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STRONG COUPLING PHASE IN LATTICE GAUGE THEORIES AT LARGE DIMENSION

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We study lattice gauge theories at large space-time dimension D. In the strong coupling (high-temperature) phase, the appropriate expansion parameter is $D^{-1/4}$. We find that this phase, which becomes metastable at a certain temperature, is limited by a singular point which looks like a second-order phase transition. For the Z_2 gauge group, this happens at $D \tanh^4\beta = \frac{128}{3125}$. At this temperature, the correlation length of two plaquettes becomes infinite. At $D = \infty$, the same result holds for any gauge group, only the value at which the transition occurs depends on the group. The corrections of order $D^{-1/4}$ are different for different gauge groups. It is not yet clear to us if the Wilson loop changes its behaviour when the plaquette-plaquette correlation function does.

1. Introduction

Very little is known unfortunately, about the long-distance behaviour of quantum chromodynamics (QCD). In particular, the two existing approximation schemes, the usual perturbation theory (which is supposed to break down at long distances) and the strong coupling expansion of the discretized version of QCD (lattice gauge theories [1–4]) exhibit a completely different behaviour. In the first case, the asymptotic states contain zero-mass excitations and colour is not confined. In the strong coupling approximation of lattice gauge theories, there are no zero-mass excitations of the gauge degrees of freedom and colour is confined.

It is reasonable to assume that there is some kind of phase transition in QCD and that one can encounter the first or the second long-distance regime, depending on the value of the gauge coupling constant. The nature and the mechanism of such a transition and, *a fortiori*, the value of the coupling constant at which it occurs are completely unknown except for some particular cases of discrete gauge groups [2] where duality arguments can be applied. This is also partly the case for the U(1) gauge group [5].

The simplest way to study the existence of different phases in statistical mechanics is the mean-field approximation. This technique gives a qualitative picture of the transition and, provided one works in sufficiently high dimensions, it also gives good quantitative results. It is nevertheless difficult to apply this approximation directly in lattice gauge theories; in its crudest version, it indeed breaks gauge invariance, while it is well-known [6] that no such breaking can occur.

An interesting tool developed by Englert, Fisher and Gaunt, Abe [7] for the case of spin systems is the $1/D$ expansion. One computes only those diagrams of the high-temperature expansion, which become dominant when the dimension D of the lattice system is large. Each dominant diagram contributes a power in $x \equiv 2\beta D$. The sum of the contributions converges for $x < x_c$. At $x = x_c$ one finds a singularity which is interpreted as a second-order phase transition (the susceptibility becomes infinite). For the spin systems the leading term in the $1/D$ expansion is equivalent to the mean-field approximation and the $1/D$ corrections have been computed [7]. First-order phase transitions cannot be seen by this method. We recall that they are characterized by a discontinuity of the first derivative of the free energy with respect to the temperature (i.e., the internal energy). Each of the two branches of the free energy is C_∞. However, according to the conventional wisdom, they are not analytic, but have an essential singularity. Unfortunately this singularity is absent in the mean-field approximation as well as in any type of perturbative expansion. If a first-order transition occurs at $x = x_1 < x_c$, the high-temperature expansion will continue to converge and will describe for $x_1 < x < x_c$ a metastable phase. Such techniques have been extended to the gauge systems [2] and corrections can also be computed [8] in this case.

In this paper we study large-dimensional pure gauge systems (no matter fields are considered) in a way which explicitly maintains gauge invariance and the other symmetries of the problem. It turns out that the relevant parameter for pure gauge fields is $x = 2\beta^4 D$ (instead of $2\beta D$ for spin systems) and we find a singularity at certain value $x = x_c$. We interpret this as a second-order phase transition at which the plaquette-plaquette correlation length becomes infinite. This transition is very similar to the condensation of branched polymers [9]. Chains of three-dimensional cubes in tree-like configurations is the analog of the polymer. The situation is less clear for the Wilson loop. It seems that the transition we found is not a deconfinement transition. The continuation to the other side of the strong coupling phase is not yet clear in this formalism. All our results are valid for any gauge group at infinite dimensions. Next we argue that the high-temperature phase is stable only in a part of the interval $0 \leqslant x < x_c$. In fact there exists rigorous lower bounds on the free energy, F, of the pure gauge system due to the convexity of the exponential function (see, for example, ref. [2]) and for large, but finite, D there exists an $x_1 < x_c$ such that the free energy we have computed violates this bound for $x > x_1$. We conclude that there exists a first-order phase transition beyond which the high-temperature phase becomes metastable. The stable phase cannot be reached by the high-temperature expansion we have considered.

