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Supergravity with and without Superspace

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We show that the superspace formalism follows from the component formalism. After constructing the supervielbeins and superconnections off-shell in second-order formalism with the minimal set of auxiliary fields, we show that the resulting supertorsions satisfy the constraints of the various equivalent superspace approaches.

I. INTRODUCTION

Two alternative descriptions of supergravity theory exist by now. The first uses fields defined over ordinary Minkovski space–time, the physical fields being the vierbein $e_\mu{}^m$ and the spin-3/2 field ψ_μ which is the gauge field of supersymmetry [1, 2]. In addition, there are three auxiliary fields: a scalar S, a pseudoscalar P, and an axial vector A_μ. In order to have a correct description of supergravity, one must use these five fields on a par. The fields S, P, and A_μ constitute the minimal set of auxiliary fields for $N = 1$ supergravity and were found by us and by Stelle and West [3]. It is a second-order formalism and leads to a tensor calculus for supergravity.

The other description uses the concept of superspace [4]. The first approach is that of Arnowitt and Nath [5] and is called gauge supersymmetry, but since it is a Riemannian geometry, it does not reproduce supergravity unless a particular singular limit is taken. In the formulation of Wess and Zumino [6] one has an affine (non-metric) geometry, with certain constraints on the supertorsion. One needs these constraints in order to be able to express the supervielbein in a particular choice of gauge in terms of e, ψ, S, P, and A_μ; a choice of gauge alone cannot achieve this since there are far fewer gauge parameters (224) than field components (1024). Ogievetski and Sokatchev [7] have developed an unconstrained geometrical formulation of superspace. By starting with a smaller group and fewer fields, they can express their superfields in terms of e, ψ, S, P, and A_μ by only choosing a gauge. Siegel and Gates [8] have also found an unconstrained one-parameter formulation of super-

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space, which for one value of this parameter leads to supergravity in ordinary space–time with the minimal set of auxiliary fields S, P, and A_μ, while for other values of this parameter they find a nonminimal set of auxiliary fields. This nonminimal set was found by Breitenlohner [9] and also by Fradkin and Vassiliev [10].

In order to show that these superspace approaches are equivalent to the ordinary space–time approach, Brink, Gell-Mann, Ramond, and Schwarz [11] have further developed a method first found in the context of gauge supersymmetry by the authors of Ref. [5] who called it "gauge completion" to construct the supervielbein and superconnection based on the requirement that the gauge algebra in superspace (general supercoordinate transformations and ordinary local Lorentz rotations) be compatible with the gauge algebra in ordinary space–time [3] (general coordinate transformations, local Lorentz rotations, and local supersymmetry transformations). By choosing a gauge in which the superparameters and the superfields with vector world indices are identified with ordinary space–time quantities, this requirement leads to an iteration procedure in successive orders of θ which allows one to determine the superfields and superparameters to all orders in θ. We remark that other methods have been used by Lindström and Roček, and Gates to construct the components of superspace quantities at higher order in θ [12]. These give results which presumably differ by only a gauge transformation in superspace.

In their series of papers [11], Brink *et al.* used the nonminimal set of auxiliary fields (the only set available at that time) for which no tensor calculus exists and obtained results up to order θ. They used first-order formalism and worked on-shell so that they did not need to distinguish between kinematical constraints which specify the geometry and the dynamical field equations (which they found by requiring compatibility with the field equations of supergravity in ordinary space–time). In this paper we begin by applying the same method to the minimal set of auxiliary fields in second-order formalism and work off-shell. At this point we would like to emphasize that the work of Brink *et al.* is quite general and direct and has been of great use to us. Having obtained the supervielbein and superparameres to order θ^2 and the superconnections to order θ, we construct supertorsions to order θ^0, and find then *tensor* relations between these supertorsions, which therefore must be constraints on the supertorsions to *all order in* θ. In this way we have found a direct derivation of the Wess–Zumino kinematical constraints [6]. The reverse of the procedure of Brink *et al.* was later used by Wess and Zumino [13], who started by postulating the constraints, and then proceeded to find a general solution of these constraints by means of the Bianchi identities. The final supertorsions and supercurvatures depend on three superfields which correspond to the three supercovariant multiplets of ordinary supergravity, and by this correspondence they were able to identify the auxiliary fields S, P, and A_μ with superspace quantities. We also show that the slightly different set of constraints which are identically satisfied in the superspace formalism of Ogievetski and Sokatchev [7], are obtained if one applies the method of Ref. [11] to the improved spin connection proposed by Townsend and ourselves [14].

Finally we give a geometrical explanation of the auxiliary fields S, P, and A_μ by using the fact that a general supercoordinate transformation is related to a supertranslation by terms proportional to supertorsions and by using that fermionic supercoordinate transformation can be identified with local supersymmetry in ordinary spacetime. As far as we know, the idea that this relation exists is due to Ne'eman and Regge [15].

The article is organized as follows. In Section 2 we define the gauge algebras of supergravity in ordinary space–time and in superspace, the composition rules of the parameters, and the gauge transformations on the superfields and ordinary fields. Then we review the iteration approach. In Section 3 we use this method to find the supervielbein and superparameters to order θ^2 in second-order formalism with minimal auxiliary fields. In Section 4 we find the superconnection to order θ and discuss how different choices for the spin connection in ordinary space–time lead to different kinematical constraints in superspace.

In the conclusion we give the geometrical meaning of the auxiliary fields and comment on the simplifications of the general supercoordinate parameters which occur when one chooses flat superindices. Related to this aspect is the remarkable factorization of our results for the supervielbein.

II. Gauge Algebras and Gauge Completion

Since we will use second- order formalism with the minimal set of auxiliary fields, we will recapitulate the properties we need. We begin with the gauge algebra of supergavity in ordinary space–time.

For second-order formalism, where the spin connection is not an independent field but a function of tetrad and spin $-3/2$ fields and is obtained by solving its field equation, the minimal set of auxiliary fields [3] needed to close the gauge algebra consists of a scalar S, a pseudoscalar P, and an axial-vector A_μ. For first-order formalism where ω_μ^{rs} is an independent field, one needs at least 18 fermionic auxiliary field components [16]. Brink *et al.* [11] used first-order formalism 2 years ago, but we will use the minimal set of auxiliary fields and hence second-order formalism.

