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Self avoiding walk and supersymmetry

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Résumé. — Nous étudions la transition de phase dans le problème de la chaîne aléatoire répulsive dans l'approximation du champ moyen. Nous introduisons des variables anticommutantes (au lieu d'utiliser l'approche conventionnelle de la limite $n \to 0$) et une supersymétrie apparaît naturellement. Cette supersymétrie est brisée spontanément au point de transition de phase.

Abstract. — We discuss the phase transition of a self-avoiding chain in the mean field approximation. We find convenient to introduce anticommuting variables (instead of using the more conventional $n \to 0$ limit) in which case a supersymmetry naturally appears. The supersymmetry breaks down spontaneously at the phase transition.

The problem of self-avoiding walk is not only interesting as a mathematical problem but presents also a physical interest because of its connection with polymer physics (for a recent review see [1]). In fact, it is known that the long scale properties of polymer chains are similar to the properties of chains constructed from self-repulsing elements in a random way. The fundamental quantity of interest is

$$G(x, 0) = \sum_{\Gamma} \mathrm{e}^{-\mu l - g n(\Gamma)} \tag{1}$$

where the sum runs over all paths Γ going from 0 to x, l is the length of the path Γ and $n(\Gamma)$ is the number of times the path intersects itself. μ and g are numerical coefficients and the effect of g is to penalize the paths intersecting themselves. ($g = 0$ for ideal random chains).

We also denote by $G(0, 0)$ the sum of all closed paths going through the point O, and by $G_2(0, x)$ the sum of the closed paths passing through both points O and x. In $G(0, 0)$ and $G_2(0, x)$ the paths are weighted with the same weight as in eq. (1).

For μ larger than some critical value μ_c, $G(0, x)$ has an exponential fall-off as x increases. As $\mu \to \mu_c$, the fall-off of $G(0, x)$ follows a power law and this is a sign of a second order phase transition. For $\mu < \mu_c$ we expect that the dominant paths will fill up the whole space. So we expect that $G_2(0, x)/G(0, 0) \to \rho$. This means that each closed path passing through the point O, will pass also through any other point x with a non-zero probability ρ.

We expect ρ to vanish near the critical point and it could play the role of the order parameter of the problem.

Let us now try to get a qualitative estimate of $G(0, 0)$ for $\mu < \mu_c$ and for small g. In order to be more precise, we consider the case of the self-avoiding walk on a finite lattice of volume V. The limit $V \to \infty$ will be taken at the end. For a given configuration, we call $\gamma(i)$ the number of elements of the chain (which are links on the lattice) passing through the point i. The length l of the chain (in lattice units) and the number of intersections $n(\Gamma)$ are given by

$$l = \frac{1}{2} \sum_i \gamma(i) \tag{2}$$

$$n(\Gamma) = \frac{1}{8} \sum_i \gamma(i)\,(\gamma(i) - 2) = \frac{1}{2}\left(\sum_i \frac{1}{4}\gamma^2(i) - l\right). \tag{3}$$

Using the Schwartz inequality

$$n(\Gamma) \geqslant \frac{1}{2}\left(\frac{l^2}{V} - l\right) \tag{4}$$

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(where the volume V is also measured in lattice units).

In the case of small g we expect the chain to fill the whole space and to cross itself several times. So $\rho = l/V$ can be larger than one, in which case the previous inequality becomes useful. We assume that the inequality (4) is saturated.

At a first approximation

$$G(0,0) \sim \sum_l e^{(\mu_c - \mu)l - gl^2/2V}. \tag{5}$$

The sum will be dominated by the value

$$l_0 \sim (\mu_c - \mu)\frac{V}{g}$$

and

$$G(0,0) \sim \exp \frac{V}{2g}(\mu - \mu_c)^2 .$$

So we get the following estimate for ρ

$$\rho \sim \frac{\mu_c - \mu}{g} V . \tag{6}$$

These estimates are meaningful when far from the critical point ($\mu \ll \mu_c$).

