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RECENT PROGRESSES IN GAUGE THEORIES

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1. - INTRODUCTION

In the last years we have seen many developments in our understanding of gauge theories, especially toward the construction of new tools for doing reliable computations in the non-perturbative region. The motivations are clear: we believe that quantum chromodynamics (QCD) is the true fundamental theory for strong interactions; this interaction is characterized by an effective running coupling constant $\alpha(q^2)$ which goes to zero when $q^2 \rightarrow \infty$, i. e. at short distances (this property of the coupling constant is called asymptotic freedom)⁽¹⁾. More precisely, if the number of quark flavours is four, we find:

$$\alpha(q^2) \rightarrow \frac{12}{25 \ln(q^2/\Lambda^2)} \quad q^2 \rightarrow \infty . \quad (1.1)$$

The parameter Λ^2 is experimentally measured (e. g. in deep inelastic scattering) and it should be in the range 0.2-0.5 GeV². QCD is a complete theory of strong interactions: using Λ^2 and the quark masses as input, we should be able to compute all the physical quantities, in particular the mass spectrum of hadrons. However perturbation theory can be used only to compute hard processes (the coupling constant being small) and non-perturbative techniques are badly needed in the soft region.

Many interesting results have been obtained; for lack of time it is impossible to mention all of them: in this talk I will speak only those ideas which are more familiar to me.

For the time being most of the theoretical effort has been concentrated on the study of pure gauge theories without fermions, where only double coloured gluons interact; fermion should be included perturbatively at a later stage. In such a simplified theory Λ is the only parameter: we want to compute the static potential between quarks ($V(r)$) and the glueball spectrum. Other quantities, like the energy dependence of the total cross section for glueball scattering, are more difficult to obtain⁽⁵⁾. In this talk I will try to give you a rough idea of how to carry on these non-perturbative calculations. In Section 2, I present the formalism of Euclidean quantum field theory, which is essential to master most of the new developments. In Section 3 I discuss

some qualitative ideas on confinement. Most of the efforts to obtain quantitative results can be divided into two categories: "brute force" computations and analytic computation. Brute force computations are usually very long and the results can be obtained only after spending a lot of human or computer time. They are mainly based on lattice gauge theories (Section 4) and can be divided into two large groups: computer simulations (Section 5) and high temperature expansions (Section 6). Some of the difficulties to deal with the high temperature expansion are connected with the possible existence of an elusive roughening transition (Section 7).

The most interesting analytic approach is based on the idea of writing equations for $W(C)$, the vacuum expectation value of the Wilson loop:

$$\overline{W(C)} = \langle W(C) \rangle ; \quad W(C) = \text{Tr} \left\{ P \left[\exp \left(i \oint_C A_\mu(x) dx_\mu \right) \right] \right\}. \quad (1.2)$$

One obtains equations which are rather difficult to be solved, however it is known that rather impressive simplifications are present in the limit in which the number of colours become infinite. This fact enable us to write simple closed equations for an $SU(N)$ in the limit $N \rightarrow \infty$ (Section 8). Although these equations seem formidable there is some hope that they can be solved, at least approximatively.

2. - EUCLIDEAN FIELD THEORY

After the first works by Schwinger⁽⁶⁾, it has strongly emphasized by Symanzik⁽⁷⁾ that there are deep similarities between quantum field theory and classical statistical mechanics: indeed a field theory defined on a D-dimensional Minkowski space (D-1 space directions, one time direction) is connected to the corresponding field theory on a D dimensional Euclidean space by an analytic continuation (Wick ro

tation): the Minkowski metric $x^2 = \sum_1^3 x_i^2 - x_0^2$ become the Euclidean metric $x^2 = \sum_1^4 x_i^2$ if we set $x_4 = ix_0$.

The equivalence of the two theories has very deep consequences: if we quantize an Euclidean field theory using the Feynman path integral formulation, we obtain a special kind of classical statistical mechanics. Now classical statistical mechanics is a much older discipline than relativistic quantum field theory and we have a much better physical intuition of it: statistical mechanics deals with probabilities, not with amplitudes as quantum mechanics and the variety of statistical systems is large also in everyday life.

Only the beginning of the seventies Symanzik ideas became popular and they started to be applied in rather different fields: after the key works of Nelson⁽⁸⁾ and of Osterwalder and Schrader⁽⁹⁾ and the beautiful results of Guerra, Rosen and Simon⁽¹⁰⁾ Euclidean field theory became an essential tool in the rigorous approach to the construction of an interacting quantum field theory⁽¹¹⁾.

In a different contest it was emphasized by Migdal and Poliakov⁽¹³⁾ that the problem of computing the critical exponents for second order phase transitions is connected to the control of infrared divergencies in a theory with massless particles. Somewhat later in a beautiful series of paper Wilson⁽¹⁴⁾ succeeded to compute the critical exponents; his approach was a combination of the block spin picture of Kadanoff⁽¹⁴⁾ and the renormalization group which was used by Gellman and Low⁽¹⁵⁾ to study the high energy limit of QED⁽¹⁶⁾.

In these last years we have seen a very fruitful cross-fertilization⁽¹⁸⁾ of statistical mechanics and quantum field theory⁽¹⁹⁾; from a conceptual point of view quantum field theory has started to be adsorbed in the general framework of classical statistical mechanics: this process is arrived to such a stage that it could be said (although it is not quite true) that quantum field theory is an high specialized branch of statistical mechanics.

To help the reader to orient himself in these recent developments I have inserted a table showing the relations between the main quantities in quantum field theory and the corresponding quantities in statistical mechanics.

In Table I the bracket indicates as usually the statistical expectation value, e. g.

$$\langle \varphi(x) \varphi(0) \rangle = \int d[\varphi] \varphi(x) \varphi(0) \exp(-\beta H) / \int d[\varphi] \exp(-\beta H) . \quad (2.1)$$

Let us see an example in details: we suppose that the field creates from the vacuum an infinite number of particles of mass m_n . In momentum space we can write:

$$G(p) = \sum_n C_n / (p^2 + m_n^2) . \quad (2.2)$$

In position space we obtain (to avoid Bessel functions let us consider the case $D=3$):

$$C(x) = \langle \varphi(x) \varphi(0) \rangle = \sum_n C_n \exp(-m_n |x|) / |x| . \quad (2.3)$$

If we know the function $C(x)$ analytically, it is trivial to compute all the m_n , however if $C(x)$ is only known numerically with some er-

TABLE I

Quantum field theory	Classical statistical mechanics
Minkowski space	Euclidean space
\hbar	$\beta^{-1} = kT$
\mathcal{L}	H
Feynman factor for amplitudes : $\exp(-i\mathcal{L}/\hbar)$	Boltzman factor for probabilities : $\exp(-\beta H)$
Sum of all vacuum to vacuum diagrams : $\int d[\varphi] \exp(-i\mathcal{L}/\hbar)$	Partition function : $\int d[\varphi] \exp(-\beta H)$
Vacuum energy	Free energy
Vacuum expectation value $\langle 0 A 0\rangle$	Statistical expectation value $\langle A \rangle$
Quantum fluctuations	Statistical fluctuations
Time ordered products	Simple products
Existence of a mass gap	Exponential decrease of correlations (Correlation length) ⁻¹
Mass	Correlation functions : $\langle \varphi(x) \varphi(0) \rangle \sim \exp - m x $
Green functions : $\langle 0 T[\varphi(x) \varphi(0)] 0\rangle \sim \sim \exp i m t $	
Changement of vacuum	Phase transition
Goldstone bosons	Spin waves
Decrease to zero the mass	Approach a second order phase transition
Free scalar bosons	Random walk (Free curves)
Scalar bosons with repulsive interaction	Self avoiding walk (Interacting curves)
Gauge theories	Interacting surfaces ?
Wightman axioms	Osterwalder and Schrader axioms
Cutoff	(Lattice spacing) ⁻¹
Hamiltonian	Transfer matrix
Instantons	Defects (vertices, dislocations)

rors, it may be not so simple to extract m_1 and the numerical evaluation of m_2 may easily present serious difficulties; however this is a practical problem in numerical computations and it does not involve questions of principles.

The Euclidean formulation of gauge theories does not present any special difficulty. In the SU(N) case (N = 3 is the physical one) the gluon field is a doubly coloured vector:

$$A_\mu^{a,b}, \quad a, b = 1, N$$

and b are the colour indices.

The lagrangian which describes a pure gluonic world is:

$$\mathcal{L} = \frac{1}{g^2} \int d^D x F_{\mu\nu}^2 \quad (2.4)$$

where g is the chromatic charge ($a = g^2/4\pi$) and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ (21). The presence of the commutator in the definition of $F_{\mu\nu}$ is the origine of the interaction among gluons. The factor $1/g^2$ seems to be unusual: using the same convention in electromagnetism, one would find:

$$\text{div } E/e^2 = \rho \quad (2.5)$$

ρ being the electron density. In other words we have set the electron charge equal to one and we have rescaled the electric field; the conventional electron field E_c can be easily obtained:

$$E_c = E/e \quad (2.6)$$

In order to define the theory in 4 dimension we must renormalize the coupling constant to avoid ultraviolet divergences; one finally obtains an effective running coupling constant which in the large q^2 region behaves as:

$$a(q^2) \sim 12\pi/(11N \ln q^2/\Lambda^2) \quad (2.7)$$

The only parameter of the theory is Λ ; dimensionless quantities are therefore fixed and cannot be changed by changing the coupling constant.

Another important feature of the theory is gauge invariance: in the electromagnetic case only the fields E and H and not the potentials A are well defined; the value of the potential in one point is arbitrary: it can be changed by a gauge transformation:

$$A_\mu^{(x)} \rightarrow A_\mu^{(x)} + \partial_\mu \lambda(x) . \quad (2.8)$$

Although in perturbation theory one usually remove this ambiguity by fixing the gauge⁽²²⁾, only gauge invariant quantities have a clear physical meaning.

All these features of the theory in the usual Minkowski formulation are also true in the Euclidean version of the theory.

As we said in the introduction our aim is to compute the glueball m_G mass and the static potential between quarks. It is easy to prove that in Euclidean space :

$$m_G = - \lim_{r \rightarrow \infty} \frac{1}{r} \ln \langle F^2(r) F^2(0) \rangle , \quad (2.9)$$

$$V(L) = - \lim_{T \rightarrow \infty} \frac{1}{T} \ln \langle W(C) \rangle , \quad C = T \times L$$

where $C = T \times L$ indicates that the circuit C is a rectangle of sizes L and T .

As we see it is rather clear how to extract the most physically interesting information from the Euclidean version of QCD.

