega) \right], \quad (36)$$

where

$$\tilde{I}_k(\omega) = i \int_{-\infty}^{+\infty} dt e^{i\omega t} I_k(t). \quad (37)$$

The probability to emit  $n_k$  photons with polarization k is

$$\begin{aligned} P_{n_1 n_2 n_3} &= |\langle n_1 n_2 n_3 | S | 0 \rangle|^2 = \\ &= \prod_{k=1}^3 \frac{1}{n_k!} \left[ \frac{\omega}{2} \tilde{I}_k^2(\omega) \right]^{n_k} \exp \left[ - \frac{\omega}{2} \tilde{I}_k^2(\omega) \right], \end{aligned} \quad (38)$$

and the average energy radiated is

$$\begin{aligned} \bar{E} &= \sum_{n_1, n_2, n_3=0}^{\infty} P_{n_1, n_2, n_3} (n_1 + n_2 + n_3) \omega = \\ &= \langle 0 | S^{-1} H_R S | 0 \rangle = \frac{\omega^2}{2} \tilde{I}_k^2(\omega). \end{aligned} \quad (39)$$

We see that photon emission follows a Poisson distribution as in the relativistic case. If the current is that of a point charge moving with velocity  $\vec{v}$  for  $t < 0$ , being kicked at  $t = 0$ , and moving with velocity  $v'$  thereafter,

$$\tilde{I}_k(\omega) = \frac{v_k - v'_k}{\omega} \sqrt{m}. \quad (40)$$

In the limit  $V \rightarrow \infty$  the probability to emit any finite number of photons vanishes, but the probability to emit any number of photons in finite and the average radiated energy is

$$\bar{E} = \frac{1}{2} m (v_k - v'_k)^2. \quad (41)$$

Radiation of soft photons is therefore described as in the relativistic case, and gives rise to finite corrections to the cross sections which can be evaluated by means of the coherent states formalism<sup>3</sup>, which suggests itself in the present context.

## 5. - CONCLUSION.

Nonrelativistic gauge theories have been formulated, and in the abelian case they have been shown to have the same low energy properties of the corresponding relativistic theories. Due to their simplicity they can possibly allow the understanding of peculiar mechanisms which are reduced to their essential features. For instance the solution of the infrared divergence would of course have been much easier in the nonrelativistic context.

The theory is uniquely determined with the exception of the equations for the constraints. However we are interested in Galieli theories as a means to get insight into quantum gauge theories. From this point of view the relevant constraints are those obtained by the limit on the Lagrangian, as shown by the quantization of the abelian case. This also been checked<sup>4</sup> by performing the limit directly on quantum gauge theories.

## REFERENCES.

1. J. M. Lévy-Leblond, Commun. Math. Phys. 6, 286 (1967).
2. J. D. Bjorken and S. D. Drell, Relativistic quantum fields (McGraw-Hill, 1965), p. 202.
3. V. Chung, Phys. Rev. B140, 1110 (1965); M. Greco and G. Rossi, Nuovo Cimento 50, 168 (1967). For the non-abelian case see M. Greco, F. Palumbo, G. Pancheri-Srivastava and Y. Srivastava, Phys. Letters 77B, 282 (1978), where other references can be found.
4. F. Palumbo, to be published.