It has been realized recently [2] that the statistics of the self-avoiding walk is closely related to the statistics of spin systems and is described by the following field theoretic model

$$G(0,x) = \lim_{n\to 0}\frac{1}{n}\int \mathfrak{D}\varphi \, e^{-\int dx \mathfrak{L}(\varphi)} \varphi_i(0)\,\varphi_i(x) \tag{7}$$

where

$$\mathfrak{L}(\varphi) = \phi(-\Delta + m^2)\,\phi + \lambda(\phi^2)^2 \tag{7a}$$

and $\phi(x)$ is a n component scalar field. The analogous expression for $G_2(0,x)$ is

$$G_2(0,x) = \lim_{n\to 0}\frac{1}{n} \times \int \mathfrak{D}\varphi \, e^{-\int dx \mathfrak{L}(\varphi)} \phi^2(0)\,\phi^2(x) . \tag{8}$$

In this expression the diagrams where the *spin* indices at the point x are contracted among themselves (and similarly at the point O) are proportional to n^2 and disappear in the limit $n \to 0$. The constant λ appearing in eq. (7*a*) is proportional to g (see eq. (1)) and m^2 is a smooth function of μ and g.

In eq. (7) the usual factor $1/Z$ where

$$Z = \int \mathfrak{D}\varphi \exp - \int dx\, \mathfrak{L}(\varphi)$$

of the correlation functions is absent. This is because

$$\lim_{n\to 0} Z = 1 . \tag{9}$$

In order to see this, it is convenient to use the identity

$$\int dx\, e^{-x^2 + i2\sqrt{\lambda}xy} = \sqrt{\pi}\, e^{-\lambda y^2} \tag{10}$$

and to write Z in the form

$$Z = c_0 \int \mathfrak{D}\varphi\, \mathfrak{D}\alpha \exp - \int dx\, \mathfrak{L}'(\varphi, \alpha) \tag{11}$$

where

$$\mathfrak{L}'(\varphi,\alpha) = \varphi(-\Delta + m^2)\varphi + 2i\sqrt{\lambda}\,\alpha(x)\varphi^2(x) + \alpha^2(x)$$

$$c_0^{-1} = \int \mathfrak{D}\alpha \exp - \int dx\alpha^2(x) . \tag{12}$$

Then the φ integration can be explicitly performed and

$$Z = c_0 \int \mathfrak{D}\alpha \exp - \int dx\, \alpha^2(x) \times \times \{ \operatorname{Det}(-\Delta + m^2 + i2\sqrt{\lambda}\,\alpha(x) \}^{-n/2} \tag{13}$$

(n is the number of components of φ), from which eq. (9) becomes obvious. $G(0,x)$ is then simply given by

$$G(0,x) = \int \mathfrak{D}\alpha\, e^{-\int dx \alpha^2(x)} \times \times \{ -\Delta + m^2 + i2\sqrt{\lambda}\,\alpha(x) \}^{-1}(0,x) . \tag{14}$$

The phase structure of this model for $n \geqslant 1$ is familiar from the study of magnetic systems.

In the mean field approximation and for $m^2 > 0$ the correlation length is proportional to $1/m$ and for $m^2 = 0$ we reach a power law decay of the correlation function. When m^2 becomes negative φ develops a non-zero expectation value $\langle \phi \rangle = \varphi_0$. This corresponds to the spontaneous breaking of the $\mathcal{O}(n)$ symmetry and to a spontaneous magnetization, and $n - 1$ long-range excitations (Goldstone modes, which are the spin waves) appear.

It is difficult to get an analogous qualitative understanding of the nature of the phase transition for the case of the self-avoiding walk, because of the $n \to 0$ limit which has to be taken. In particular, a negative number of spin waves will appear [4].