The set of transformation rules which forms a closed gauge algebra consists of general coordinate transformations $\delta_g(\xi^\mu)$, local Lorentz rotations $\delta_l(\lambda^{mn})$, and local supersymmetry transformations $\delta_s(\epsilon^a)$. The latter are given by (putting the gravitational coupling constant κ equal to 1 and normalizing ϵ to $\delta\psi_\mu = \partial_\mu\epsilon$ + more):

$$\begin{aligned}
\delta e_\mu{}^a &= \tfrac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu\,,\\
\delta\psi_\mu &= \left(D_\mu(\omega) + \frac{i}{2}A_\mu\gamma_5\right)\epsilon - \frac{1}{2}\gamma_\mu\eta\epsilon,\\
\delta S &= \tfrac{1}{4}\bar{\epsilon}\gamma\cdot R^{\text{cov}},\\
\delta P &= -\frac{i}{4}\bar{\epsilon}\gamma_5\gamma\cdot R^{\text{cov}},\\
\delta A_m &= \frac{3i}{4}\bar{\epsilon}\gamma_5\left(R_m^{\text{cov}} - \frac{1}{3}\gamma_m\gamma\cdot R^{\text{cov}}\right),\\
\eta &= -\tfrac{1}{3}(S - i\gamma_5 P - i A\!\!\!/\gamma_5).
\end{aligned} \qquad (1$$

Note that we take $A_m = A_\mu e_m{}^\mu$ with flat index as independent field. The spin connection ω_μ^{ab} in the derivative D_μ is given by

$$\omega_{\mu ab} = \tfrac{1}{2}(R_{\mu b,a} - R_{\mu a,b} + R_{ab,\mu}), \tag{2}$$

$$R_{\mu\nu,a} = \partial_\nu e_{a\mu} - \partial_\mu e_{a\nu} + \tfrac{1}{2}\bar{\psi}_\mu \gamma_a \psi_\nu\,, \tag{3}$$

and is clearly minimally supercovariant. It varies under local supersymmetry into [17]

$$\delta_s \omega_{\mu ab} = \tfrac{1}{4}\bar{\epsilon}(\gamma_b \psi_{\mu a}^{\text{cov}} - \gamma_a \psi_{\mu b}^{\text{cov}} - \gamma_\mu \psi_{ab}^{\text{cov}}) + \tfrac{1}{2}\bar{\epsilon}(\sigma_{ab}\eta + \eta\sigma_{ab})\,\psi_\mu\,. \tag{4}$$

The symbol $R^{\mu,\text{cov}}$ is the supercovariant spin-3/2 field equation $R^{\mu,\text{cov}} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\gamma_5 \gamma_\nu \psi_{\rho\sigma}^{\text{cov}}$ and the supercovariant spin $-3/2$ curl follows easily from (1):

$$\psi_{\mu\nu}^{\text{cov}} = \left(D_\mu \psi_\nu - \frac{i}{2} A_\nu \gamma_5 \psi_\mu + \frac{1}{2}\gamma_\nu \eta \psi_\mu\right) - (\mu \leftrightarrow \nu). \tag{5}$$

The only nontrivial relation of the gauge algebra is the anticommutator of two local supersymmetries (the local equivalent of the $\{Q, Q\} \sim P$ relation of global supersymmetry):

$$\begin{aligned}[\delta_s(\epsilon_1), \delta_s(\epsilon_2)] &= \delta_g(\xi^\mu) + \delta_s(-\xi^\mu \psi_\mu{}^a) \\ &\quad + \delta_l(\xi^\mu \hat{\omega}_\mu^{mn} + \tfrac{1}{3}\bar{\epsilon}_2 \sigma^{mn}(S - i\gamma_5 P)\,\epsilon_1); \qquad \xi^\mu = \tfrac{1}{2}\bar{\epsilon}_2\gamma^\mu\epsilon_1\,.\end{aligned} \tag{6}$$

[We recall that our normalization $\delta\psi_\mu = \partial_\mu \epsilon$ differs from Ref. [3] by a factor of 2.] The symbol $\hat{\omega}_\mu^{mn}$ is the new spin connection which we will discuss shortly. The other commutation relations are summarized by

$$\begin{aligned}&[\delta_g(\xi_1{}^\mu) + \delta_l(\lambda_1^{mn}) + \delta_s(\epsilon_1{}^a), \delta_g(\xi_2{}^\mu) + \delta_l(\lambda_2^{mn}) + \delta_s(\epsilon_2{}^a)] \\ &\quad = \delta_g(\xi_2{}^\nu \partial_\nu \xi_1{}^\mu + \tfrac{1}{4}\bar{\epsilon}_2\gamma^\mu\epsilon_1) + \delta_s(\xi_2{}^\nu \partial_\nu \epsilon_1{}^a + \tfrac{1}{2}(\lambda_2 \cdot \sigma\epsilon_1)^a - \tfrac{1}{4}\bar{\epsilon}_2\gamma^\mu\epsilon_1\psi_\mu) \\ &\qquad + \delta_l(\xi_2{}^\nu \partial_\nu \lambda_1^{mn} + \lambda_2^{mp}\lambda_1^{pn} + \tfrac{1}{4}\bar{\epsilon}_2\gamma^\mu\epsilon_1\hat{\omega}_\mu^{mn} + \tfrac{1}{6}\bar{\epsilon}_2\sigma^{mn}(S - i\gamma_5 P)\,\epsilon_1) - 1 \leftrightarrow 2.\end{aligned} \tag{7}$$

For later purposes we rewrite these parameter composition rules as follows:

$$\begin{aligned}\xi_{12}^\mu &= \xi_2{}^\nu \partial_\nu \xi_1{}^\mu + \tfrac{1}{4}\bar{\epsilon}_2\gamma^\mu\epsilon_1 - 1 \leftrightarrow 2, \\ \epsilon_{12}^a &= \xi_2{}^\nu \partial_\nu \epsilon_1{}^a + \tfrac{1}{2}(\lambda_2 \cdot \sigma\epsilon_1)^a - \tfrac{1}{4}\bar{\epsilon}_2\gamma^\mu\epsilon_1\psi_\mu{}^a - 1 \leftrightarrow 2, \\ \lambda_{12}^{mn} &= \xi_2{}^\nu \partial_\nu \lambda_1^{mn} + \lambda_2^{mr}\lambda_1^{rn} + \tfrac{1}{4}\bar{\epsilon}_2\gamma^\mu\epsilon_1\hat{\omega}_\mu^{mn} + \tfrac{1}{6}\bar{\epsilon}_2\sigma^{mn}(S - i\gamma_5 P)\,\epsilon_1 - 1 \leftrightarrow 2.\end{aligned} \tag{8}$$

The new spin connection

$$\hat{\omega}_\mu^{mn} = \omega_\mu^{mn} - (i/3)\,\epsilon_\mu^{mnp} A_p \tag{9}$$

was found [14] to simplify the description of conformal supergravity [18] and Einstein supergravity [1, 2]. In conformal supergravity the introduction of the new spin connection $\hat{\omega}_\mu^{mn}$ showed that one of the constraints (the one which originally was found as the model-dependent field equation of the conformal boost field) was actually equal to the Einstein equations. In Einstein supergravity it explained why in some matter coupling models the four-fermion contact terms were pure torsion, while in other models only the tensor calculus could reproduce these terms, but first-order formalism could not. We will show that also in superspace the new spin connection plays an important role. We will need the gravitino transformation law in terms of $\hat{\omega}^\mu_{mn}$:

$$\delta_s \psi_\mu = (\partial_\mu + \tfrac{1}{2}\hat{\omega}_\mu^{mn}\sigma_{mn})\,\epsilon - \tfrac{1}{2}\hat{\eta}\gamma_\mu\epsilon, \\ \hat{\eta} = -\tfrac{1}{3}(S + i\gamma_5 P - 2i A\!\!\!/\gamma_5). \tag{10}$$