3. - CONFINEMENT

It is clear that the Wilson loop $W(C)$ plays a very important role in the study of gauge theories : it is the most natural gauge invariant observable. Indeed in the abelian case $\Phi(C) = i \ln W(C)$ (i. e. $W(C) = \exp -i\Phi(C)$) is the flux concatenated with the circuit C .

The quantity $A(C) = \ln \langle W(C) \rangle$ ⁽²⁴⁾ is the contribution of the chromodynamic field to the action of a coloured particle having C as trajectory. $A(C)$ is well defined in perturbation theory⁽²⁵⁾ apart from a linear divergence proportional to the length of the circuit C , which corresponds to the classical self energy of the electron.

At the first order in perturbation theory (or in the abelian case), i. e. neglecting gluons self couplings, $A(C)$ is the standard electromagnetic self induction of the circuit C :

$$A(C) = e^2 \oint_C dx_\mu \oint_C dy_\mu \frac{1}{(x - y)^2} . \quad (3.1)$$

Perturbation theory will obviously break down at large distances. If the potential $V(L)$ increase at infinity like σL (confinement), we find that :

$$\overline{W}(C) = \langle W(C) \rangle \sim \exp -\sigma L \cdot T = \exp -\sigma S . \quad (3.2)$$

The decrease of the expectation value of the Wilson loop like the surface may be considered as a criterion for confinement. In such a situation we expect the formation of a physical string between the two quarks where the energy is concentrated. In the time evolution the string will describe a surface: in the Euclidean space we expect that there will be a region of space on which the increase in action will be concentrated. We can introduce the parameter $a(x, C)$ defined by⁽²⁶⁻²⁸⁾:

$$a(x, C) = \langle F^2(x) W(C) \rangle / \langle W(C) \rangle . \quad (3.3)$$

Intuitively $a(x, C)$ has the meaning of the increase of the action at the point x as effect of the Wilson loop: more precisely:

$$d/d(1/g^2) W(C) = \int d^4x W(C) \cdot a(x, C) . \quad (3.4)$$

If the theory confines we expect that there will be a region in which $a(x, C)$ is substantially different from zero this region become a surface of some thickness⁽²⁹⁾ having C as boundary in the limit in which the loop C becomes very large.

Perturbation theory tell us that at short distances $V(L)$ defined in eq. (2.9) behaves as $\alpha(1/L^2)/L$ and it does give us no informations on the large distance behaviour.

"Has anybody proved confinement in QCD?" is the standard question to the expert. However this is not the most important question; there are general arguments showing that or QCD confines, or gluons take mass like in the Higgs mechanism, or there are long range forces⁽³⁰⁾. This statement is very similar in spirit to the sentence "a material is solid or liquid or gas"; we need an explicit computation to find if Helium is liquid at zero temperature and it would very difficult to decide the issue using general theorems. Here the situation is the same: what we need in QCD, is efficient way to do computations in the low energy region: the output should be the whole mass spectrum.

Let me present for completeness a simplified argument which shows how confined may be realized. I will neglect for simplicity the non abelian character of the theory. We consider a flat surface of area S . We divide it in N smaller surfaces of area S/N . We obtain:

$$W(C) = \exp i\Phi(C) = \exp(i \sum_1^N \Phi_i) = \prod_1^N \exp(i\Phi_i) , \quad (3.5)$$

where Φ_i is the flux going through each of the smaller surfaces. We have to compute the statistical average of $W(C)$. Let us assume that the Φ_i^2 are statistically independent (so happens in two dimensional theories):

$$\langle \Phi_i \rangle = 0, \quad \langle \Phi_i^2 \rangle = f, \quad \langle \Phi_i \Phi_j \rangle = 0, \quad i \neq j \quad (3.6)$$

and let us postpone the discussion on the origine of (3.6).
Now confinement is trivial; indeed:

$$\begin{aligned} \langle \exp -i \Phi_i \rangle &\simeq \exp(-f/2), \\ \langle W(C) \rangle &= \prod_i^N \langle \exp i \Phi_i \rangle \simeq \exp - \frac{Nf}{2}. \end{aligned} \quad (3.7)$$

In other words the total flux $\Phi(C)$ is the sum of N statistically uncorrelated variables of zero mean and fixed variance; the central limit theorem tell us that the probability distribution of Φ is a gaussian with variance proportional to S :

$$P(\Phi) = \frac{1}{(2\pi f)^{1/2}} \exp\left(-\frac{N\Phi^2}{2f}\right). \quad (3.8)$$

We finally get:

$$\langle W(C) \rangle = \int d\Phi P(\Phi) \exp i\Phi = \exp - \frac{fN}{2}. \quad (3.9)$$

Confinement is a simple consequence of the large fluctuations of the flux concatenated to large circuits and the statistical independence hypothesis naturally lead to (3.8-9). Of course (3.6) is not true in perturbation theory, where the conservation law for $\nabla_\mu F_{\mu\nu} = 0$ gives strong constraints; beyond perturbation theory everything is possible, as it has been advocated by many authors^(31, 32), the main difference being in mechanism producing eq. (3.6) chromomagnetic; monopoles, dense instantons, merons, condensations of flux tubes have been suggested; all these approaches share have one common point: the practical impossibility of using them to obtain reliable quantitative answers⁽³³⁾.

In the next Section we shall see other approach which should be able to give quantitative predictions.

4. - LATTICE GAUGE THEORIES

In the standard formulation of the theory ultraviolet divergences are present; although these ultraviolet divergences can be removed in perturbation theory, in order to give a non perturbative definition of the theory it is better not to introduce them from the beginning. This can be easily done by using a cutoff M and send M to infinity only at the end (momenta greater than M are disregarded). This can be done by discretizing the Euclidean space introducing a lattice, the fields will be defined only on the points or links of the lattice. In any computation the momenta will be bounded inside the first Brillouin zone; if we consider an hypercubic lattice of spacing a , each component p will belong to the interval

$$\left[-\frac{\pi}{a}, \frac{\pi}{a} \right], \quad M \text{ being equal to } \frac{\pi}{a}.$$

In principle it is also possible to work in the real Minkowski space and to discretize only the space and not the time. This approach has been suggested long time ago, but only recently it has been strongly developed. The introduction of a lattice is a device for dealing only with a finite number of degrees of freedom: it is very similar to the introduction of a mesh of points for solving differential equations⁽³⁴⁾.

There are many ways in which one can write a field theory on the lattice, however it is better to conserve the symmetries of the original problem as far as possible. The symmetry we want to preserve here is gauge invariance. As a first step we must define the gauge fields A_μ and the Wilson loop $W(C)$. We associate to each link (i, k) of the lattice⁽³⁵⁾ a variable U_{ik} belonging to the group^(36,37):

it is the lattice equivalent of $\exp(i \int_i^k A_\mu dx_\mu)$ ⁽³⁸⁾. The Wilson loop is simple given by:

$$W(C) = \text{Tr} \prod_{(i,k) \in C} U_{ik}, \quad (4.1)$$

where the product runs over all the links belonging to the path C .

We notice that for a small loop $C_{\mu\nu}$ of area a^2 laying in the $\mu\nu$ plane we have in the continuum case:

$$W(C_{\mu\nu}) = 1 + ia^2 \text{Tr} F_{\mu\nu} - \frac{a^4}{2} \text{Tr} F_{\mu\nu}^2 + O(a^6). \quad (4.2)$$

We find:

$$F_{\mu\nu}^2 \simeq \frac{1}{a^4} [1 - W(C_{\mu\nu}) + \text{h. c.}] , \quad (4.3)$$

$$\int (\sum_{\mu, \nu} F_{\mu\nu}^2) d^D x = a^{D-4} \sum_P [1 - W(P) + \text{h. c.}] ,$$

where the sum runs over all the plaquettes P (faces of the cubes) of the lattice (W(P) is the Wilson loop associated to the plaquette P).

The final expression for the Partition function and the Wilson loop are:

$$\begin{aligned} Z &= \int dU \exp(-H/g_B^2) , \\ \langle W(C) \rangle &= \int dU \exp(-H/g_B^2) W(C) / Z , \\ H &= a^{D-4} \sum_P [1 - W(P) + \text{h. c.}] . \end{aligned} \quad (4.4)$$

Formally g_B^2 and $1/g_B^2$ play the role of the temperature and β respectively. The low coupling expansion (i. e. in powers of g_B) can be done using the saddle point method: Feynman rules can be derived^(40, 41); they are more complex than the usual ones: the interaction is non polynomial but no ultraviolet divergences are present, all momenta being bounded. In the limit a going to zero, one recovers the standard Feynman rules.

The bare coupling constant ($\alpha_B = g_B^2/4\pi$) is approximately the running coupling constant evaluated at $q^2 = \pi^2/a^2$. More precisely one obtains for the SU(2) group

$$\begin{aligned} \alpha(M^2) &= \alpha(\pi^2/a^2) = \alpha_B + H\alpha_B^2 + O(\alpha_B^3) , \\ \frac{11}{6\pi} \alpha(q^2) &= \frac{1}{\ln q^2/\Lambda^2 + \frac{102}{121} \ln \frac{11}{6\pi} \alpha(q^2)} , \end{aligned} \quad (4.5)$$

where the running coupling constant is defined in the momentum subtraction scheme and the constant H has been computed by Hasenfratz and Hasenfratz⁽⁴⁰⁾; ($H \simeq 3.39$).

From eq. (4.5) we trivially get:

$$\Lambda = M \left(\frac{11}{6\pi} \alpha(M^2) \right)^{-\frac{51}{121}} \exp \left(- \frac{3\pi}{11 \alpha(M^2)} \right) \simeq$$

$$\approx M \left(\frac{11}{6\pi} a_B \right)^{-\frac{51}{121}} \exp \left[-\frac{3\pi}{11} \left(\frac{1}{a_B} - H \right) \right]. \quad (4.6)$$

It is clear that the continuum approximation can be good only in the region $a_B H \ll 1$. We need of such a small value of a_B in order to apply perturbation theory: indeed a simple computation⁽⁴²⁾ shows that the mean value of the plaquette (U) (normalized to 1)⁽⁴³⁾ is equal to

$$U = 1 - \frac{3\pi}{4} a_B. \quad (4.7)$$

For a_B as small as 0.2, U is equal to only 60% of its free value. The presence of terms proportional to πa_B is typical of lattice gauge theories: the relevant expansion parameter is g_B^2 and not a_B/π .