In this note, we want to stress the fact that the $n \to 0$ limit corresponds to a supersymmetry which is spontaneously broken at the phase transition point. The fact that one can avoid the $n \to 0$ limit by introducing anticommuting variables has been noticed independently by McKane [5].

In order to see this, we have to introduce anticommuting variables $\bar{\psi}^i(x)$, $\psi^i(x)$ (1). We define a new Lagrangian

(1) Anticommuting variables are well defined mathematical objects. Differentiation and integration over anticommuting variables are also well defined. See Ref. [3].

$$\mathcal{L}_{ss} = \phi(-\Delta + m^2)\phi + \bar{\psi}^i(-\Delta + m^2)\psi^i + + 2i\sqrt{\lambda}\,\alpha(x)\{\varphi^2(x) + \bar{\psi}^i(x)\psi^i(x)\} + \alpha^2(x) \quad (15)$$

and we claim that

$$G(0, x) = \int \mathcal{D}\varphi\, \mathcal{D}\psi\, \mathcal{D}\alpha\; e^{-\int dx \mathcal{L}_{ss}}\, \varphi^i(0)\, \varphi^i(x) \quad (16)$$

(for the definition of the integration over anticommuting variables see ref. [3]).

In order to verify eq. (16) the following property is useful :

$$\int \mathcal{D}\psi\; e^{\bar{\psi}^i_n M_{nm} \psi^i_m + \bar{\eta}^i_n \psi^i_n + \bar{\psi}^i_n \eta^i_n} = (\mathrm{Det}\, M)^N\, e^{-\bar{\eta}^i_n (M)^{-1}_{nm} \eta^i_m} \quad (17)$$

where N is the number of components of the field $\bar{\psi}^i$ ($i = 1, ..., N$) and of the field ψ^i. Eq. (16) becomes equivalent to eq. (14) provided the total number of anticommuting fields $2N$ equals the number of commuting fields n.

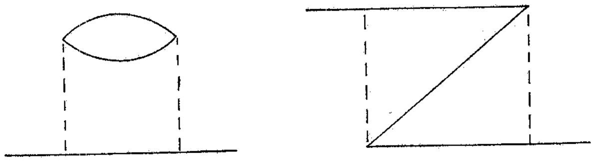
The Lagrangian $\mathcal{L}_{ss}$ is invariant under the transformations (generalized *rotations*)

$$\varphi^i \to a_{ij}\,\varphi^j + c_{ij}\,\psi^j\,, \quad \psi^i \to d_{ij}\,\varphi^j + b_{ij}\,\psi^j \quad (18)$$

which leave $\varphi^2 + \bar{\psi}\psi$ invariant.

In eq. (18) a and b are commuting numbers and c and d are anticommuting ones. These transformations, which mix commuting and anticommuting variables, are called supersymmetry transformations. For a review of supersymmetry see Ref. [6].

It is easy to see that the critical exponents obtained from this supersymmetric theory are the same as those obtained from the $n \to 0$ prescription. We illustrate this point with the lowest order diagrams which contribute to the anomalous dimension of φ (see Fig. 1).



(*a*) (*b*)

Fig. 1. — Feynman diagrams contributing to the anomalous dimensions (see text).

In figure 1 continuous lines represent φ propagators and dashed lines represent α propagators. The $n \to 0$ limit kills diagrams in which closed lines appear (diagr. (*a*) in figure 1) but has no effect on the other diagrams (like diagr. (*b*)).

Now, in the supersymmetric model no $n \to 0$ limit is involved but one has to add the diagrams with closed fermionic loops to the above bosonic diagrams. (These are the only diagrams in the absence of external fermionic sources). It is well known that each fermionic loop carries a (-1) sign and thus all the φ loops cancel against the fermionic loops and we obtain the same result as in the $n \to 0$ limit. So the m fermionic degrees of freedom are equivalent to $-2m$ bosonic degrees of freedom [7] (2).