We now turn to the gauge algebra in superspace. In the Wess–Zumino approach [6] to superspace there are two local symmetries: general supercoordinate transformations $\delta_G(\Xi^\Lambda)$ with $\Lambda = (\mu, \alpha)$ and ordinary local Lorentz rotation $\delta_L(\Lambda^{rs})$ with capital indices to distinguish them from g, l, and s. The index convention is

	Bosonic	Fermionic	Both
Flat space	$m, n,\ldots,$	$a, b,\ldots,$	$A, B,\ldots,$
Curved space	$\mu, \nu,\ldots,$	$\alpha, \beta,\ldots,$	$\Lambda, \Pi,\ldots,$

The parameters $\Xi^\Lambda(x, \theta)$ and $\Lambda^{rs}(x, \theta)$ depend on x^μ and θ^α and $\alpha = 1,4$ since we consider $N = 1$ superspace supergravity. The fields in superspace are the supervielbein $E_\Lambda{}^A(x, \theta)$ and the superconnection $\Omega_\Lambda^{mn}(x, \theta)$. They transform as in ordinary general relativity

$$\delta_G(\Xi^\Lambda)\, E_\Pi{}^A = \Xi^\Lambda \partial_\Lambda E_\Pi{}^A + (\partial_\Pi \Xi^\Lambda)\, E_\Lambda{}^A, \qquad \text{idem } \Omega_\Lambda^{mn}, \\ \delta_L(\Lambda^{rs})\, E_\Pi{}^m = \Lambda_n{}^m E_\Pi{}^n, \qquad \delta_L(\Lambda^{rs})\, E_\Pi{}^a = \tfrac{1}{2}(\Lambda\cdot\sigma)^{ab}\, E_\Pi{}^b, \\ \delta_L(\Lambda^{rs})\, \Omega_\Pi^{mn} = -(\partial_\Pi \Lambda^{mn} + \Omega_\Pi^{mt}\Lambda^{tn} + \Omega_\Pi^{nt}\Lambda^{mt}). \tag{11}$$

Covariant derivatives are defined by

$$D_\Lambda E_\Pi{}^m = \partial_\Lambda E_\Pi{}^m + \Omega_\Lambda^{mn} E_\Pi{}^n, \qquad D_A \equiv E_A{}^\Lambda D_\Lambda \tag{12}$$

with a similar relation for $D_\Lambda E_\Pi{}^a$. Supertorsions T_{AB}^C are defined by

$$2T_{AB}^C = (-)^{\Lambda(B+\Pi)}\, E_A{}^\Lambda E_B{}^\Pi (D_\Lambda E_\Pi{}^C - (-)^{\Lambda\Pi}\, D_\Pi E_\Lambda{}^C) \tag{13}$$

and supercurvatures are defined by

$$R^{mn}_{AB} = (-)^{A(B+\Pi)} E_A{}^A E_B{}^\Pi (\partial_A \Omega^{mn}_\Pi - (-)^{A\Pi} \partial_\Pi \Omega^{mn}_A + \Omega^{ms}_A \Omega^{sn}_\Pi - (-)^{A\Pi} \Omega^{ms}_\Pi \Omega^{sn}_A). \tag{14}$$

The gauge algebra in superspace is the same as the space–time part of the gauge algebra of ordinary supergravity. However, since at higher orders in θ the superparameters will become dependent on the fields of ordinary supergravity, one must also vary these fields inside the superparameters according to the transformation rules of ordinary supergravity [Eq. (1)]. Hence, one has

$$[\delta_G(\Xi_1{}^A) + \delta_L(\Lambda_1^{rs}), \delta_G(\Xi_2{}^A) + \delta_L(\Lambda_2^{rs})]$$
$$= \delta_G(\Xi_2{}^\Pi \partial_\Pi \Xi_1{}^A + \delta_1 \Xi_2{}^A) + \delta_L(\Xi_2{}^\Pi \partial_\Pi \Lambda_1^{rs} + \Lambda_2^{rt} \Lambda_1^{ts} + \delta_1 \Lambda_2^{rs}) - 1 \leftrightarrow 2, \tag{15}$$

where contractions are defined by $A^A B_A = A^\mu B_\mu + A^\alpha B_\alpha$ with $\mu = 1{,}4$ and $a = 1{,}4$.

We now summarize the method of finding that form of $E_A{}^A$, Ω^{mn}_A, Ξ^A, and Λ^{rs}, which is compatible with ordinary supergravity. It consists of two steps: first, a gauge is chosen such that the $\theta = 0$ components are identified with the fields and parameters of ordinary supergravity. The choice we make is

$$\begin{aligned} &E_\mu{}^m(x, \theta = 0) = e_\mu{}^m, \quad &&E_\mu{}^\alpha(x, \theta = 0) = \psi_\mu{}^\alpha, \quad &&\Omega^{mn}_\mu(x, \theta = 0) = \omega^{mn}_\mu \\ &\Xi^\mu(x, \theta = 0) = \xi^\mu, \quad &&\Xi^\alpha(x, \theta = 0) = \epsilon^\alpha, \quad &&\Lambda^{rs}(x, \theta = 0) = \lambda^{rs}. \end{aligned} \tag{16}$$

Then the higher order in θ components follows by requiring consistency between the gauge algebras of superspace supergravity and of ordinary space supergravity. To make the principle clar, we give one simple example. According to the superspace algebra, one has for the order $\theta = 0$ components in $\delta_G(\Xi^A)\, E_\Pi{}^A$ for $A = m$ and $\Pi = \mu$

$$\delta_G(\Xi^A)\, E_\mu{}^m = \xi^\nu(x)\, \partial_\nu e_\mu{}^m + \epsilon^\alpha(x) \frac{\partial}{\partial\theta^\alpha} E_\mu{}^m(x, \theta) + \left(\frac{\partial}{\partial x^\mu} \xi^\nu(x)\right) e_\nu{}^m + \left(\frac{\partial}{\partial x^\mu} \epsilon^\beta(x)\right) E_\beta{}^m(x, \theta = 0). \tag{17}$$

On the other hand, $E_\mu{}^m(x, \theta = 0) = e_\mu{}^m$ and since in ordinary supergravity $\delta_s e_\mu{}^m = \frac{1}{2}\bar{\epsilon}\gamma^m\psi_\mu$, one finds upon equating these results that

$$E_\mu{}^m(x, \theta) = \tfrac{1}{2}\bar{\theta}\gamma^m\psi_\mu, \qquad E_\beta{}^m(x, \theta = 0) = 0. \tag{18}$$

The great advantage of this method is that it is linear in the unknown and that at no stage does one need to invert the supervielbein. This is due to the choice of parameters Ξ^A to have curved indices. For other applications, it may be more convenient [13] to use field-independent parameters Ξ^A, in which case $\Xi^A = \Xi^A E_A{}^A$ will be field dependent. In the conclusion we come back to this point.

The scheme of iteration proceeds now as follows. First, one calculates the parameters Ξ^A and Λ^{rs} to the next order in θ. Then one calculates $E_\mu{}^m$ and $E_\mu{}^a$ to this same order, and finally one obtains $E_\alpha{}^m$ and $E_\alpha{}^a$. Similarly for Ω_Λ^{ab}. The construction of the complete parameters can be done before one determines the supervielbein and superconnection, but certain regularities in $\Xi_n{}^\Lambda E_\Lambda{}^A$, discussed in the conclusions, provide checks.