H is a fundamental constant in the comparison of the results of the lattice theory with the continuum version; let us present a rough qualitative computation of H. We first notice that the renormalized charge is different from the free one also in the pure electromagnetic case, the origine of this difference is the non linearity of the lattice action. If a is not zero, the thermal fluctuations renormalize the charge; let us try to estimate this effect. In order to compute the renormalized charge, we must know the variation of the action with respect to an external perturbation; let us decompose the field A as $A_f + A_e$: A_f is the fluctuating part and A_e is the external field; $F_{\mu\nu}$ is essentially given by:

$$\frac{1}{2} \cos A = \frac{1}{2} (\cos A_f \cos A_e - \sin A_f \sin A_e). \quad (4.8)$$

If we do the mean over the fluctuating field A_f we get

$$\frac{1}{2} \cos A \sim \frac{\langle \cos A_f \rangle}{2} \cos A_e = \frac{\cos A_e}{\tilde{g}_B^2}, \quad (4.9)$$

$$\tilde{g}_B^2 = g_B^2 / U.$$

In other words the more appropriate expansion parameter should be:

$$\tilde{\alpha}_B = \alpha_B / U = \alpha_B + 0.75 \alpha_B^2 + \dots \quad (4.10)$$

This elementary computation give an estimate of H which is 70% of the correct value. Using the new variable $\tilde{\alpha}_B$ one gets:

$$\alpha(M^2) = \tilde{\alpha}_B + 1.03\tilde{\alpha}_B^2 + O(\tilde{\alpha}_B^3) , \quad (4.11)$$

$$A = 2.4 M \left(\frac{6\pi}{11\tilde{\alpha}_B} \right)^{-\frac{51}{121}} \exp\left(-\frac{3\pi}{11\tilde{\alpha}_B}\right) .$$

A good choice of the expansion parameter is very important: the two expression for A in eq. (4.6); which are equivalent in the limit $\alpha_B \rightarrow 0$, differ by a factor 10 for $\alpha_B \approx 0.15$. Although we need a formidable two loop computation to have reliable results, it may be useful to investigate the problem in an abelian theory where two loops computations are much simpler than in the non abelian case (in other words we should compute those diagrams giving contributions proportional to powers of $\frac{N^2-1}{N}\alpha_B$)(43).

5. - COMPUTER SIMULATION

In the last years the most spectacular results have been obtained doing computer simulations using the Montecarlo technique. Although the Montecarlo technique is time honoured in the framework of statistical mechanics(44), only recently Wilson(45) suggested to apply it to the study of gauge theories; let me spend some time to give a physical picture of the method.

Suppose that we consider a finite piece of the lattice: a cube of size d ($d \gg \lambda$). In this situation we have practically reached the thermodynamic limit $d \rightarrow \infty$; if periodic boundary conditions are used the corrections to the thermodynamic limit should be small as $\exp(-d/\lambda)$ (46).

Using the integral representation (4.4) the expectation value of the Wilson loop can be written as the integral over all the configurations of the fields in the cube. The number of fields N is of order $(d/a)^4$, so that also for small values of d the evaluation of the integral using the Simpson rule is practically impossible (i. e. the computer time needed is greater than the age of the universe). We need a method such that the time computer increase like N (or a small power of N) and not as $\exp(N)$.

In order to find the method we must go back in time and undo what Boltzmann and Maxwell did. Equilibrium statistical mechanics was introduced to study the large time behaviour of the system. Let us consider a classical example: we study the time behaviour of N

particle in a box, whose trajectories $x_i(t)$ ($i = 1 - N$) satisfy the Newton law :

$$m \ddot{x}_i = - \frac{\partial U}{\partial x_i} = F_i \quad (5.1)$$

where $U(x_1, \dots, x_N) = U[\mathbf{x}]$ is the interparticle potential.

Standard arguments based on ergodic theorems, tell us that in most of the cases after enough time the system will reach equilibrium; for large N the microcanonical distribution will be equivalent to the canonical distribution, given by the Boltzman factor. We finally find the highly non trivial result :

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau f[\mathbf{x}(t)] dt = \overline{f[\mathbf{x}]} = \frac{\int dx_1 \exp - \beta U[\mathbf{x}] f[\mathbf{x}]}{\int dx_1 \exp - \beta [U]} \quad (5.2)$$

The temperature T ($\beta = 1/kT$) can be computed as function of the energy E (which is a conserved quantity) using a thermometer ; in this case the momenta themselves may play the role of the thermometer ; indeed we know that :

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau \frac{p_i^2(t)}{2m} dt = \frac{\overline{p_i^2}}{2m} = \frac{3}{2} kT \quad (5.3)$$

Now the point of view of a computer is the opposite of Boltzman; the number of steps needed to solve the coupled Newton equations is of order $N \times T$, i. e. it increase linearly with N : the right hand side of eq. (5.2) is much easier to compute then the right hand side⁽⁴⁸⁾.

What happens for finite times ? Also at equilibrium random thermodynamic fluctuations are present ; we finally obtain :

$$\frac{1}{T} \int_0^\tau f(x) dt = \langle f \rangle + O(1/\tau^{1/2}) \quad (5.4)$$

The $\tau^{-1/2}$ law come from the mean of independent random fluctuations (the practically random behaviour of the deterministic system (5.1) is the basis of thermodynamics).

Of course we must understand how the time τ_e for which equilibrium is reached (i. e. the time for which $\tau^{-1/2}$ corrections are small), depends on N . The physical intuition tell us that if N is increased at fixed density and the potential is not pathological, equilibrium is reached locally in a time which is independent from N ⁽⁴⁹⁾.

Near a second order phase transition (when long range correlations, i. e. zero mass particles, are present) the $\tau^{-1/2}$ law is no more valid and the pace of the approach to equilibrium is much lower, this phenomenon being called critical slowing down⁽⁵⁰⁾. Large times are also needed to reach the equilibrium near a first order transition; the system may be locked into a metastable state until a fluctuation greater than a critical size, is formed and becomes the germ of the condensation⁽⁵¹⁾; this difficulty may be avoided by fixing the initial conditions in such a way that in half of the box there is one phase (gas) and in the other half there is the other phase (liquid) and studying the movement of the interphase boundary⁽⁵²⁾.

In other words, instead of using the Boltzmann integral representation, it is more convenient to introduce a fictitious time t , to write appropriate equations of motion and to study the large time behaviour of the system.

The equation of motion can be freely chosen, provided that thermal equilibrium is asymptotically reached. For example we can consider the same particles as in eq. (5.1) moving in highly viscous liquid at temperature T . One finds the Langevin equation^(50, 53):

$$\eta \dot{x}_i(t) = F_i[x(t)] + b_i(t) \quad (5.6)$$

$$\langle b_i(t) \rangle = 0, \quad \langle b_i'(t) b_j'(t') \rangle = \delta_{ik} \delta(t - t') B,$$

$$B = 2KT\eta,$$

where η is the viscosity, $b_i(t)$ are random gaussian variables, uncorrelated in time and represent the effect of the Brownian motion. The relation among the viscosity, the temperature and B dates back to Einstein⁽⁵⁴⁾.

It is intuitive that at large time the particles must go to thermal equilibrium, they are in contact with the liquid that plays the role of heat reservoir. The formal proof of this statement can be done using the Fokker Plank equation^(53, 55).

It is possible to do computer simulations based on the Langevin equation, however for practical purpose we must discretize the time interval and some errors are introduced. It is also possible to write random equation of motion for discrete times whose solution goes to equilibrium at large times: in this way we obtain the Montecarlo technique; in the limit of small steps the Montecarlo technique reduces to the Langevin equation: it can be considered as a discretized form (discretization in time) of the Langevin equation, such to preserve the asymptotic limit. From the physical point of view there is no substantial difference between the Langevin equation and the Montecarlo pro-

cedure⁽⁵⁷⁾: My impression is that the Montecarlo method seems to be faster while the Langevin equation have the advantage of having a simpler analytic form⁽⁵⁹⁾.

Now in the last year many computer simulations have been done and many very interesting results have been obtained; we start to have a good understanding of the physics with finite groups and of the problem of approximating a continuous group with a discrete subgroup⁽⁶¹⁾. For lack of time I will not discuss here this very interesting problematics, and I will present the results for the continuous groups (mainly SU(2) and U(1)).

Let me describe a typical computer simulation: we start with a lattice having cubic shape⁽⁶²⁾, the length of the cube range from 4 to 16 lattice spacings in most of the computations.

The first thing to do is to look for phase transitions by studying the temperature dependence of the internal energy (the expectation value of the Wilson loop around one plaquette) on the temperature⁽⁶³⁾. Phase transitions can be divided into two groups; first order: U is discontinuous but it is infinitely differentiable from both sides; it is believed that U can be approximatively analitically continued from each side beyond the transition point⁽⁶⁴⁾, giving the results for the metastable phase⁽⁶⁴⁾. If a second order phase transition the internal energy is continuous, there is no metastability, but only a "critical slowing down", and the internal energy has a singularity proportional to $|T - T_c|^{-\alpha+1}$ $\alpha = 2 - D\nu$ ⁽⁶⁵⁾, the energy connected correlation function (in the case of gauge theories the connected correlation functions of two plaquettes) decrease to zero like $r^{-2D+2/\nu}$. If α is negative it is very easy to see the transition by plotting U against T. For α strongly positive it is not so easy; it is also difficult to distinguish an higher power decrease of the correlation function⁽⁶⁶⁾ from an exponential decrease; it is therefore possible to miss a second order transition.

Let see some of the results: in Fig. 1 we have the internal energy of 5 dimensional SU(2) theory⁽⁴²⁾, some of the points are obtained decreasing the temperature, other by increasing it, so that we see an hysteresis loop which can be interpreted as a first order phase transition⁽⁶⁷⁾. The high temperature expansion (described in the next Section) tell us that the theory is confined at

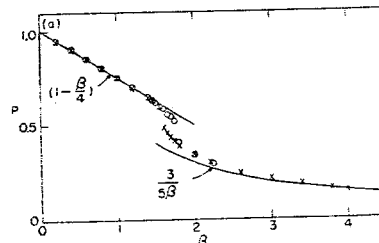


Fig. 1. The expectation value of a single plaquette P (defined in the text as $1-U$) as function of β in the 5-dimensional SU(2) gauge theory⁽⁴²⁾. Crosses heating; circles cooling.

high temperature, the absence of infrared divergences in the low perturbative expansion (the standard perturbative expansion) suggest us that we see the transition from the confined and the unconfined phase.

In Fig. 2 we have shown the same plot for the 4 dimensional U(1) theory: we still have a transition between the confined and the unconfined phase; a careful analysis shows that the hysteresis loop is not due to metastability, but to critical slowing down: the transition is a second order one $\nu \sim 1/3$ (69, 70).

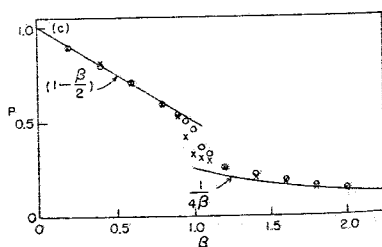


Fig. 2. The expectation value of a single plaquette P (defined in the text as $1-U$) as function of β in the 4-dimensional U(1) theory⁽⁴²⁾. Crosses heating; circles cooling.