The study of gauge theories in the $D \to \infty$ limit has also been proposed independently by Polyakov [10]. However, this approach differs from ours.

This paper is organized as follows. In sect. 2, we review the $D \to \infty$ approximation for lattice spin systems. Sect. 3 contains a brief review of the high-temperature expansion of lattice gauge theories and a discussion of the $D \to \infty$ approximation. All technical details have been left to the appendices. The conclusions are presented in sect. 4.

Appendix A explains in more technical details the diagrammatic rules of the high-temperature expansion. In appendix B, we discuss which are the dominant diagrams in the $D \to \infty$ limit and in appendix C we proceed to their counting. Corrections are investigated in appendix D. Some inequalities concerning the free energy are recalled in appendix E.

2. Ising model

For illustrative purposes, we start by summarizing the situation for the Ising model on a hypercubical lattice in D dimensions.

The partition function Z and the susceptibility are given by

$$Z(\beta) = \sum_{\{\sigma_i = \pm 1\}} \exp[\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j] \,, \tag{2.1}$$

$$\chi(\beta) = \frac{1}{Z} \sum_k \sum_{\{\sigma_i = \pm 1\}} \sigma_0 \sigma_k \exp[\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j] \,, \tag{2.2}$$

where $\langle ij \rangle$ represents all the nearest neighbours in the lattice. The first formula can be written as

$$Z(\beta) = (\cosh \beta)^N \sum_{\{\sigma_i\}} \prod_{\langle ij \rangle} (1 + t\sigma_i \sigma_j) \,, \tag{2.3}$$

with $t = \tanh \beta$, and where N is the number of sites in the lattice and the product runs over all the Nd links $\langle ij \rangle$. At sufficiently high temperature (small t), the expansions of Z and χ in powers of t are convergent. From the high-temperature expansion, one can compute the location of the singularities of the "free energy" $F = (1/N) \ln Z$ and extract the critical temperature.

These expansions have obvious diagrammatic representations. In case of the partition function or of the free energy F, we sum over closed paths on the lattice, while for the susceptibility we consider the open paths going from the site 0 to any possible site k. Note that, in the t-expansion (2.3), each link appears at most once while, in the β expansion (2.1), it can appear several times (with different weights) [11].

We proceed as follows. We start by drawing all the possible diagrams $\mathcal{G}$ (defined by the topological properties of a set of links) and multiply their contributions by the number of times they can be mapped as a graph on the lattice. An estimation of this number is easily done provided the diagram does not intersect with itself. The simplification at $D = \infty$ comes from the fact that the probability of such an intersection vanishes at $D \to \infty$ and thus the incorrect treatment of diagrams with such intersections becomes unimportant.

The limit $D \to \infty$ may be taken in several ways and this may lead to apparently contradictory results. We illustrate this point with the following two estimations.

(A) We consider the high-temperature expansion of the free energy

$$F = \sum_n F_n t^{2n} \tag{2.4}$$

(which is an even expansion in t since any closed path contains an even number of links). F_n is the number of all closed connected paths of length $2n$ (up to a translation), evaluated for large D. From every such path we can get another one of length $2(n + 1)$ in the following way. Cut the original path (at two arbitrary points) into two pieces (there are $n(2n + 1)$ possibilities), then translate one of the two pieces by one unit on the lattice ($2D$ possibilities at large D) and finally tie the pieces again by adding two links. The same new path (of length $2(n + 1)$) can be obtained in this way from $n + 1$ different paths of length $2n$ (because it involves $n + 1$ different, but equivalent, pairs of links); so finally we get

$$F \propto \sum_n n!(4t^2D)^n\,. \tag{2.5}$$

(B) In the same expansion for the susceptibility,

$$\chi = \sum_n \chi_n t^n\,, \tag{2.6}$$

χ_n is now the number of open paths of length n, with a fixed origin. We get a path of length $n + 1$ by adding a link at its end. This can be done in $2D$ ways again neglecting $1/D$ corrections). Therefore $\chi_{n+1} = 2D\chi_n$ and

$$\chi \propto \sum_{n=0}^{\infty} (2Dt)^n = \frac{1}{1 - 2Dt}\,. \tag{2.7}$$

In case A, we have taken the limit $D \to \infty$, keeping t^2D fixed while, in case B, tD is kept fixed. The treatment A of the free energy is incorrect and this can be seen from the fact that the series in t^2D has a zero radius of convergence ⋆. The con-