It should be stressed that *at all stages one should consider the whole group*. For example, both $\Xi^A(x, \theta)$ and $\Lambda^{rs}(x, \theta)$ will in general depend on $\epsilon^a(x)$, and since (as we shall see) $\Omega_\alpha^{mn}(x, \theta = 0) = 0$, one has

$$\delta_s \Omega_{\alpha ab}(x, \theta = 0) = 0 = (\epsilon^a) - \text{terms in } (\Xi^\beta \partial_\beta \Omega_\alpha^{rs}(x, \theta) - \partial_\alpha \Lambda^{rs}(x, \theta)). \qquad (19)$$

In the iteration procedure for the parameters one should also be careful to consider at all stages the whole group, but in addition there is one more point to be explained. To make this point clear, we give a second example. The parameter composition law is given by

$$\Xi_{12}^\Pi(\xi_{12}, \epsilon_{12}, \lambda_{12}, \theta, \phi) = \Xi_2{}^\Lambda(\xi_2, \epsilon_2, \lambda_2, \theta, \phi)\, \partial_\Lambda \Xi_1{}^\Pi(\xi_1, \epsilon_1, \lambda_1, \theta, \phi)$$
$$+ \delta_1(\phi)\, \Xi_2{}^\Pi(\xi_2, \epsilon_2, \lambda_2, \theta, \phi) - 1 \leftrightarrow 2. \qquad (20)$$

Suppose one has given all superspace quantities at order θ^k. Then, on the right-hand side the term $\Xi_2{}^\alpha \delta_\alpha \Xi_1{}^\Pi - 1 \leftrightarrow 2$ allows one to integrate to obtain the quantity Ξ^Π at order θ^{k+1}. However, the composite superparameters to order θ^k on the left-hand side are obtained by replacing in the superparameters to order θ^k the ordinary parameters ξ, ϵ and λ by the composite ordinary parameters ξ_{12}, ϵ_{12}, and λ_{12}.

III. The Supervielbein and Superparameters in Second-Order Formalism with Minimal Auxiliary Fields

We begin by determining the parameters to order θ. For Ξ^μ one finds with Eq. (15)

$$\Xi_{12}^\mu(x, \theta) = \xi_{12}^\mu = \xi_2{}^\nu \partial_\nu \xi_1{}^\mu + \epsilon_2{}^\alpha \partial_\alpha \Xi_1{}^\mu - 1 \leftrightarrow 2 \qquad (21)$$

There are no $\delta(\phi)$ terms since at $\theta = 0$ the parameters are field independent. Substituting ξ_{12}^μ from (8), one finds that the terms without ϵ cancel identically, while for the ϵ-dependent terms one finds

$$\tfrac{1}{2}\bar{\epsilon}_2 \gamma^\mu \epsilon_1 = \epsilon_2{}^\alpha \partial_\alpha \Xi_1{}^\mu - \epsilon_1{}^\alpha \partial_\alpha \Xi_2{}^\mu. \qquad (22)$$

Clearly, the natural solution is

$$\Xi^\mu(x, \theta) = \xi^\mu - \tfrac{1}{4}\bar{\epsilon}\gamma^\mu\theta + \mathcal{O}(\theta^2). \qquad (23)$$

This means that the integrability conditions are satisfied. In general, *there are severe nontrivial integrability conditions on such equations, and they are always satisfied.* A second feature of this equation which we will encounter repeatedly below is the nonuniqueness of the solution: the homogeneous equation has solutions, for example, $\Xi^\mu = (\bar{\theta}\epsilon)\, H^\mu$, where H^μ is some arbitrary field. We will consistently put such extra solutions equal to zero and choose only the natural solution. In each case considered, there was always one typical natural solution. *For the supervielbein and superconnection, no integration constants occur,* so that these solutions are unique, once one has given the superparameters.

Having said this, one finds easily the other parameters to order θ. For $\Xi^\alpha(x, \theta)$ one finds, replacing $\Xi^\alpha_{12}(x, \theta = 0)$ by ϵ^α_{12} and substituting for ϵ^α_{12} the three terms in (8), that the term with $\xi_2{}^\nu \partial_\nu \epsilon_1{}^\alpha$ cancel identically while the two remaining terms appear in $\Xi^\alpha(x, \theta)$

$$\Xi^\alpha(x, \theta) = \epsilon^\alpha + \tfrac{1}{4}\psi_\mu{}^\alpha \bar{\epsilon}\gamma^\mu\theta - \tfrac{1}{2}(\lambda \cdot \sigma\theta)^\alpha. \qquad (24)$$

For the Lorentz parameter $\Lambda^{rs}(x, \theta)$ one finds that it is proportional to the Lorentz parameter which appears in the anticommutator $\{\delta_s, \delta_s\}$

$$\Lambda^{rs}(x, \theta) = \lambda^{rs} - \tfrac{1}{4}\bar{\epsilon}[\gamma^\mu \hat{\omega}_\mu^{rs} + \tfrac{1}{3}\sigma^{rs}(S - i\gamma_5 P)]\,\theta. \qquad (25)$$

We now turn to the order θ components of the supervielbein. The two relevant equations are

$$(\delta_s + \delta_g)\begin{pmatrix} e_\mu{}^m \\ \psi_\mu{}^a \end{pmatrix} = \Xi^A \partial_A E_\mu{}^A + (\partial_\mu \Xi^A)\, E_A{}^A, \qquad A = \begin{pmatrix} m \\ a \end{pmatrix} \qquad (26)$$

since the Lorentz parts cancel separately. *One must always start with $E_\mu{}^A$ at a given order in θ and then continue with $E_\alpha{}^A$* because one has started to define $E_\mu{}^A(x, \theta = 0)$ and because this last equation gives information about $E_\mu{}^A$ at order θ and $E_\alpha{}^A$ at order θ^0. The last step is then to use the equations

$$(\delta_s + \delta_g)\, E_\alpha{}^A(x, \theta = 0) = \Xi^A \partial_A E_\alpha{}^A + (\partial_\alpha \Xi^A)\, E_A{}^A \qquad (27)$$

to also obtain $E_\alpha{}^A$ to order θ. Note that it is only in this equation that one needs the parameters to order θ.

Proceeding as discussed above, one finds

$$\delta_s e_\mu{}^m = \tfrac{1}{2}\bar{\epsilon}\gamma^m\psi_\mu = \epsilon^\alpha \partial_\alpha E_\mu{}^m + (\partial_\mu \epsilon^\alpha)\, E_\alpha{}^m(x, \theta = 0). \qquad (28)$$

From this equation one easily obtains two results

$$E_\alpha{}^m(x, \theta = 0) = 0 + \mathcal{O}(\theta), \qquad E_\mu{}^m(x, \theta) = e_\mu{}^m - \tfrac{1}{2}\bar{\psi}_\mu\gamma^m\theta + \mathcal{O}(\theta^2). \qquad (29)$$

From the gravitino law one finds a similar result. Since, however, $\delta_s\psi_\mu$ contains a $\partial_\mu\epsilon$ term, one finds that $E_\alpha{}^a$ is nonzero

$$E_\alpha{}^a(x, \theta = 0) = \delta_\alpha{}^a, \qquad E_\mu{}^a(x, \theta) = \psi_\mu{}^a + \tfrac{1}{2}(\hat{\omega}_\mu \cdot \sigma - \tilde{\eta}\gamma_\mu)\,\theta^a + \mathcal{O}(\theta^2). \qquad (30)$$

This completes the order θ evaluation of the parameters and supervielbein.