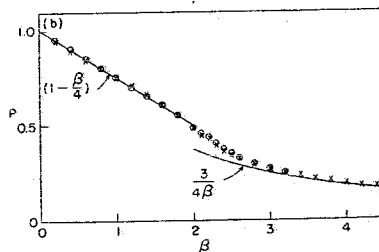


Fig. 3. The expectation value of a single plaquette P (defined in the text as $1-U$) as function of β in the 4-dimensional SU(2) theory⁽⁴²⁾. Crosses heating; circles cooling; no evident hysteresis loop is appreciable.

In Fig. 3 we see the internal energy for the more interesting case: the 4 dimensional SU(2) theory; there is no evidence for a phase transition, although the glitch around $\beta = 2$ ($\alpha = 1/2$) may suggest a second order transition with a strong negative value of α .

Let us see what happens to the Wilson loop in order to decide if the theory is confined also for $\beta > 2$. Let me recall what we expect in the confined phases for large r : for the static potential $V(r)$ and for the expectation value of a Wilson loop of size $L \times T$

$$V(r) \simeq \sigma r + \mu + \frac{\lambda}{r} \quad , \quad (5.7)$$

$$W(L, T) \simeq LT\sigma + \mu(L+T) + \ln(L+T) + f_c(L/T) + \dots$$

for $r \gg \Lambda^{-1}$. In the perturbative region $r \ll \Lambda^{-1}$ we get:

$$V(r) \sim \mu + \frac{\alpha}{r}, \quad (5.8)$$

$$W(L, T) \sim \mu(L+T) + \alpha f_p(L/T).$$

The presence of the last term in the confined case has been suggested in a beautiful fundamental paper of Lüscher, Symanzik and Weitz(71) and it is connected to the presence of oscillations of the string (as we shall discuss in the Section 7).

A careful analysis is needed to separate the term proportional to the surface from the other ones(72). This has been done by Rebbi(39) in the case of a discrete subgroup of SU(2) and by Creutz for the group SU(2)(73). According to our policy of not discussing the results for finite groups for lack of space (although the analysis is more accurate) we present only the results for the SU(2) group.

Let us consider the quantity:

$$R(L, \alpha) = \ln \left[\frac{\bar{W}(L, L) W(L-1, L-1)}{W^2(L, L-1)} \right]^2. \quad (5.9)$$

The contribution linear with L has been eliminated we expect that

$$\begin{aligned} R(L, \alpha) &\sim (a^2 \sigma + \lambda^2/L^2), & L \gg A^{-1}(\alpha), \\ R(L, \alpha) &\sim \alpha(\pi^2/L^2), & L \ll A^{-1}(\alpha). \end{aligned} \quad (5.10)$$

The results are shown in Fig. 4. The straight line is a rough approximation to \bar{A} . The value of σ can be reasonably estimated in the region $\beta \ll 2.3$.

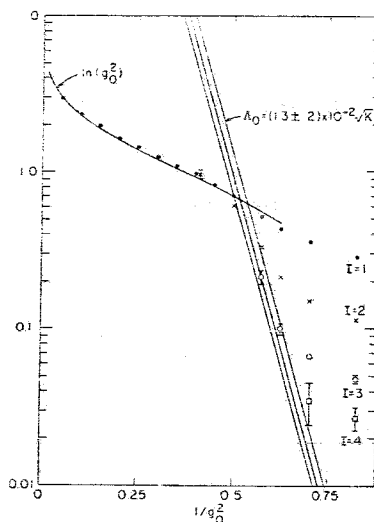


Fig. 4. The quantity $R(I, \alpha)$ defined in the text against $1/g_0^2 = \beta/4$ (73) for the 4-dimensional SU(2) theory.

We notice that in the region of small $\alpha_B = 1/\beta\pi$ and large L , the function $R(L, \alpha)$ should depend only on a renormalization invariant quantity⁽²⁷⁾:

$$\begin{aligned} \frac{L^2}{\alpha} R(L, \alpha) &= f(z), \\ z &= \frac{L^2}{\pi^2} \left(\frac{6\pi}{11} \frac{1}{\alpha(M^2)} \right) \exp\left(\frac{6\pi}{11\alpha(M^2)} \right) \end{aligned} \quad (5.11)$$

Standard perturbation theory allow us to compute the coefficients of the expansion in power of $\log(z)$ around $z=0$, while if eq. holds, in the large z region we have:

$$f(z) \rightarrow z + \text{Const} . \quad (5.12)$$

The function $f(z)$ control the cross over from perturbation theory to confinement: it would be rather interesting to extract it from the Montecarlo data⁽⁷⁴⁾ and to see if $\text{Const} \neq 0$.

It is rather unfortunately that the smallness of the lattice (8^4) prevents us from studying Wilson loop greater than 4 without feeling the effects of the periodic boundary conditions⁽⁷⁵⁾. However from these relatively small lattices we can definitely say that the SU(2) theory confines also in the small coupling region as it was predicted by the Migdal Kadanof recurrence equation⁽⁷⁶⁾: the exponential decrease of σ with a slope similar to the one suggested by the renormalization group is a greater succes of the theory. The drastic changement in the behaviour around $\beta=2$ ($\alpha = 1/2\pi$) can be easily understood: the quadratic terms dominates over the linear ones in eq. (4.5) and perturbation theory breaks down. The ratio σ/Λ^2 is found to be of order 1, although higher values are not excluded, given our ignorance on the precise form of Λ .

The glue-ball mass could be extracted by studying the decrease of the gauge invariant plaquette-plaquette connected correlation function

$$C(r) = \langle W(P(0))W(P(r)) \rangle - \langle W(P) \rangle^2 . \quad (5.13)$$

There are two difficulties. The connected term is defined as the difference of two terms of order 1 and it is very difficult to measure at distances greater than one⁽⁷⁷⁾; at high distances one sees no signal, only noise, unless one waits a very large time. Perturbation theory tell us that this correlation function decrease like a^2/r^8 at small distances⁽⁷⁸⁾; one should see a cross over from $1/r^8$ to $\exp(-M_{Gr})/r^{3/2}$. This cross over is invisible unless one knows the function with high accuracy.

Before changing argument let me mention a very interesting computation⁽⁷⁹⁾. It was suggested^(80, 81) that QCD in Minkowski

space has a phase transition from a confined to an unconfined phase when the temperature become greater than T_c (Here the temperature is really the temperature, measured in Kelvin). In the interpretation of ref. (80) this phenomenon is connected to the exponential increase of the hadronic mass spectrum and the Hagedorn limiting temperature should be interpreted as a phase transition⁽⁸²⁾.

Finite temperature field theories can also simulated on the computer using the same Montecarlo approach. This was done in ref. (79) where some estimate of T_c are presented; at this preliminary stage it is unclear if the transition is first or second: a more detailed analysis and more computer time is needed; the results are however very promising.

6. - THE HIGH TEMPERATURE EXPANSION

The high temperature expansion is a familiar technique in statistical mechanics⁽⁸³⁾: at infinite temperature the theory is trivial: entropy dominates over energy and no correlation is present between different variables, i. e. all connected correlation functions are zero. If the lattice hamiltonian is enough simple (nearest neighbour interaction is the ideal) is easy to develop any physical quantity (e. g. $f(\beta)$) in powers of β .

$$f(\beta) = \sum_k f_k \beta^k . \quad (6.1)$$

Under general conditions it is possible to prove that this expansion has a finite radius of convergence (that is not true with the usual perturbative expansion), i. e. the function $f(\beta)$ is analytic around $\beta = 0$ ⁽⁸⁴⁾. A well known theorem tell us that the knowledge of the function $f(\beta)$ around $\beta = 0$ (i. e. all the f_k) is sufficient to fix the function in the whole analyticity domain. The boundaries of the domain of analyticity are phase transitions for real temperatures and other physical uninteresting singularities for complex temperatures. If no phase transitions are present for real positive temperature⁽⁸⁵⁾, the behaviour at low temperatures can be extracted from the high temperature expansion!⁽⁸⁶⁾.

So long with the theorems; let us come back with the reality. One could think that it would be enough to compute some of the f_k , construct the Padé approximations to $f(\beta)$ and look to the computer output. Although this procedure is convergent it is highly inefficient; it is a general rule that it is difficult to extract "quantitative" informations from the high temperature expansion without a "qualitative" input⁽⁸⁸⁾. Let me give classical examples: we consider the three dimensional Ising model; the magnetic susceptibility $\chi(\beta)$ is believed

to have a singularity of the form $(\beta_c - \beta)^{-\gamma}$. It is pure nonsense to use the first 10 or 20 χ_k known to construct Padé approximant to χ and to fit the output in order to find β_c and γ . A Padé approximant has only simple poles and it does not approximate well a cut near the tip of the cut. It is much wiser to construct the logarithmic derivative

$$\text{ld}(\beta) = \frac{d}{d\beta} \ln \chi(\beta) . \quad (6.2)$$

$\text{ld}(\beta)$ has a simple pole at β_c (plus a subdominant cut) and the residuum is the critical exponent γ .

This example shows that it is much better to use approximants⁽⁸⁹⁾ which have automatically the correct singularity structure. The informations on the nature of the singularities should be deduced from physical arguments. If we do not have this informations, we should try to extract it from the k dependence of the f_k . The following Appel comparison theorem is usual very useful.

If the nearest singularity to the origine has the form $(\beta_c - \beta)^{-\gamma}$, the asymptotic behaviour of the f_k is given by :

$$f_k \propto k^{\gamma-1} |\beta_c|^{-k} \exp -i n \theta , \quad \beta_c = |\beta_c| \exp i \theta . \quad (6.3)$$

If the function is real and β_c is complex we must have a pair of complex conjugate singularities and $\exp -i n \theta$ is substituted by $\cos(n\theta + \varphi_0)$. If all terms are positive the nearest singularity is on the positive real axis. We can therefore use the ratio test :

$$\beta_c = \lim_{k \rightarrow \infty} R_k , \quad \gamma = \lim_{k \rightarrow \infty} \left[1 + k \left(\frac{R_k}{\beta_c} - 1 \right) \right] , \quad (6.4)$$

$$R_k = \frac{f_k}{f_{k-1}} .$$

We must now extrapolate the values for R_k and γ_k we have computed, up to $k = \infty$. If the sequence is smooth a fit with inverse powers of k normally give the correct result⁽⁹⁰⁾, if the sequence is not smooth, you are in trouble.