⋆ This is not always the case in spin systems. For instance, the spherical model with frozen random interactions (spin glasses in the Edwards-Anderson model) has a critical inverse temperature which behaves as $1/\sqrt{D}$. The series (2.6) have an infinite radius of convergence and similarly (2.4) is convergent. However, the sum (2.4) does not lead, in this particular case, to the correct free energy, due to an additional singularity in the $1/D$ correction [12].

vergence of (2.4) is governed by the large-n behaviour of F_n and one can suspect that the two limits $n \to \infty$ and $D \to \infty$ do not commute in the expression (2.4). That this is indeed the case, can be seen by a more careful counting argument at large (but not infinite) and fixed D. When $n \ll D$, the path of length $2D$ will contain links pointing in n directions which are all different with a high probability; this is the origin of our $n!$ factor. Now, for $n \gg D$, each direction will be used several times. Let $2n_i$ be the number of links pointing in the ith direction (half of them point forwards, half backwards). The freedom we are left in constructing the path is the permutation of only those links which point in different directions. Hence a factor $(2n)!/[(n_1!)^2 \dots (n_D!)^2]$. When $n \gg D$, this factor is important only for $n_i \sim n/D$, and behaves indeed as $(2D)^{2n}$ instead of $n!$. Therefore, we again recover a behaviour in $(2tD)^{2n}$, in agreement with the case B of the susceptibility. The expansion in t^2D is indeed not appropriate.

In the case of the susceptibility, we get a convergent expansion with a singularity at $t_c = 1/2D$. This is exactly the mean-field approximation result. Thus $t_c^2 D \to 0$ and this is why the simple-minded method A has a zero radius of convergence. One could expect *a priori* that the proposed method fails for the free energy because we expect (from scaling arguments) $F \sim (\beta_c - \beta)^{D/2}/\sin(\frac{1}{2}\pi D)$ and the singularity becomes softer and softer with increasing dimension.

3. Gauge theories

We shall now apply the same method to the slightly more complicated case of pure gauge theories, not coupled to other fields. For completeness, we start by recalling well-known results for the high-temperature expansion of lattice gauge theories. More details can be found in appendix A and in ref. [2]. One associates one element R_{ij} of the group G with each link ij of the D-dimensional hypercubical lattice. The free energy is given in this case by

$$F = \frac{1}{N} \ln Z \,, \tag{3.1}$$

$$Z = \int \Big(\prod_{ij} \mathcal{D}R_{ij}\Big)\, e^{A} \,, \tag{3.2}$$

$$A = \frac{1}{2g^2} \sum_{p} \chi(U_p) \,, \tag{3.3}$$

where $\mathcal{D}R$ is the invariant group measure, g is the gauge coupling constant ($\beta = 1/2g^2$), U_p is the product of fields R_{ij} (gauge group elements) around the plaquette p, χ is any real class function on the group (i.e., a real function of the traces of the representations) and the sum Σ_p runs over all the elementary plaquettes of the lattice.

We shall take the "Fourier" transform of e^A, using [4]

$$e^{\beta\chi(U_p)} = \sum_r \widetilde{\beta}_r \chi_r(U_p) = \widetilde{\beta}_0 \sum_r d_r \beta_r \chi_r(U_p) . \tag{3.4}$$

The sum runs over all irreducible representations r of the group, including the trivial one $r = 0$; d_r and χ_r denote, respectively, the dimension of the representation r and its character (trace of U_p in this representation). The coefficients $\beta_r = \beta_r(\beta)$ are given by

$$\widetilde{\beta}_r = \widetilde{\beta}_0 d_r \beta_r = \int \mathcal{D}U \chi_r^*(U)\, e^{\beta\chi(U)} . \tag{3.5}$$

We shall argue later that the range of validity of the high-temperature expansion is for β very small at large D (more precisely $\beta = O(D^{-1/4})$). Therefore $\widetilde{\beta}_0 = \text{cst} + O(\beta^2)$, $\beta_r \sim \beta^{\nu_r}$ where ν_r is some positive integer. If we restrict to the case where $\chi = \chi_1$, the fundamental character (or more generally where χ is chosen as a trace of an irreducible representation) ν_r is the minimum number of times the fundamental representation has to be taken in order to obtain by composition the representation r. Because of this property, only the β_1 coefficient of the fundamental representation of a group will play a role as D goes to infinity, the other ones contributing only to the corrections. In this case, our argument is valid for any gauge group at large D, only the $D^{-1/4}$ corrections will depend on the group.