TABLE I

Supervielbein and Parameters to Order θ^2

$$
\begin{aligned}
E_\mu{}^m &= e_\mu{}^m + \tfrac{1}{2}\bar\theta\gamma^m\psi_\mu + \tfrac{1}{8}\bar\theta\gamma^m N_\mu\theta \\
E_\alpha{}^m &= 0 - \tfrac{1}{4}(\bar\theta\gamma^m)_\alpha + 0 \\
E_\alpha{}^a &= \delta_\alpha{}^a + 0 + \tfrac{1}{24}\bar\theta M_\alpha{}^a\theta \\
E_\mu{}^a &= \psi_\mu{}^a + \tfrac{1}{2}(N_\mu\theta)^a + \tfrac{1}{8}\bar\theta M_\beta{}^a\psi_\mu{}^\beta\theta + \text{“curl”} \\
\Xi^\mu &= \xi^\mu + \tfrac{1}{4}\bar\theta\gamma^\mu\,[\epsilon + \tfrac{1}{4}(\bar\epsilon\gamma^\nu\theta)\,\psi_\nu] \\
\Xi^\alpha &= [\epsilon + \tfrac{1}{4}(\bar\epsilon\gamma^\mu\theta)\psi_\mu]^\alpha - \bar\theta(M^\alpha{}_\beta\epsilon^\beta)\theta - \tfrac{1}{8}(\bar\theta\gamma^\mu\epsilon)(N_\mu\theta)^\alpha \\
&\quad - \tfrac{1}{2}(\lambda\cdot\sigma\theta)^\alpha + \tfrac{1}{16}(\bar\theta^\mu\gamma\psi_\nu)(\bar\theta\gamma^\nu\epsilon)\psi^{\mu\alpha} \\
\Lambda^{rs} &= \lambda^{rs} + \tfrac{1}{4}\bar\theta[\gamma\cdot\hat\omega^{rs} + \tfrac{1}{3}\sigma^{rs}(S - i\gamma_5 P)][\epsilon + \tfrac{1}{4}(\bar\epsilon\gamma^\mu\theta)\psi_\mu] \\
&\quad - \tfrac{1}{24}\,(\bar\theta\gamma_a\epsilon)[\bar\theta\gamma^a\psi^{rs}_{\text{cov}} + \tfrac{1}{2}\bar\theta\gamma_5\gamma_b\gamma\cdot R^{\text{cov}}\epsilon^{nsab} - \tfrac{2}{3}\bar\theta\gamma^a\sigma^{rs}\gamma\cdot R^{\text{cov}}]
\end{aligned}
$$

The symbols N_μ, $\bar\theta M_\alpha{}^a\theta$ and “curl” are defined by

$$
\begin{aligned}
N_\mu &= \hat\omega_\mu\cdot\sigma - \tilde\eta\gamma_\mu \\
\bar\theta M_\alpha{}^a\theta &= (\bar\theta\gamma^r)_\alpha(\tilde\eta\gamma_r\theta)^a + \tfrac{2}{3}(\bar\theta\sigma^{rs})_\alpha[\sigma_{rs}(S - i\gamma_5 P)\theta]^a \\
\text{“curl”} &= \tfrac{1}{6}\bar\theta\sigma^{rs}(R_\mu^{\text{cov}} - \tfrac{1}{3}\gamma_\mu\gamma\cdot R^{\text{cov}})(\sigma_{rs}\theta)^a \\
&\quad - \tfrac{1}{8}(\bar\theta\gamma_5\gamma^\lambda\theta)(\tfrac{1}{6}\gamma_5\gamma_\mu\gamma_\lambda\gamma\cdot R^{\text{cov}} + \gamma_5\psi^{\text{cov}}_{\mu\lambda})^a
\end{aligned}
$$

We now turn to the order θ^2 evaluation of the same quantities. We will not give all details but only mention a few points, and summarize all results in Table I.

For the order θ^2 part of Ξ^μ one finds the equation

$$
\begin{aligned}
\Xi^\mu_{12}(\xi_{12}, \epsilon_{12}, \lambda_{12}, \theta) &= -\tfrac{1}{4}\bar\epsilon_{12}\gamma^\mu\theta \\
&= \xi_1{}^\nu\partial_\nu\Xi_2{}^\mu + \Xi_1{}^\nu\partial_\nu\xi_2{}^\mu + \epsilon_1{}^\alpha\partial_\alpha\Xi_2{}^\mu(\theta^2) \\
&\quad + \Xi_1{}^\alpha(\theta)\,\partial_\alpha\Xi_2{}^\mu(\theta) + \delta_1(-\tfrac{1}{4}\bar\epsilon_2\gamma^\mu\theta) - 1\leftrightarrow 2. \qquad (31)
\end{aligned}
$$

Here one encounters for the first time the need to *vary fields inside parameters with respect to the whole group*: $\gamma^\mu = \gamma^a e_a{}^\mu$ and $\delta_1 e_a{}^\mu = -\tfrac{1}{2}\bar\epsilon\gamma^\mu\psi_a + \lambda_{ab}e_\mu{}^b - \xi^\nu\partial_\nu e_a{}^\mu + (\partial_\nu\xi^\mu)\,e_a{}^\nu$. The equation reduces to

$$
\epsilon_2{}^\alpha\partial_\alpha\Xi_1{}^\mu(\theta^2) - 1\leftrightarrow 2 = -\tfrac{1}{8}(\bar\theta\gamma^\mu\psi_\nu)(\bar\epsilon_2\gamma^\nu\epsilon_1) - \tfrac{1}{16}(\bar\epsilon_2\gamma^a\theta\bar\epsilon_1\gamma^\mu\psi_a - 1\leftrightarrow 2). \qquad (32)
$$

Again there is a natural solution, namely,

$$
\Xi^\mu(\theta^2) = -\tfrac{1}{16}(\bar\theta\gamma^\mu\psi_\nu)(\bar\theta\gamma^\nu\epsilon).
$$

For the order θ^2 part of Ξ^α one finds in the relevant equation on the left-hand side the symbol $\Xi^\alpha_{12}(\xi_{12}, \epsilon_{12}, \lambda_{12}, \theta)$. We know at this point that $\Xi^\alpha(\theta) = \tfrac{1}{4}\psi_\mu{}^\alpha(\bar\epsilon\gamma^\mu\theta) -$