Without a preliminary estimation of the effective radius of convergence of the series there is the danger of using it in the region where they are not convergent or slowly convergent. Unfortunately in many paper I have seen on the high temperature expansion for lattice gauge theories this elementary precaution has not been taken and unreliable results have been obtained.

Life is not always so easy. Sometimes the nearest singularity is on the negative axis: however a simple conformal mapping like $\beta = \frac{z}{b+z}$ (91) may map far away the singularity on the negative axis and the physical singularity on the real positive axis become the nearest to the origine.

We must also add a word of caution; first order transitions are normally invisible in the high temperature expansion, one obtains automatically the analytic continuation of the free energy in the metastable region. Of course the combined use of the high and the low temperature expansion is very efficient to locate first order transitions.

Let us come back to gauge theories. At infinite temperature there is no correlation among variables at different points. The mass of the glue-ball (the inverse of the correlation length) is infinite in this limit. Condition (3.6) is clearly satisfied and the theory is confined. The construction of the high temperature expansion is straightforward: in order to compute the term proportional to β^k in the free energy, we must count how many closed surfaces of k plaquette can be imbedded on the lattice and weight each surface with a group theoretical factor depending on the topology of the surface (92).

Before analyzing real interesting series for 4 dimensional models it is wise to study what happens on an hypercubic lattice when the dimension of the space (D) goes to infinity. In this situation the high temperature expansion can be exactly summed up (93); in the $SU(2)$ case (and also in the Z_2 case) one finds (94)

$$U(\beta) = \beta \left[u_0 (\beta^2/D)^{1/2} + \frac{1}{D^{1/2}} u_1 (\beta^2/D)^{1/2} + \dots \right] \quad (6.5)$$

The functions u_0, u_1, \dots can be explicitly computed. When D goes to infinity one finds two singularity $\beta^2 = \pm 1/D^{1/2}$; the $1/D$ corrections show that the negative β^2 singularity is the nearest to the origine. As explained in details in ref. (93), near the transition point, the surface, on which is concentrated the energy in presence of a large Wilson loop, is no more flat, there are many tree like deformations, which form a branched polymer of cubes. At the transition point, the length of this tubes arrives to infinity, their thickness remaining fixed; we can call this transition a "local roughening" transition, because the deformation can start from a small size of the surface. The critical exponent α is $1/2$.

A more careful analysis shows that there is a first order phase transition at $\beta = 1/D$ which separate the confined and the unconfined phases (39, 93). The second order transition is in the metastable phase and it is a virtual transition not directly relevant for the physics.

Decreasing the dimensions the two transitions become rather near; the analysis is very clear in the 4 dimensional Z2 theory: there is a first order transition at the self dual point(37, 95, 96) and there are two virtual second order transition very near: about $\beta_{II} \approx 0.49$ and 0.40 respectively, as can be easily seen using the ratio test and conformal mapping(97).

I have analyzed the 4-dimensional high temperature expansion for $U(\beta)$ in $SU(2)$ up to β^{15} (39). A first sight one finds that a transition at $\beta \approx 2.15$ with $\alpha \approx 1/2$ is strongly suggested; however this result is in strong contrast with Montecarlo: if we resume the high temperature expansion according to this hypothesis the expression for $U(\beta)$ violently disagree with the Montecarlo results in the region near the critical temperature. In order to reach compatibility of the high temperature expansion with computer simulations we must decrease to 2 and to $-1/2$ (98). This drastic solutions still compatible with the first fifteen orders of ref. (37), however a preliminary analysis of the 21 orders of Wilson shows that the higher orders have the tendency of preferring higher still values of β and α (100). Indeed a simple minded analysis would suggest $\beta_c \sim 3$ and $\alpha = 2$ which is a pure nonsense. The only deceiving conclusion is that it is impossible to use the high temperature expansion to extrapolate beyond $\beta \approx 2.1$ unless 50 orders are computed. Similar conclusions can be extracted from the high temperature expansion for σ (101) and from $a(x)$ (28) defined in eq. (3.3).

The reason for this debacle is clear. In the mean field theory the deformations of the surface have Hausdorff dimensions 4(93, 102), as ordinary branched polymers, in other words there radius increase like $N^{1/4}$, N being the number of steps(103). This means that these polymers of cubes, if self-repulsion is neglected, would be strongly overlapping. The self-repulsion, due to the non linear superposition of fluxes, decrease the phase space allowed to these deformation and probably forbids the transition(93). In the high temperature expansion this effect comes from excluded volume effects: however to construct a diagram of a tube which bends back with the two ending point touching we must go to the 17th order for $U(\beta)$. In other words the effect that is potentially able to stop the transition appears only at rather high orders and low order computations do not contain enough informations on what happens beyond the would be transition point.

It may be possible that better results are obtained if we use all the informations we have on the high temperature expansion at arbitrary non integer dimensions; e. g. we set $D = 2 + \frac{2\beta^2}{1+\beta^2}$ and we extrapolate at $D = 4$ to the point $\beta^2 = \infty$ by changing D together with β^2 . This procedure avoid us to pass near the region $D = 4$, $\beta \approx 2.1$ which is not smooth. We can also try more fancy parametrizations like $D = 4 - \frac{8\beta^2}{16+\beta^4}$ to outflank the non existing local roughening

transition.

The impossibility of extracting informations for the behaviour in the low coupling region from the high temperature expansion, should dissuade from using shorter series in the same region where longer series for $U(\beta)$ do not give the correct result.

As usually happens, it is possible that using unjustified procedures one gets the correct order of magnitude, only because it is very difficult to make errors in the order of magnitude; however I want to recall the attention of the reader on ref. (104, 105): in these two papers the same high temperature expansion is used to extrapolate at zero temperature and the results for a quantity like σ/Λ^2 differ of a few orders of magnitude. A blind use of the matching conditions may give rather serious errors.

7. - THE ROUGHENING TRANSITION

In the strong coupling limit the surface associated to the Wilson loop is very rigid, the surface tension being very high ($\sigma \sim -\ln\beta$); it coincides with the minimal area surface and no fluctuations are present. If we consider a Wilson loop on the x_1, x_2 plane the parameter $a(x)$ introduced in eq. (3.3) will be different from zero only on this plane:

$$a(x_1, x_2) = \delta(x_{\perp}). \quad (7.1)$$

By decreasing the temperature, the surface tension decreases and the surface is no more flat. The simplest defect consists of shifting one lattice element in one of the $2(D-2)$ directions and adding four more plaquettes. The probability for this deformation is proportional to $2(D-2)\beta^4 \approx \lambda$. When the parameter λ becomes of order 1 the surface full of defects; each of them corresponding to placing a three dimensional cube on the surface in one of the $2(D-2)$ allowed directions⁽¹⁰⁶⁾.

If two near by cubes are parallel they gain a factor β^2 in energy but they loose a factor $2(D-2)$ in entropy⁽⁹⁶⁾. The probability of being parallel and vanish when the dimension go to infinity⁽¹⁰⁷⁾ at fixed λ , for lower dimensions it is possible that the cube organize themselves and their directions start to be correlated on large scale⁽¹⁰⁸⁾. In this case we can speak of "global roughening" while if the direction of the cubes are not correlated we have a "local roughening".

A local roughening transition happens when adding cubes on the top of cubes the deformation may arrive to infinity (see Fig. 5). At the transition point one would find for Wilson loops of any size:

$$a(x) = \delta(x_1) + \tilde{a}(x), \quad (7.2)$$

$\tilde{a}(x)$ going slowing to zero at infinity.

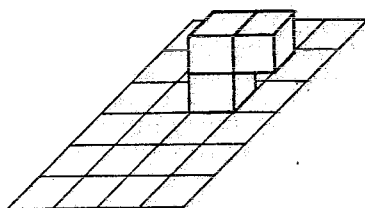


Fig. 5. A typical deformation of a surface (hangover) near a local roughening transition.

For a local roughening transition all thermodynamic quantities, in particular the expectation value of Wilson loops of any size, are singular.

A global roughening transition^(28, 30) can be defined only in the limit of infinite size of the Wilson loop: the surface tension is singular but no singularity is present in the expectation value of the Wilson loop of finite size. In the case of a Wilson loop of size $L \times L$, the function $a(x)$ should be substantially different from zero inside a region having transversal dimensions $R(L)$, $R(L)$ going to infinity with L .

Both transitions have their correlative in dual models. The local roughening transition correspond to the tachion of the conventional dual models⁽⁹³⁾ and the global roughening transition is responsible of the last term in eq. (5.7) which was found in ref. (71) using a new solution of the equations of motions of the string, which should not suffer of the drawbacks of the conventional solution; according to this analysis the function $R(L)$ should behave like $\ln L$ ⁽³⁰⁾.

In 4-dimensional $SU(2)$ theories the high temperature expansion suggest the presence of a local roughening transition at $\beta \approx 2.15$, however as we said in the previous Section, this indication cannot be taken seriously. It is also possible that the self-repulsion effect transform an incipient local roughening transition in global one⁽¹¹⁰⁾. This issue may decided only if good quality compute simulations for the quantity $a(x)$ are available.

In 3-dimensional $SU(2)$ theories the situation is much clearer. No singularity is seen in the high temperature expansion for the internal energy⁽¹¹¹⁾, although the extrapolation at zero temperature is problematic, while the surface tension clearly shows a singularity at about $\beta = 1.4$ ⁽¹¹²⁾. Summarizing, we have a good evidence for a global roughening transition in $D = 3$, a similar transition is likely present in the 4-dimensional case, but there is no serious evidence point.

8. - THE LARGE N EXPANSION

Many progresses have been done using the large N expansion. This technique has been introduced long time ago⁽¹¹³⁾ in statistical mechanics: the main idea is that if the number of colours (N) goes to infinity, it is possible to use statistical theorems also in colour space. In the simplest situation fields have only one colour index and the problem can be exactly solved⁽¹¹⁴⁾: field theory can be reduced to an integral equation (the Hartree-Fock approximation is correct). This large N expansion is a wonderful laboratory to study the formal properties of field theory such as infrared finiteness⁽¹¹⁵⁾, existence of non renormalizable interactions⁽¹¹⁶⁾, analyticity in the Borel plane⁽¹¹⁷⁾, etc.

It was remarked by t'Hooft⁽¹¹⁸⁾ if the field is double coloured (as the gluon)⁽¹¹⁹⁾, the theory can still be defined in the infinite N limit: remarkable simplifications are present, only planar diagrams survive⁽¹²⁰⁾. Although in this case the theory cannot be solved (exception done for some notable cases), it has been argued that in this limit only zero width resonances are present⁽¹²¹⁾. In other words if we write formally:

$$SU(N) = SU(\infty) + A_1/N^2 + A_2/N^4$$

the mass spectrum is contained in the leading term ($SU(\infty)$) the width of the resonances in the first correction (A_1/N^2) and the third term (A_2/N^4) gives (for $N=3$) 1% corrections to the mass spectrum, which hopefully can be neglected⁽¹²²⁾.