We will now discuss some properties of the broken symmetry phase. For convenience's sake we consider the ground state structure for $n = 2$. In the mean field approximation the fields do not depend on the position and we get

$$Z = \int d_2\varphi\, d\psi\, d\bar{\psi}\; e^{-\mathcal{U}(\varphi^2 + \bar{\psi}\psi)} = \int d_2\varphi\; e^{-\mathcal{U}(\varphi^2)}\, \mathcal{U}'(\varphi^2) = 1 \quad (19)$$

$$\langle \varphi^2 \rangle = \int d_2\varphi\, d\psi\, d\bar{\psi}\; e^{-\mathcal{U}(\varphi^2 + \bar{\psi}\psi)}\, \varphi^2 = \int d_2\varphi\; e^{-\mathcal{U}(\varphi^2)} \simeq e^{-\mathcal{U}_{\min} V} \quad (20)$$

$$\langle \varphi^2\, \varphi^2 \rangle = \int d_2\varphi\, d\psi\, d\bar{\psi}(\varphi^2)^2\; e^{-\mathcal{U}(\varphi^2 + \bar{\psi}\psi)} = 2\int d_2\varphi\varphi^2\; e^{-\mathcal{U}(\varphi^2)} \simeq 2\,\varphi^2_{\min}\; e^{-\mathcal{U}_{\min} V}\,. \quad (21)$$

In the above equations the *potential* $\mathcal{U}$ is given by

$$\mathcal{U}(x) = \frac{1}{2} m^2\, x + \frac{\lambda}{4} x^2\,,$$

$$\mathcal{U}'(x) = \frac{d}{dx}\mathcal{U}(x)\,,$$

the integrals have been computed by the saddle point approximation and $\varphi^2_{\min}$ is the value of φ^2 which minimizes $\mathcal{U}$ and $\mathcal{U}_{\min} = \mathcal{U}(\varphi^2_{\min})$.

For $m^2 < 0$,

$$\varphi^2_{\min} = \frac{|m^2|}{\lambda} \quad \text{and} \quad \mathcal{U}_{\min} = -\frac{(m^2)^2}{4\lambda}\,.$$

We see in particular that $G(0, 0)$ diverges exponentially as the volume $V \to \infty$ and that $\dfrac{G_2(0, x)}{G(0, 0)} \sim \dfrac{m^2}{\lambda}$.

(2) It is amusing to notice that $n = -2$ is a pure fermionic theory. The interaction $(\bar{\psi}\psi)^2$ vanishes identically because of the anticommutation properties so that the 2-point function is given by free field theory. This has already been remarked by Balian and Toulouse [8].

So this supersymmetric model reproduces the results we were expecting from the qualitative discussion at the beginning of this paper.

If we consider the full model again and develop around the saddle point, we see the appearance of long-range bosonic and fermionic excitations (Goldstone bosons and Goldstone fermions).

Let us consider the case where the supersymmetry is explicitly broken by the presence of an external field h

$$Z = 1 + \frac{h}{\varphi_{\min}} \, e^{-\mathcal{U}_{\min} V} .$$

It is easy to see that for the free energy

$$F = -\frac{1}{V} \log Z$$

$$\lim_{V \to \infty} \left(\lim_{h \to 0} F \right) = 0$$

$$\lim_{h \to 0} \left(\lim_{V \to \infty} F \right) = \mathcal{U}_{\min} .$$

So the free energy is not a continuous function of the parameter h : we have a *zeroth* order phase transition, which is forbidden for conventional statistical mechanical systems. In principle this picture can be realized by introducing $2\,m$ bosonic fields and m anticommuting pairs. It is not clear whether $m = 1$ is the most convenient choice.

Let us discuss the simplest case of $m = 1$. When the spontaneous breaking of the supersymmetry occurs we will get one bosonic and two fermionic Goldstone excitations. What is the underlying physical interpretation of these excitations is not very clear.

Finally we would like to point out the existence of another approach to the polymer problem which also avoids any $n \to 0$ limit [9].

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