$\frac{1}{2}(\lambda \cdot \sigma\theta)^\alpha$ and hence one obtains $\epsilon_1\epsilon_2$ terms from the Lorentz parameter λ_{12}. All terms with ξ and λ^{rs} cancel easily and one finds for the remainder after heavy algebra:

$$\begin{aligned}\epsilon_2{}^\beta\partial_\beta\Xi_1{}^\alpha(\theta^2) - 1 \leftrightarrow 2 = {} & [\tfrac{1}{16}\psi_\mu{}^\alpha(\bar{\theta}\gamma^\mu\psi^\nu)(\bar{\epsilon}_2\gamma_\nu\epsilon_1) \\ & + \tfrac{1}{16}(\bar{\epsilon}_1\gamma^\nu\theta)(\bar{\psi}_\nu\gamma^\mu\epsilon_2)\,\psi_\mu{}^\alpha - \tfrac{1}{4}(\bar{\epsilon}_2\gamma^\nu\theta)(\tfrac{1}{2}\omega_\nu \cdot \sigma\epsilon_1 + (i/2)\,A_\nu\gamma_5\epsilon_1 \\ & - \tfrac{1}{2}\gamma_\nu\eta\epsilon_1)^\alpha - \tfrac{1}{8}(\bar{\epsilon}_2\gamma^\mu\epsilon_1)(\hat{\omega}_\mu \cdot \sigma\theta)^\alpha \\ & - \tfrac{1}{12}\bar{\epsilon}_2\sigma^{\mu s}(S - i\gamma_1 P)\,\epsilon_1(\sigma^{rs}\theta)^\alpha] - [1 \leftrightarrow 2]. \end{aligned} \tag{33}$$

The integration is not particularly difficult, and for reasons to be explained below, we write the result in terms of $(S - i\gamma_5 P)$, $\tilde{\eta}$, and the combination $\hat{\omega}_\mu \cdot \sigma - \tilde{\eta}\gamma_\mu$ which appears in the gravitino transformation law.

After the order θ^2 superparameters, we turn to the order θ^2 supervielbein components with world index μ. First, we find rather easily $E_\mu{}^m(\theta^2)$ from the order θ terms in

$$\delta E_\mu{}^m(\theta) = \Xi^A\partial_A E_\mu{}^m + (\partial_\mu\Xi^A)\,E_A{}^m + \Lambda^{mn}E_\mu{}^n, \tag{34}$$

where the variation $\delta E_\mu{}^m(\theta)$ is again over all symmetrices g, l, and s.

The order θ^2 terms in $E_\mu{}^a(\theta^2)$ are less easy to obtain, since $\delta E_\mu{}^a(\theta)$ involves here variation of $\hat{\omega}_\mu \cdot \sigma - \tilde{\eta}\gamma_\mu$ and because on the right-hand side of the relevant equation terms from many different sources build up a supercovariant gravitino curl $\psi_{\mu\nu}^{\text{cov}}$. One finds

$$\begin{aligned}\epsilon^\alpha\partial_\alpha E_\mu{}^a(\theta^2) = {} & -\tfrac{1}{4}(\bar{\epsilon}\gamma^\nu\theta)(\psi_{\mu\nu}^{\text{cov}})^a - \tfrac{1}{8}(2\bar{\epsilon}\gamma_r\psi_{\mu s}^{\text{cov}} + \bar{\epsilon}\gamma_\mu\psi_{rs}^{\text{cov}})(\sigma^{rs}\theta)^a \\ & - \tfrac{3}{8}(\gamma_5\theta)^a\,\bar{\epsilon}\gamma_5(R_\mu^{\text{cov}} - \tfrac{1}{3}\gamma_\mu\gamma \cdot R^{\text{cov}})^a + \tfrac{1}{24}(\gamma_\mu\theta)^a\,(\bar{\epsilon}\gamma \cdot R^{\text{cov}}) \\ & - \tfrac{1}{24}(\gamma_\mu\gamma_5\theta)^a\,(\bar{\epsilon}\gamma_5\gamma \cdot R^{\text{cov}}) + \tfrac{1}{8}(\gamma_\mu\gamma^\lambda\gamma_5\theta)^a\,\bar{\epsilon}\gamma_5(R_\lambda^{\text{cov}} - \tfrac{1}{3}\gamma_\lambda\gamma \cdot R^{\text{cov}}) \\ & + \tfrac{1}{6}\bar{\psi}_\mu\sigma_{rs}(S - i\gamma_5 P)\,\epsilon(\sigma^{rs}\theta)^a + \tfrac{1}{12}\bar{\epsilon}\sigma_{rs}(S - i\gamma_5 P)\,\theta(\sigma^{rs}\psi_\mu)^a \\ & + (i/12)(\bar{\psi}_\mu\gamma_q\epsilon)\,\epsilon^{rstq}A_t(\sigma^{rs}\theta)^a + (i/4)(\bar{\epsilon}\not{A}\psi_\mu)(\gamma_5\theta)^a \\ & - \tfrac{1}{4}(\bar{\epsilon}\gamma^r\psi_\mu)(\gamma_r\eta\theta)^a - \tfrac{1}{4}(\bar{\epsilon}\gamma^\nu\theta)(\tfrac{1}{2}\tilde{\eta}\gamma_\mu\psi_\nu). \end{aligned} \tag{35}$$

We now comment on the integrability of this equation. All terms of the forms $\bar{\epsilon}\gamma^\nu\theta$ should cancel, and one easily shows that $\bar{\epsilon}\gamma^\nu\theta\psi_{\rho\sigma}^{\text{cov}}$-type terms indeed cancel. For the S, P, and A_μ terms the solution is manifest upon inspection. For the terms proportional to the gravitino curl one uses

$$\begin{gathered}\tfrac{1}{6}\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu\psi_{\rho\sigma} - \tfrac{2}{3}\gamma^\nu\psi_{\mu\nu} = R_\mu - \tfrac{1}{3}\gamma_\mu\gamma \cdot R, \\ 2\sigma^{\mu\nu}\psi_{\mu\nu} = \gamma \cdot R, \\ \gamma^\rho\psi_{\rho\mu} = R_\mu - \tfrac{1}{2}\gamma_\mu\gamma \cdot R, \\ \gamma_\alpha\psi_{\beta\gamma} + \gamma_\beta\psi_{\gamma\alpha} + \gamma_\gamma\psi_{\alpha\beta} = \epsilon_{\mu\alpha\beta\gamma}\gamma_5 R^\mu, \\ \psi_{\mu\nu} + \tfrac{1}{2}\gamma_5\epsilon_{\mu\nu}^{\rho\sigma}\psi_{\rho\sigma} = -\gamma_\alpha\sigma_{\mu\nu}R^\alpha, \\ R^\mu = \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu\psi_{\rho\sigma}\,, \end{gathered}$$

and we determined the solution by general means because the solution was n manifest. Since there is no ambiguity in the equations for superfields but only f superparameters, this will not lead to a less desirable solution.