If we want to solve the $SU(\infty)$ theory, we must find something better than to sum all the planar diagrams; we can take advantage of the fact that statistical (quantum) fluctuations are strongly depressed in the limit $N \rightarrow \infty$. Indeed if we consider two quantities having a finite expectation value when N goes to infinity, we find the factorization property⁽¹²⁵⁾:

$$\langle AB \rangle = \langle A \rangle \langle B \rangle + O\left(\frac{1}{N^2}\right) . \quad (8.1)$$

No fluctuations are present (in other words the commutator A, B in the standard operatorial approach can be neglected). Although we are inclined to think that the saddle point method (or the classical equations of motion) must give the correct result, the situation is more complex. Let me spend some time to present the results for the quantum mechanics which, together with 2-dimensional QCD, is the better understood model.

In the first case we consider the hamiltonian

$$H = \frac{1}{2}P^2 + V(X^2), \quad P^2 = \frac{1}{N} \sum_i p_i^2, \quad X^2 = \frac{1}{N} \sum_i x_i^2 \quad (8.2)$$

i. e. central potential for a particle moving in N dimensions.

As can be checked in the harmonic oscillator, the variables are well defined in the limit $N \rightarrow \infty$ however the commutator P^2, X^2 vanishes when N goes to infinity.

The ground state energy is given by ⁽¹²³⁾:

$$E_0 = \min \left(\frac{1}{x^2} + V(x^2) \right), \quad (8.3)$$

i. e. the classical result, plus the centrifugal term for a N dimensional s wave.

In the second case the variables X and P are $N \times N$ matrices: the Hamiltonian is:

$$H = \frac{1}{N} \text{Tr} \frac{P^2}{2} + \text{Tr} [V(X)]. \quad (8.4)$$

The ground state energy is given by

$$E = \frac{1}{2\pi} \int h(p, x) \theta(\epsilon - h(p, x)), \quad h(p, x) = \frac{p^2}{2} + V(x), \quad (8.5)$$

where ϵ is fixed by the condition:

$$\frac{1}{2\pi} \int \theta(\epsilon - h(p, x)) = 1. \quad (8.6)$$

Eqs. (8.5) and (8.6) can be considered as a variation of the conventional WKB-Thomas-Fermi approximation.

Quantum effects do not disappear in the limit $N \rightarrow \infty$ but they can be easily computed. A better understanding of the physical origin of eqs. (8.5) and (8.6) (it is not self evident that they describe the sum of all the planar diagrams), would be very useful, especially for extending these results to real scalar field theories in higher dimensions ⁽¹²⁵⁾.

Fortunately the simple geometrical interpretation of gauge theories allow us to write directly useful equation in the limit $N \rightarrow \infty$. The fundamental variables are the expectation values of the Wilson loops associated to an arbitrary path C . It is convenient to introduce the functional derivative $\frac{\delta}{\delta \sigma_{\mu\nu}(x)}$ $W(C)$, which quantifies the variation of $W(C)$ under an infinitesimal deformation of the path C

around the point x ⁽¹²⁷⁾. It was shown by Mandelstam⁽¹³¹⁾ that:

$$\frac{\delta W}{\delta \sigma_{\mu\nu}(x)} = \text{Tr} \left[P(F_{\mu\nu}(x) \exp i \oint_x^x A_\mu(z) dz_\mu) \right]. \quad (8.7)$$

We can now transcribe the "Maxwell" equations ($D_\mu F_{\mu\nu} = J_\nu$, $D_\mu \tilde{F}_{\mu\nu} = 0$) in functional equations for $W(C)$ ⁽¹³²⁾; we find in the limit $N \rightarrow \infty$ ⁽¹²⁸⁾:

$$\begin{aligned} \partial_\mu \frac{\delta}{\delta \sigma_{\mu\nu}(x)} \bar{W}(C) &= \int dy_\nu \delta(x-y) W(C_{xy}) W(C_{yx}) \\ \partial_\rho \frac{\delta W}{\delta \sigma_{\mu\nu}} \varepsilon_{\rho\mu\nu\lambda} &= 0. \end{aligned} \quad (8.8)$$

The δ -function implies that the points x and y coincides: the loop C looks like an eight, C_{xy} and C_{yx} are the two smaller loops into which the eight may be decomposed.

It is very important to understand the physical meaning of eq. (8.8).

A serious step in this direction has been done by Foester and by Migdal and Makeenko. These last two authors have been shown that an approximate solution of the eqs. (8.8) satisfy the integral equation⁽¹³⁴⁾:

$$\begin{aligned} \frac{\delta}{\delta \sigma_{\mu\nu}(x)} \ln \bar{W}(C) &= \int dy_\nu dt dP_{xy}^t [\omega] \omega_\mu(t) \cdot \\ &\cdot \frac{\bar{W}(C_{xy\omega}) \bar{W}(C_{\omega xy})}{\bar{W}(C)} - (\mu \leftrightarrow \nu), \end{aligned} \quad (8.9)$$

where $dP_{xy}^t [\omega]$ stands for the sum over all the paths $\omega(t')$ going from x to xy in "time" t (Wiener measure); the point x and y divide the closed path C into two open paths C_1 and C_2 , $C_{xy\omega}$ and $C_{\omega xy}$ denote the two closed path obtained by adding ω to C_1 and C_2 .

The meaning of eq. (8.9) is clear⁽¹³⁵⁾; $\delta \ln \bar{W} / \delta \sigma_{\mu\nu}$ is roughly speaking the field induced at the point x by the Wilson loop: gluons coming from any point (y) of the loop contribute to it; in the abelian case, the factor $\bar{W}(C_{xy\omega}) \bar{W}(C_{\omega xy}) / \bar{W}(C)$ is substituted by 1 and the photon trajectories are free one, in the non abelian case the trajectories of the gluons are strongly influenced by the presence of

the Wilson loop.

The most interesting result is contained in the last paper of Migdal⁽¹³⁶⁾: he shows that the solution of eq. (8.8) can be written as a two dimensional field theory involving both bosons and fermions⁽¹³⁷⁾. For lack of space I cannot enter in the details and the reader is strongly recommended to read the original literature on the subject.

I believe that this field is still in its infancy and that many very spectacular results are waiting us in the next future.

FOOTNOTES AND REFERENCES

- (1) - Long time ago, Landau⁽²⁾, using general arguments, suggested that strong interaction should be asymptotically free: however the only asymptotically free theory known at that time was the $g\phi^4$ with negative coupling (Landau's arguments were too advanced for the time and they have been forgotten; in modern times the first example of an asymptotic free theory was given by Symanzik⁽³⁾: he also stressed the importance of being asymptotically free). A modern discussion of Landau philosophy can be found in ref. (4).
- (2) - L. D. Landau, A. A. Abrikosv and I. M. Khalatnikov, Dokl. Akad. Nauk USSR 95, 773, 1177 (1954); 96, 261 (1954); L. D. Landau and I. Pomeranchuk, Dokl. Akad. Nauk USSR 102, 489 (1955); L. D. Landau, Niels Bohr and the Development of Physics, ed. by W. Pauli (Pergamon, 1955).
- (3) - K. Symanzik, Lett. Nuovo Cimento 6, 420 (1973).
- (4) - G. Parisi, Phys. Rev. D11, 909 (1975); L. Maiani, G. Parisi and R. Petronzio, Nuclear Phys. B136, 115 (1978); N. Cabibbo, L. Maiani, G. Parisi and R. Petronzio, Nuclear Phys. B158, 295 (1979).
- (5) - V. Alessandrini and A. Krzywicki, Orsay Preprint LPTHE 80/24 (1980).

- (6) - J. Schwinger, Proc. Nat. Acad. Sci. (U.S.) 44, 956 (1958).
- (7) - K. Symanzik, J. Math. Phys. 7, 510 (1966); Rendiconti della Scuola Internazionale di Fisica "E. Fermi", Corso 45, Varenna 1968, ed. by R. Jost (Academic Press, 1969), pag. 152.
- (8) - E. Nelson, in "Constructive Quantum Field Theory", ed. by K. Velo and A. S. Wighman (Springer, 1973); See also: F. Guerra, Phys. Rev. Letters 28, 1213 (1972), and F. Guerra and P. Ruggero, Phys. Rev. Letters 31, 1022 (1973).
- (9) - K. Osterwalder and J. Schrader, Comm. Math. Phys. 31, 83 (1973).
- (10) - F. Guerra, L. Rosen and B. Simon, Comm. Math. Phys. 27, 10 (1972).
- (11) - Technically speaking the fact that probabilities (not amplitudes) are positive definite allow us to write very powerful inequalities (the use of inequalities is a very common procedure in statistical mechanics), e. g. the intuitive absence of a two particle bound state in presence of a repulsive interaction turns out to be a rigorous consequences of the Lebowitz inequalities⁽¹²⁾ for Ising spin system.
- (12) - For a review see: J. Glimm and A. Faffe, Cargese Summer School 1976 (Plenum Press, in press).
- (13) - A. M. Poliakov, Soviet Phys. -JEPT 28, 533 (1969); A. A. Migdal, Soviet Phys. -JEPT 22, 1036 (1969).
- (14) - K. G. Wilson and M. E. Fisher, Phys. Rev. Letters 28, 234 (1972); K. G. Wilson, Phys. Rev. Letters 28, 548 (1972); K. G. Wilson and J. Kogut, Phys. Rep. 12C, 77 (1974); L. P. Kadanoff, W. Götze, D. Hamblen, R. Hecht, E. A. S. Lewis, V. V. Palcianskas, M. Rayl, J. Swift, D. Aspnes and J. Kane, Rev. Mod. Phys. 39, 395 (1967).
- (15) - M. Gellman and F. Low, Phys. Rev. 95, 1300 (1954).
- (16) - The relevance of the renormalization group in the study of second order phase transitions was also stressed by Di Castro and Jona-Lasinio⁽¹⁷⁾.
- (17) - C. Di Castro and G. Jona-Lasinio, Phys. Letters 29A, 332 (1969); F. de Pasquale, C. Di Castro and G. Jona, Rendiconti della Scuola Internazionale di Fisica "E. Fermi", Corso 51, Varenna 1970, ed. by M. S. Green (Academic Press, 1971), pag. 123.
- (18) - D. Amit, The role of statistical mechanics in contemporary physics, presented at Camerino (1979), unpublished.
- (19) - For a more detailed discussions of the relations between statistical mechanics and quantum field theory at the beginning of the seventies see refs. (18, 20).
- (20) - A. Baracca, G. Parisi, L. Peliti, M. Rasetti e M. Valdacchino, Le transizioni di fase e i problemi attuali della fisica delle particelle elementari, in: A. Baracca: Manuale Critico di Meccanica Statistica (CULP, Catania), in press.