The factor $\widetilde{\beta}_0$ associated with the trivial representation gives a simple volume factor and does not play any important role.

Using the orthogonality properties of the group characters, we get

$$\int \mathcal{D}U \chi_r(SU) \chi_s^*(TU) = \frac{\delta_{rs}}{d_r} \chi_r(ST^{-1}) . \tag{3.6}$$

Because of this orthogonality property, any link must belong to, at least, two plaquettes and only closed two-dimensional surfaces (built up from elementary plaquettes) will contribute in the diagrammatic expansion of F. This generalizes the expansion (2.3) for the Ising model. Here again, because we construct the diagrammatic expansion once the Fourier transform (3.4) has been performed, each plaquette of the lattice has to be taken at most once, when constructing the diagrams. The temperature-dependent part of each diagram is β_1^S, where S is the number of plaquettes (area) of which the surface is made. The smallest contributing diagram is the surface of a three-dimensional cube.

The high-temperature expansion of the free energy has been computed for various groups in ref. [2], up to order β^{16}, for any space-time dimension. Retaining from these results the dominant terms as D goes to infinity, we see that they have a very simple structure; the degree in D increases by one unit every time the degree in β increases by four units. It seems therefore that $\beta_1^4 D$ would be the appropriate variable in the large-dimension limit. The interested reader can find a detailed analysis in appendix B, and we restrict here to a qualitative discussion. The important point is that the dominant diagrams have a structure of three-dimensional cubes arranged in a tree. As in sect. 2, we look for a recursive construction of the dominant diagrams. Adding a new dimension may be done by cutting the surface along a closed curve, then by shifting one of the pieces to a new dimension, and finally, by joining the two pieces together again with a chain of plaquettes. (In sect. 2, the analog operation needed two links). The number of new plaquettes equals the length of the cutting curve (in lattice units); its minimum is 4 (for an elementary square). Starting now from the lowest-order diagram (a 3-dimensional cube), we thus obtain trees of cubes.

The appearance of the variable $\beta_1^4 D$ is also easily understood. The construction consists of "sticking"a cube on a plaquette (yielding a factor β_1^6 since a cube has six faces) and of suppressing the frontier made of the two contact plaquettes (yielding a factor β_1^{-2}). The new cube may point out in any of the $2D-4$ ($\sim 2D$ for large dimensions) directions orthogonal to the selected plaquette. We thus see that the variable $x = 2\beta_1^4 D$ arises naturally. We have neglected the possibility that the added cube touches any other cube of the parent configuration. The probability of such an accident, however, vanishes as D goes to infinity.

The next problem is the counting of these configurations. As a cube has only six faces, a tree of cubes may be associated with a connected graph on a Cayley tree of coordination number 6. The counting problem is easily solved using standard methods. In appendix C, we proceed to this evaluation, using the generating function for the connected trees. Here we quote the result

$$F = \frac{D^{3/2} d_1^2}{12\sqrt{2}} u^{3/2}(1-3u)(1+\mathrm{O}(D^{-1/4}))\,, \tag{3.7}$$

with

$$x = 2D\beta_1^4 = u(1-u)^4\,. \tag{3.8}$$

Let us recall that β_1 is related to β by (3.5) and, for instance, $\beta_1 = \tanh\beta$ in the case of the Z_2 gauge group. A graphical representation of the curve is given in figs. 1 and 2. F starts at point A, increasing from 0 regularly until $x = x_c = 4^4/5^5$ (part AB of the curve) and shows a curious behaviour. The point B is a cusp and the curve comes back (part BCD of the curve). Also, a complex branch starts at B. This would suggest a "zeroth-order" transition at $x = x_c$, with a jump in the free energy. However, as one knows, thermodynamics forbids such a transition and it is well-known that F is a convex function of β. (We give in appendix E a proof of this well-known result). Therefore only the branch AB may be physically acceptable.