IV. The New Spin Connection and Superspace Constraints

The method discussed so far can be extended to the superspin connection $\Omega_\Lambda^{rs}(x, \theta)$ We consider the transformations of Ω under supercoordinate transformations ar local Lorentz rotations

$$\delta\Omega_\Pi^{rs} = \Xi^\Lambda \partial_\Lambda \Omega_\Pi^{rs} + (\partial_\Pi \Xi^\Lambda)\, \Omega_\Lambda^{rs} - D_\Pi \Lambda^{rs}, \qquad (3$$

where $\delta_L \Omega_\Pi^{mn} = -D_\Pi \Lambda^{mn}$ as in (11). Comparison with the transformation rules the ordinary fields inside Ω according to the ordinary gauge algebra then allows o to compute the supertorsion and supercurvatures to all orders in θ. These quantiti are gauge independent and since they are also covariant, it is sufficient to calcula only their $\theta = 0$ components in order to deduce the constraints satisfied by t complete superfields. Moreover, since the supertorsions do not contain derivatives the superconnections, it is sufficient to know the superconnection at $\theta = 0$ in order obtain the gauge-invariant constraints on the supertorsions to all orders in θ. Wi these constraints on the supertorsions one may then show that the affine geometry superspace reproduces the supergravity theory of ordinary space–time.

We begin by identifying the superconnection with vector world index

$$\Omega_\mu^{rs}(x, \theta = 0) = \omega_\mu^{rs}(e, \psi), \qquad (3$$

where $\omega_\mu^{rs}(e, \psi)$ is the minimal supercovariant spin connection of ordinary sup gravity. From $\delta_s \omega_{\mu ab}$ in (4) one then finds

$$\Omega_\alpha^{rs}(x, \theta = 0) = 0, \qquad (3$$

$$\Omega_{\mu r,s} = \omega_{\mu,rs} + \tfrac{1}{4}\bar\theta(\gamma_s \psi_{\mu r}^{\text{cov}} - \gamma_r \psi_{\mu s}^{\text{cov}} - \gamma_\mu \psi_{rs}^{\text{cov}}) + \tfrac{1}{2}\bar\theta(\sigma_{rs}\eta + \eta\sigma_{rs})\, \psi_\mu . \qquad (4$$

From $\delta\Omega_\alpha^{rs}(x, \theta = 0) = 0$ one finds then that

$$\Omega_\alpha^{rs}(x, \theta) = 0 + \tfrac{1}{6}\bar\theta\sigma^{rs}(S - i\gamma_5 P)_\alpha - (i/12)\, \bar\theta\gamma_{mn} e^{rsmn}. \qquad (4$$

From $\Omega_\Lambda^{rs}(x, \theta = 0)$ one can deduce $T_{AB}^C(x, \theta = 0)$ and from $\Omega_\Lambda^{rs}(x, \theta)$ one c deduce $R_{AB}^{rs}(x, \theta = 0)$. Since a tensor which vanishes at $\theta = 0$ vanishes identically all order in θ, we can now derive the constraints on the supertorsions. It is straig forward to derive

$$T_{ab}^r = -\tfrac{1}{4}(C\gamma^r)_{ab}, \qquad T_{rs}^t = T_{ab}^c = T_{ar}^s = 0 \qquad (4$$

For the nonvanishing components we have

$$T^c_{ar} = -\tfrac{1}{4}(\gamma_r\eta)^c_a + (i/4)\, A_r(\gamma_5)^c_a\,, \tag{43}$$

$$T^a_{rs} = \tfrac{1}{2}e_r{}^\mu e_s{}^\nu(\psi^{\text{cov}}_{\mu\nu})^a. \tag{44}$$

An interesting different identification of the $\theta = 0$ components of Ω^{rs}_μ makes contact with the work of Ogievetski and Sokatchev [7]. If one identifies

$$\Omega^{rs}_\mu(x, \theta = 0) = \hat{\omega}^{rs}_\mu, \tag{45}$$

where $\hat{\omega}^{rs}_\mu = \omega^{rs}_\mu - (i/3)\,\epsilon^{rst}_\mu A_t$ is the new spin connection of Ref. [14] discussed in Section 2, then one still has $\Omega^{rs}_\alpha(x, \theta = 0)$, and again

$$T^r_{ab} = -\tfrac{1}{4}(C\gamma^r)_{ab}\,, \qquad T^c_{ab} = T^s_{ar} = 0, \qquad T^a_{rs} = \tfrac{1}{2}(\psi^{\text{cov}}_{rs})^a. \tag{46}$$

However, now

$$T^t_{rs} = -(i/3)\,\epsilon^{tu}_{rs}A_u\,, \qquad T^c_{ar} = -\tfrac{1}{4}(\tilde{\eta}\gamma_r)^c_a \tag{47}$$

and projecting out A_μ from T^c_{ar} and substituting in the result for T^t_{rs}, one finds that the constraint $T^t_{rs} = 0$ is replaced by

$$T^t_{rs} = \tfrac{1}{2}\epsilon^{tu}_{rs}(\gamma_5)^a_c\, T^c_{au}\,. \tag{48}$$

For the order θ components of the superconnection one now finds

$$\Omega^{rs}_\mu(x, \theta) = -\tfrac{1}{2}\bar{\theta}\gamma_\mu(\psi^{rs})^{\text{cov}} - \tfrac{1}{12}(\bar{\theta}\gamma_5\gamma^\nu\gamma\cdot R^{\text{cov}})\,\epsilon^{rs}_{\mu\nu} + \tfrac{1}{3}\bar{\psi}_\mu\sigma^{rs}(S - i\gamma_5P)\,\theta. \tag{49}$$

$$\Omega^{rs}_\alpha(x, \theta) = \tfrac{1}{6}\bar{\theta}\sigma^{rs}(S - i\gamma_5P)_\alpha\,. \tag{50}$$

With these order θ results one can now obtain the curvatures, and we find in particular that

$$(1 + \gamma_5)_{ab}\, R^{rs}_{bc}(1 - \gamma_5)_{cd} = 0, \tag{51}$$

or, in two-component notation, that $R^{rs}_{a\dot{a}} = 0$. These are exactly the same results as obtained by Ogievetski and Sokatchev. The connection between their results and ours can be understood from their definition of the vector covariant derivative D_r in terms of spinor derivatives D_a

$$\{D_a\,, D_{\dot{a}}\} = \tfrac{1}{2}(\sigma^r)_{a\dot{a}}\, D_r\,. \tag{52}$$

This is consistent with the usual definition

$$\{D_a\,, D_{\dot{a}}) = R^{rs}_{a\dot{a}}\,\tfrac{1}{2}\Sigma_{rs} - 2T^A_{a\dot{a}}D_A\,. \tag{53}$$

if and only if $R^{rs}_{\alpha\dot{\alpha}}$ vanishes since $T^{A}_{\alpha\dot{\alpha}} = (\sigma^{\mu}_{\alpha\dot{\alpha}}, 0)$. Although these authors do not admit independent Lorentz rotations but lock them to general coordinate transformations, their definition of D_r is equivalent to our choice

$$D_r = \partial_r + \tfrac{1}{2}\hat{\omega}_r^{mn}\sigma_{mn}\,. \tag{54}$$

V. Conclusions

We have found a direct method to derive the constraints for superspace, once the auxiliary fields of ordinary supergravity are known. For $N = 1$ supergravity we thus derived straightforwardly the constraints of the Wess–Zumino approach [6] while we also reobtained the slightly different constraints found by Ogievetski and Sokatchev [7] by using the new spin connection proposed some time ago by Townsend and us [14]. Now that the auxiliary fields for $N = 2$ supergravity have been obtained by Fradkin and Vassiliev [10] and de Wit and van Holten [19], our method should also straightforwardly lead to the constraints for $N = 2$ superspace, and hence to $N = 2$ superspace geometry. Also the constraints of conformal supergravity and the constraints for $N = 8$ on-shell should follow directly. Work in this direction is in progress.