- (21) - I will use a matrix notation and I will not write the colour indices in most of the cases; here $[,]$ indicates the commutator.
- (22) - The Langevin equation formulation of field theory (see Sect. 5) may be used to construct a perturbative diagrammatic approach to gauge theories in which no gauge fixing is needed⁽²³⁾.
- (23) - G. Parisi and Wu Yong-shi, Preprint of the Institute of Theoretical Physics of the Academia Sinica ASITP-80-004 (1980); Scientia Sinica, to be published.
- (24) - Please notice that $\ln \langle W(C) \rangle$ is very different from $\langle \ln W(C) \rangle$.
- (25) - J. L. Gervais and A. Neveu, Nuclear Phys. B163, 189 (1980); Cargese Summer School 1979 (Plenum Press, in press); Phys. Rep., to be published.
- (26) - J. Groenvelde, J. Jurkiewicz and C. P. Korthals Altes, Phys. Letters 92B, 312 (1980).
- (27) - G. Mack and E. Pietarinen, Phys. Letters 94B, 397 (1980).
- (28) - A. Hasenfratz, E. Hasenfratz and P. Hasenfratz, CERN Preprint TH 2890 (1980); C. Itzykson, M. Peskin and J. B. Zuber, Saclay Preprint, to be published; M. Lüscher, G. Münster and P. Weisz, DESY Preprint 80/63 (1980); G. Münster and P. Weisz, DESY Preprint 80/74 (1980).
- (29) - As we shall see in the Sect. 7 we have two options: the thickness of the surface may go to a limit when the diameter L of the circuit goes to infinity, or it may go to infinity with L .
- (30) - G. 't Hooft, Nuclear Phys. B138, 1 (1978); B153, 141 (1979).
- (31) - C. Callan, R. Dashen and D. Gross, Phys. Rev. D17, 2717 (1978); A. Mandelstam, Phys. Rep. 23C, 237 (1977); A. M. Polyakov, Nuclear Phys. B120, 429 (1977); T. Banks, R. Myerson and J. Kogut, Nuclear Phys. B129, 493 (1977); G. Parisi, Frascati Preprint LNF-76/15 (1976); B. Glimm and A. Jaffe, Phys. Letters 73B, 167 (1978); A. Jaffe, Introduction to Gauge Theories, presented at Helsinki Conference on Mathematics on Physics, 1978; Harvard Preprint, unpublished; M. Stone and P. R. Thomas, Phys. Rev. Letters 41, 351 (1979); D. Foerster, Phys. Letters 76B, 597 (1978); T. Yoneya, Nuclear Phys. B144, 195 (1978).
- (32) - N. K. Nielsen and P. Olesen, Nuclear Phys. B144, 376 (1978); Phys. Letters 79B, 304 (1978); J. Ambjörn, N. K. Nielsen and P. Olesen, Nuclear Phys. B152, 75 (1979); H. B. Nielsen and N. Ninomiya, Nuclear Phys. B156, 1 (1979); H. B. Nielsen and P. Olesen, Nuclear Phys. B160, 380 (1979); J. Ambjörn and P. Olesen, Nuclear Phys. B170 (FS1), 60 (1980).
- (33) - The most serious effort to transform this qualitative model into a quantitative one has been done by the Copenhagen School (see refs.(32)).
- (34) - In the study of partial differential equations one approximates the derivative with a finite difference operator, however in many cases the most efficient way for solving differential equa

- tions is the finite element method; it would be nice to see if and how this method can be transferred to the field theory framework.
- (35) - i and k are two next neighbour points of the lattice.
 - (36) - K. G. Wilson, Phys. Rev. D10, 2445 (1974).
 - (37) - R. Balian, J. M. Drouffe and C. Itzykson, Phys. Rev. D10, 3376 (1974); D11, 2098 (1975); D11, 2104 (1975); D19, 2514 (1979).
 - (38) - The variables A belong to the Lie algebra, they are the generators of the Lie group; their exponent belongs to the group. Notice that in the lattice formulation of the gauge theory all variables belong to the group: it is possible to construct theories based on discrete groups for which no continuum formulation is possible (e. g. the group of rotations of a cube)⁽³⁹⁾. Although in many cases no theory is obtained in the continuum limit (the mass gap is always proportional to the cutoff), a notable exception is the three dimensional Z_2 theory⁽³⁷⁾.
 - (39) - C. Rebbi, Brookhaven Preprint BNL-27203 (1980); Phys. Rep., to be published; D. Petcher and D. H. Weingarten, Indiana Preprint (1980).
 - (40) - A. Hasenfratz and P. Hasenfratz, CERN Preprint TH 2827 (1980).
 - (41) - V. F. Müller and W. Rühl, Kaiserlautern Preprint (1980).
 - (42) - M. Creutz, Phys. Rev. Letters 43, 553 (1979).
 - (43) - We notice that for general N , $H = \frac{N^2 - 1}{2N} \pi + 0.51N$ and the pre factor 2.4 in eq. (4.11) is N independent.
 - (44) - N. Metropolis, A. W. Rosenbluth, A. H. Teller and E. Teller, J. Chem. Phys. 21, 1087 (1953); for recent applications see for example: K. Binder, in Phase Transition and Critical Phenomena, ed. by C. Domb and M. Green (Academic Press, 1976), vol. 5B; Montecarlo Methods, ed. by K. Binder (Springer, 1979); S. Kirpratick, Les Houches 1978, ed. by R. Balian, J. Meynard and G. Toulouse (North Holland, 1980).
 - (45) - K. Wilson, talk given at the Crete Summer School 1977.
 - (46) - More precisely they should be of order $\exp(-d/M_G)$ where M_G is the glueball mass. If we use the experimental information that M_G is at least $2-3 \text{ \AA}$ (the value $M_G = 1500 \text{ MeV}$ was suggested in ref. (47)), finite volume effects should be rather small for $d=4$).
 - (47) - G. Parisi and R. Petronzio, Phys. Letters 94B, 51 (1980).
 - (48) - This procedure is currently used and it is called molecular dynamics.
 - (49) - That is true if we do not store energy in coherent motion of the particles (i. e. macroscopic motion): it would take some time to dissipate it into heat via turbulence.
 - (50) - L. Van Hove, Phys. Rev. 93, 1374 (1954); See for a review: B. Halperin and P. C. Hoemberg, Rev. Mod. Phys. 49, 435 (1977).
 - (51) - J. S. Langer, Ann. of Phys. 41, 108 (1967).

- (52) - Notice that the speed of the interphase boundary goes to zero, near the critical temperature T_c , like a power of $|T - T_c|$.
- (53) - P. Langevin, Comptes Rendus 146, 530 (1908); A.D. Fokker, Ann. d. Physik 43, 812 (1914); M. Planck, Sitz. der Preuss. Akad. 324 (1917); For more recent references see for example: Noise and Stochastic Processes, ed. by N. Wax (Dover, 1954); R. F. Fox, Phys. Rep. 48C, 179 (1978).
- (54) - A. Einstein, Ann. d. Physik 17, 549 (1905); 19, 371 (1906).
- (55) - I want to profit of my position to suggest to the constructivists that stochastic differential equations (see in this respect ref. (56)) may be an alternative technique to construct an interacting field theory in a mathematically rigorous way.
- (56) - G. Parisi and N. Surlas, Phys. Rev. Letters 43, 744 (1979).
- (57) - A detailed description of the Montecarlo procedure technique can be found in refs. (44, 58).
- (58) - K. Wilson, Cargese Summer School 1979 (Plenum Press, in press).
- (59) - It has been shown in ref. (60) that the computation of correlation functions can be done with much higher accuracy using the Langevin equation.
- (60) - G. Parisi, Correlation functions and computer simulations, Frascati Preprint LNF-80/54 (1980).
- (61) - For example if we substitute to the group of rotation in the space $O(3)$ the 60 elements group of rotations of the icosaedrum, there will be a first order transition at a temperature T_c , i. e. at a coupling $\alpha = \alpha_1$; at high temperature ($\alpha > \alpha_c$) the two theories will be very similar, while at low temperature ($\alpha > \alpha_c$) the two theories will behave very differently⁽³⁷⁾.
- (62) - However for special problems it would be more convenient to use rectangular lattices.
- (63) - We will use the following definitions $T = g_B^2/4 = \pi \alpha_B$; $\beta = T^{-1}$
- (64) - Some care must be used in doing the analytic continuation; in reality the transition point is an essential singularity and the analytic continuation of U has an exponentially small imaginary part, proportional to the inverse of the mean life of the metastable state⁽⁵¹⁾.
- (65) - According to the original Erenfest classification second order transition are characterized by $\alpha \geq 0$, if $0 > \alpha \geq -1$ the transition is third order, the general rule is that the transition is of order k is the k 'th derivative of the free energy is discontinuous, the $(k - 1)$ 'th being continuous. However one often use the words "second order phase transitions" to indicate any transition of order higher than the first.
- (66) - ν must satisfy the bounds $1/D < \nu \leq \infty$
- (67) - Let me explain this mysterious sentence. In order to fasten the approach to equilibrium in Montecarlo simulations it is usual to start at an high temperature, reach the equilibrium, change the temperature and restart the Montecarlo simulation using

as starting point an equilibrium configuration of the previous temperature. It is also possible to go in the opposite direction, i. e. to start from low temperature and gradually increasing the temperature. In special cases it is convenient to have some oscillations in the temperature in order to anneal the defects⁽⁶⁸⁾ (a well known procedure in metallurgy).

