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G. Parisi: ON THE RELATION BETWEEN THE RENORMALIZED AND THE BARE COUPLING CONSTANT ON THE LATTICE.

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ABSTRACT.

We discuss how to compare the lattice regolarized field theory with the continuum limit: the finite renormalization of the coupling constant must be computed on the lattice at the one loop level.

In recent times some computation have been done for renormalizable asymptotically free field theories in the low bare coupling region using lattice regolarization[1,2]. The theories under consideration (the non linear σ-model and gauge theories) are formally massless and scale invariant; however there is a spontaneous generation of mass: one can define an effective running coupling constant $\alpha_R(\mu)$, μ being the renormalization point, satisfying the renormalization group equation:

(1) $$\frac{d\alpha_R(\mu)}{d\mu^2} = \beta_2 \alpha_R^2 + \beta_3 \alpha_R^3 + O(\alpha_R^4) \qquad (\beta_2 < 0)$$

Eq. 1 implies that any quantity ρ having the dimensions of a mass squared must depend on μ and $\alpha_R(\mu)$ in the following way:

(2) $$\rho = C\rho\ \alpha^2 \alpha_R(\mu)\ \beta_3/\beta_2^2 \exp(1/\beta_2 \alpha_R)\ [1 + O(\alpha_R)]$$

The quantity $C\rho$ cannot be computed in perturbation theory and depends on the definition of α_R. If we use the standard parametrization of high energy physics:

(3) $$\frac{-1}{\beta_2 \alpha_R(\mu)} - \frac{\beta_3}{\beta_2^2} \ln \alpha_R(\mu) \simeq \ln(\mu^2/\Lambda^2) \qquad (\mu \longrightarrow \infty)$$

eq. 2 can be written as:

(4) $$\rho = C\rho\ \Lambda^2$$

If the same quantity is computed on the lattice we get:

(5) $$\rho = L\rho\, a^{-2} \exp(\frac{1}{\beta_2 \alpha_B} + \frac{\beta_3}{\beta_2^2} \ln \alpha_B)$$

where α_B is the bare coupling constant and a is lattice spacing; $L\rho$ is a new constant which can be obtained by direct non perturbative computation on the lattice.

We can compare eq. 2 and eq. 5 in the region:

(6) $$\alpha_B \ll 1 \qquad \mu a \gg 1 \qquad \alpha_B \ln \mu a \ll 1 \quad .$$

In this region one get with good accuracy:

(7) $$\frac{1}{\alpha_R(\mu)} = \frac{1}{\alpha_B}\ \beta_2 \ln \mu^2 a^2 S \quad .$$

S being a computable constant; by substituting in the previous equations we finally find for any ρ :

(8) $$C\rho = S\, L\rho$$

Eq. 8 is crucial for doing an absolute comparison between the lattice and the continuum theories: S can be obtained doing a simple one loop computation on the lattice.

Let us compute S for the two dimensional non linear σ-model invariant under the O(N) group defined on a quadratic lattice.

Following Brézin and Zinn-Justin[3] we introduce an N-1 component field and we eliminate the remaining component of the field using the non linear constraint. Using their notation the propagator of the renormalized field is:

$$G_R^{-1}(p^2,t,H) = \frac{Z}{Z_1 t}\left(p^2 + H\frac{Z_1}{Z_2}\right) + \text{(loop corrections)} \tag{9}$$

where t and H are the renormalizedcoupling constant (temperature) and magnetic field, Z and Z_1 are the field and coupling constant renormalization constants respectively. The renormalization constants are fixed by the condition:

$$G_R^{-1}(p^2,t,\mu^2) = \frac{1}{t}(p^2 + H) + O(p^4) \tag{10}$$

On the lattice the free propagator is:

$$tG_0^{-1}(p,t,H) = \left[2 - \cos(p_x a) - \cos(p_y a)\right] 2\, a^{-2} + H \tag{11}$$

All the integral must be done in the first Brioullin zone $-(\pi/a) \leq p_x \leq (\pi/a)$, $(\pi/a) \leq p_y \leq (\pi/a)$. At the one loop level one finds:

$$\begin{aligned} G^{-1}(p,t,H) = G_0^{-1}(p,t,H) - \frac{\pi}{2} \int_B d^2K \; \{ G_0(K,t,H)/t \,. \\ \cdot\left[\frac{N+1}{2} + 2a^{-2}(2 - \cos(ap_x + aK_x) - \cos(ap_y + aK_x)\right] - 1\} \end{aligned} \tag{12}$$

Using the asymptotic relation[4]:

$$\int_B d^2K \frac{G_0}{t}(K,t,H) \Longrightarrow \pi \ln(Ha^2/32), \qquad H \longrightarrow 0$$

we get:

$$Z_1 = 1 + t\left[(N-2)\ln(32/\mu a^2) + \pi\right] \Longrightarrow S^{-1} = 32\ \text{esp}\left(\frac{\pi}{N-2}\right)$$

Notice that in the continuum limit the exact formula is:

$$G^{-1}(p,t,H) = G_0^{-1}(p,t,H) - \frac{\pi}{2}\left(p^2 + \frac{N-1}{2}\right) S \frac{d^2K}{K^2 + H}$$

If one simply substitute to the formulae valid in the continuum the lattice propagator without quantizing the theory on the lattice, one would miss the factor , which, for not too marge N, is very important.

From this example it should be clear that the value of S in QCD can be extracted only doing a straightforward, but long computation.

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