Our results for the supervielbein and superparameters display several striking regularities. For example, the terms involving only ϵ and ψ in Ξ^{α} and Ξ^{μ} can be written as

$$\epsilon + (-\tfrac{1}{4}\bar{\theta}\gamma^{\nu}\epsilon\psi_{\nu}) + (-\tfrac{1}{4}\bar{\theta}\gamma^{\nu}(-\tfrac{1}{4}\bar{\theta}\gamma^{\gamma}\psi_{\gamma})\,\psi_{\nu}) + \cdots$$

and the pattern is clear. The same phenomenon is found in Λ^{rs}. Also, one finds the same functions M and N appearing over and over again. There is an explanation of these results, and that is that the flat space parameters Ξ^{A} are to a large extent field independent. Thus, using our results for Ξ^{A} and $E_A{}^{A}$, if one constructs $\Xi^{A} = \Xi^{A}E_A{}^{A}$ and requires that $\Xi^{A}(x, \theta = 0)$ be field independent

$$\Xi^{m}(x, \theta = 0) = \xi^{\mu}e_{\mu}{}^{m} = \hat{\xi}^{m}, \qquad \hat{\xi}^{m} = \text{field independent},$$

$$\Xi^{a}(x, \theta = 0) = \xi^{\mu}\psi_{\mu}^{\alpha\mu} + \epsilon^{\alpha}\delta_{\alpha}{}^{a},$$

$$\epsilon^{a} = \hat{\epsilon}^{a} - \xi^{\mu}\psi_{\mu}{}^{a}, \qquad \hat{\epsilon}^{a} = \text{field independent},$$

then in terms of $\hat{\xi}^{m}$ and $\hat{\epsilon}^{a}$ one finds indeed remarkable cancellations which lead to

$$\begin{aligned}\Xi^{m}(x, \theta) &= \hat{\xi}^{m} + \tfrac{1}{2}\bar{\theta}\gamma^{m}\epsilon + \tfrac{1}{8}\bar{\theta}\gamma^{m}(\hat{\xi}\cdot\hat{\omega}\cdot\sigma - \hat{\eta}\xi - \lambda\cdot\sigma)\,\theta,\\ \Xi^{a}(x, \theta) &= \hat{\epsilon}^{a} + \tfrac{1}{2}(\hat{\xi}\cdot\hat{\omega}\cdot\sigma - \hat{\eta}\xi - \lambda\cdot\sigma)\,\theta + \text{“curl terms”}\\ &\quad - \tfrac{1}{8}(\bar{\theta}\gamma^{\mu}\hat{\epsilon})(\hat{\eta}\gamma_r\theta)^{a} + \tfrac{1}{12}(\bar{\theta}\sigma^{rs}\hat{\epsilon})(\sigma_{rs}(S - i\gamma_5 P)\,\theta)^{a}.\end{aligned}$$

Thus, if one defines $\lambda = \hat{\lambda} + \hat{\xi}\cdot\hat{\omega}$, then in terms of $\hat{\epsilon}$, $\hat{\lambda}$, and $\hat{\xi}$ it appears that the flat parameters only depend on auxiliary fields and covariant curls. Apparently, our gauge flat superparameters are field dependent, even on-shell, but only through supercovariant expressions. It is an interesting question whether a gauge exists in which the flat parameters are field independent. In this connection it may be useful to remark that we have defined the gauges in superspace to all order by our criterion of "natural solution." Perhaps one may define other gauges by requiring that certain components of the supervielbein vanish to all order in θ. Typical candidates are $E_\alpha{}^a$ propoertional to $\delta_\alpha{}^a$ and $E_\alpha{}^m = 0$.

Another result which appears from our explicit calculation of the supervielbein is that it depends on-shell only on the covariant gravitino curl. This prompts us to speculate that from the order θ^2 level onward, the complete supervielbein may be only a function of the three covariant gauge-invariant superfields and the superconnection. Our four-dimensional results also agree with the two-dimensional results of Howe [20], who makes the ansatz for the supervielbein on-shell

$$E_\mu{}^m = e_\mu{}^m + \tfrac{1}{2}\bar\theta\gamma^m\psi_\mu\,, \qquad E_\alpha{}^a = \delta_\alpha{}^a$$
$$E_\mu{}^a = \psi_\mu{}^a - \omega_\mu(\gamma_5\theta)^a, \qquad E_\alpha{}^m = -\tfrac{1}{4}(\bar\theta\gamma^m)_\alpha\,.$$

This equivalence follows from the fact that in two dimensions the gravitino curl itself vanishes "on-shell."

Finally we come to the geometrical aspects of the auxiliary fields S, P, and A_μ in ordinary space–time. In general, the relation between a general supercoordinate transformation and supertorsions and supercurvatures is given by

$$\delta_G(\Xi^\Lambda)\, h_\Lambda{}^A = (D_\Lambda{}^G H)^A - \Xi^\Pi R^A_{\Pi\Lambda}\,,$$

where $H^A = \Xi^\Lambda h_\Lambda{}^A$ and $h_\Lambda{}^A$ are the gauge fields associated with the gauged group G. The derivative $D_\Lambda{}^G$ is the covariant derivative, covariant with respect to the group G. For the vierbein $E_\mu{}^m(x, \theta = 0)$ one has for local supersymmetry no extra terms since $T^n_{am} = 0 \cdot R^C$ are the curvatures with the same structure constants as appear in the covariant derivative $(D^G H)^A$. (If one defines this derivative to contain only a spin connection a la Cartan, then for $A = (m, a)$, $R^A_{\Pi\Lambda} = E_\Lambda{}^B E_\Pi{}^A T^C_{AB}$ where T are the flat supertorsions. In that case the tetrad law is given by $R^m_{\alpha\mu}$. If one prefers to consider for G the whole super Poincare algebra, then the full P-curvature $R^m_{\alpha\mu}$ vanishes ("supertorsion is zero") and $D_\Lambda H$ yields the tetrad law.) However, for the gravitino $E_\mu{}^a$ $(x, \theta = 0)$ one does find an extra term, namely, the supertorsion $R^b_{\alpha\mu} = \delta_\alpha{}^a e_\mu{}^m T^b_{am}$. With the result of (43)

$$T^c_{ar} = -\tfrac{1}{4}(\gamma_r\eta)^c_a + (i/4)(A_r\gamma_5)^c_a\,,$$

one may verify that these extra terms are precisely the S, P, and A_μ terms in $\delta\psi_\mu = (D_\mu + (i/2)\, A_\mu\gamma_5)\,\epsilon - \tfrac{1}{2}\gamma_\mu\eta\epsilon$. These results are in general agreement with Ref. [15] and mean that the auxiliary fields appear in ordinary supergravity because in superspace one must convert the supertranslation gauge transformation into general supercoordinate transformations.

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