- (68) - S. Kirpatrick and D. Sherrington, Phys. Rev. 17B, 4385 (1978).
- (69) - B. Lautrup and M. Nauenberger, CERN Preprint TH 2873 (1978).
- (70) - Although Figs. 1 and 2 look similar, longer computer simulations show a difference between the two cases^(42, 69).
- (71) - M. Lüscher, K. Symanzik and P. Weisz, DESY Preprint 80/31 (1980). It would be interesting to compare the results of this paper with D. J. Wallace and R. K. Zia, Phys. Rev. Letters 43, 808 (1971) and M. J. Lowe and D. Wallace, Edinburgh Preprint 80/110 (1980).
- (72) - For small α , only σ is exponentially small.
- (73) - M. Creutz, Phys. Rev. Letters 45, 313 (1980).
- (74) - It is also rather interesting to compute at least the first terms in the development of $f(z)$ in powers of $\ln z$ and to compare it with the Monte Carlo data.
- (75) - I am convinced that asymmetric lattices like $6 \times 6 \times 12 \times 12$ or $4 \times 4 \times 12 \times 12$ are more appropriate to obtain information on large Wilson loops. Moreover if one finds an observable dependence of the Wilson loop on the perpendicular boundary conditions, it would be rather interesting to analyze it.
- (76) - A. A. Migdal, Z. Eksper. Theoret. Fiz. 69, 810 (1975); 69, 1457 (1975); L. P. Kadanoff, Ann. Phys. 100, 359 (1976); S. Caracciolo and P. Menotti, Ann. Phys. 122, 74 (1979); CERN Preprint TH 2899 (1980); G. Martinelli and G. Parisi, CERN Preprint TH 2882 (1980).
- (77) - K. Wilson, C. Rebbi and M. Creutz, private communication.
- (78) - It would be rather interesting to compute the perturbative expansion also for this function.
- (79) - J. Polonyi, K. Szlachanyi and J. Kuti, Talk presented at this Conference.
- (80) - N. Cabibbo and G. Parisi, Phys. Letters 59B, 67 (1975).
- (81) - A. Poliakov, ICTP Lectures Notes taken by E. Gava, Report IC/78/4 (1978); W. Fischler, J. Kogut and L. Susskind, Phys. Rev. D19, 1188 (1979); T. Banks and E. Rabinovici, Nuclear Phys. B160, 349 (1979); E. Gava, R. Jengo and C. Omero, ICTP Preprint IC/80/77 (1980).
- (82) - To study the developments of the limiting temperature hypothesis would be a nice subject for an historical reconstruction of the attitude toward physical laws which was shared by many physicists at that time.
- (83) - For a review see: Phase Transitions and Critical Phenomena, ed. by C. Domb and M. Green (Academic Press, 1972), vol. 3.
- (84) - See for example: D. Ruelle, Statistical Mechanics (Benjamin, 1972).

- (85) - Negative temperature are well defined for bounded hamiltonian when the "field" variables are bounded: they are interesting because they correspond to antiordering.
- (86) - This approach has been advocated by ref. (87).
- (87) - L. I. Shiff, Phys. Rev. 92, 766 (1952); J. B. Kogut and L. Susskind, Phys. Rev. D11, 395 (1975); C. J. Hamer, J. B. Kogut and L. Susskind, Phys. Rev. Letters 41, 1337 (1978); T. Banks, S. Rabi, L. Susskind, J. Kogut, D. R. T. Jones, P. N. Sharback and D. H. Sinclair, Phys. Rev. D15, 1111 (1977).
- (88) - The Lord, whose oracle is at Delphes, neither says nor hides, but hints (Heracleitus).
- (89) - An approximant of order k ($f_k(\beta)$) is the only function, inside a given class, such that $f_k(\beta) - f(\beta) = O(\beta^{k+1})$. If the class of function is given by the polynomial of order k , we have the standard Taylor expansion; if it is given by rational functions, we have the Padé approximants.
- (90) - For a simple and efficient procedure to extrapolate see: J. Zinn Justin, Journ. de Physique 40, 969 (1979).
- (91) - The value of b can be chosen such to fasten the convergence. The choice of the appropriate conformal mapping is very important.
- (92) - J. M. Drouffe, Phys. Rev. D18, 1174 (1978).
- (93) - J. M. Drouffe, G. Parisi and N. Sourlas, Nuclear Phys. B161, 397 (1979); G. Parisi, in Proc. Third J. Hopkins Workshop on Current Problems in High Energy Particle Theory, Florence 1979, ed. by R. Casalbuoni et al. (J. Hopkins Univ., 1979), pag. 179; Cargese Summer School 1979 (Plenum Press, in press); J. M. Drouffe, Saclay Preprint (1980).
- (94) - In any dimensions $U(\beta)$ is an odd function of β for these two groups.
- (95) - M. Creutz, L. Jacobs and C. Rebbi, Phys. Rev. Letters 42, 1390 (1979); Phys. Rev. D20, 1915 (1979).
- (96) - J. M. Drouffe, Nuclear Phys. B170 (FS1), 79 (1980).
- (97) - I have done this computations using a pocket calculator. Using Padé approximants the higher singularity has been estimated at in the same region $\beta_{II} \approx 0.55^{(96)}$ and $\beta_{II} \approx 0.46^{(98)}$.
- (98) - N. Kimura, Hokkaido Univ. Preprint (1980).
- (99) - As clearly shown by eq. (6.4) the values of β_c and α are strongly correlated.
- (100) - I am grateful to K. Wilson for having communicated to me his results prior to publication. I believe that he has reached the same conclusions as me; however the responsability of any errors is only mine.
- (101) - G. Münster, DESY Preprint 80/44 (1980).
- (102) - P. G. De Gennes, Academie des Sciences (Paris) Preprint (1980); G. Parisi, Phys. Letters 81B, 327 (1979).
- (103) - B. Zimm and W. Stockmayer, J. Chem. Phys. 17, 301 (1949); T. Lubensky and J. Isaacson, Phys. Rev. Letters 41, 829 (1978).

- (104) - D. S. Fisher and D. R. Nelson, Phys. Rev. B16, 2300 (1977).
- (105) - G. Parisi, Phys. Letters 90B, 111 (1980).
- (106) - We can also add a cube on the top of an other cube, but let us neglect this possibility for the time being.
- (107) - In high dimensions this discussion is purely academic because λ is of order 1 only in the metastable phase.
- (108) - The argument on the infinite dimensional case should be more refined. The system of cubes is equivalent to a 2 dimensional Pott model with $q = 2(D - 2)$ states and it is known that for q greater than 4 a first order transition is present⁽¹⁰⁹⁾.
- (109) - R. G. Baxter, J. Phys. C6, L445 (1973); J. B. Kogut, Illinois Preprint (1980), to appear in Phys. Rev. ; P. Ginsparg, Y. Y. Goldschmidt and J. B. Zuber, Saclay Preprint DPh-T/63/80 (1980).
- (110) - The analogy with the Potts model suggests that the spontaneous orientation of cube-like deformations of the surface is a first order transition for $D > 4$ and it is a second order transition for $D \leq 4$.
- (111) - A. Duncan and H. Waidya, Phys. Rev. D20, 903 (1979).
- (112) - In this case after conformal mapping the ratio test gives good results in agreement with Padé approximants.
- (113) - A. Stanley, Phys. Rev. 176, 718 (1968); R. Abe, Progr. Theor. Phys. 48, 1414 (1972); G. Parisi and L. Peliti, Phys. Letters 41A, 331 (1972); M. Suzuki, Phys. Letters 45A, 5 (1972); E. Brezin and D. J. Wallace, Phys. Rev. B7, 1967 (1973); R. A. Ferrel and D. J. Scalapino, Phys. Rev. Letters 29, 413 (1972); S. Ma, Phys. Rev. Letters 29, 1361 (1972); K. Wilson, Phys. Rev. D4, 2911 (1973).
- (114) - The field transform as the fundamental representation of the $O(N)$ or $SU(N)$ group.
- (115) - A. Jevick, Phys. Letters 71B, 327 (1977).
- (116) - G. Parisi, Nuclear Phys. B100, 368 (1975); K. Symanzik, DESY Preprint 77/05 (1977); Y. Araf'eva, Theor. Math. Phys. 31, 279 (1977).
- (117) - G. Parisi, Phys. Letters 76B, 65 (1978); Phys. Rep. 49C, 215 (1978); Nuclear Phys. B150, 153 (1979).
- (118) - G. 't Hooft, Nuclear Phys. B72, 461 (1974).
- (119) - The field transform as the adjoint representation of the $O(N)$ or $SU(N)$ group.
- (120) - J. Koplizk, A. Neveau and S. Nussinov, Nuclear Phys. B123, 109 (1977).
- (121) - G. Veneziano, Nuclear Phys. B117, 719 (1976); A. A. Migdal, Ann. of Phys. 109, 365 (1977).
- (122) - In the quantum mechanics case studied in ref. (123), one find that the expansion parameter is about $1/N^2\pi^2$ (124). The first two terms give the value of the ground state energy with a relative error of 10^{-4} .

- (123) - E. Brezin, C. Itzykson, G. Parisi and J. Zuber, *Comm. Math. Phys.* 59, 35 (1978).
- (124) - E. Brezin, *Phys. Rep.* 49C, 221 (1978).
- (125) - E. Witten, *Nuclear Phys.* B160, 57 (1979).
- (126) - A. Jevicki and B. Sakita, *Nuclear Phys.* B165, 511 (1980); A. Jevicki and H. Levine, *Brown Preprints HET-418 and HET-419* (1980); T. Yoneya, *Tokyo Preprint* (1980).
- (127) - The precise definition of $\delta W / \delta \sigma_{\mu\nu}(x)$ can be found in refs. (128-130): the whole analysis becomes simpler on a lattice.
- (128) - Yu. M. Makeenko and A. A. Migdal, *Phys. Letters* 88B, 135 (1979); A. A. Migdal, *Chernolovka Preprint* 27 (1979); Yu. M. Makeenko, *ITEP Preprint* 141 (1979); Yu. M. Makeenko and A. A. Migdal, *ITEP Preprint* 23 (1980).
- (129) - D. Foerster, *Phys. Letters* 87B, 87 (1979); T. Eguchi, *Phys. Letters* 87B, 91 (1979).
- (130) - V. Volterra, *Rend. Lincei* 4a, III, 274 (1887), and 4a, VI, 127 (1890); A. T. Ogielski, *Brookhaven Preprint* (1980).
- (131) - S. Mandelstam, *Phys. Rev.* 175, 1580 (1968).
- (132) - This program was started in ref. (133).
- (133) - G. De Angelis, D. De Falco and F. Guerra, *Lett. Nuovo Cimento* 19, 55 (1977); Y. Nambu, *Phys. Letters* 80B, 372 (1978); A. M. Poliakov, *Phys. Letters* 82B, 247 (1978); J. L. Gervais and A. Neveu, *Phys. Letters* 80B, 255 (1979); E. Corrigan and H. Hasslacher, *Phys. Letters* 81B, 181 (1979).
- (134) - Yu. M. Makeenko and A. A. Migdal, *ITEP Preprint* 23, Sections 6-8 (1980), private communication from A. A. Migdal.
- (135) - I am trying to reproduce here what Migdal kindly explained to me more than one year ago at the Alustha Conference (1979).
- (136) - A. A. Migdal, *The Elf of the String*, to be submitted to *Phys. Letters*, private communication.
- (137) - The need of introducing Fermionic degrees of freedom in the description of the string in QCD was stressed by Foerster⁽¹³⁸⁾.
- (138) - D. Foerster, *Nuclear Phys.* 170B, 107 (1980).