What may be the physical significance of point B? It is easy to verify that the second derivative of F,

$$\frac{\partial^2 F}{\partial\beta_1^2} = \tfrac{5}{2}D^2 \frac{u}{1-5u} \sim (\beta_{1c}-\beta_1)^{-1/2}\,,$$

has a singularity. But this expression is the plaquette-plaquette "susceptibility"

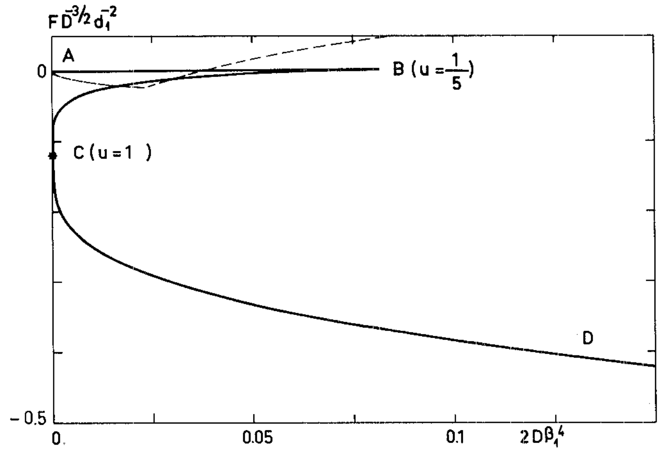

Fig. 1. The free energy (minus the trivial term $\frac{1}{2}D(D-1)\ln\cosh\tilde{\beta}_0$ discussed in appendix B) as a function of the coupling constant $x = 2D\beta_1^4$. The dashed curve is Peierls' lower bound for $D = 8$ and the Z_2 gauge group.

(the plaquette-plaquette correlation function summed over the positions of the plaquettes). Our system undergoes, therefore, a second-order phase transition, since the corresponding correlation length becomes infinite. It is indeed proportional to $(\beta_{1c} - \beta_1)^{\nu}$ with $\nu = \frac{1}{4}$, as was conjectured in ref. [13]. In the language of ref. [14], this corresponds to a zero-mass boxciton excitation.

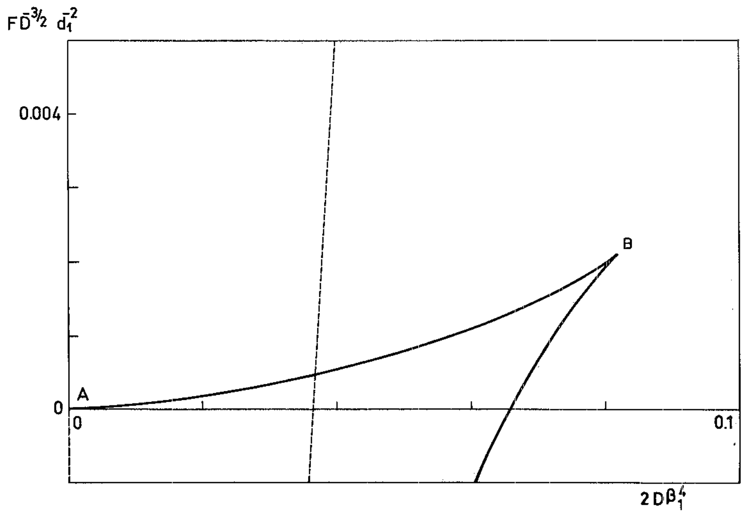

Fig. 2. Fig. 1, on a larger scale.

The next object we want to discuss is the Wilson loop [1]. We treat the case where the loop is contained in a coordinate 2-plane of the lattice. The lowest contribution is obtained by filling with plaquettes the plane surface enclosed by the loop and hence is equal to β_1^S, where S is the area of this minimal surface. However, this is not the general case; for any other orientation of the loop, the minimal surface is not unique and furthermore their number may go to infinity as D and S increase. This fact throws doubt on the reliability of our results for the generic orientation of the loop. The general case is certainly difficult and we were unfortunately unable to solve it. In our particular case, the minimal surface can be deformed as follows. Each plaquette may be replaced by a "tube" made of three-dimensional cubes as in the case of the free energy. The number of these tubes is computed in the appendix C and $f(t)$ denotes the corresponding generating function. Therefore the total contribution is

$$W(C) \simeq [\beta_1 f(x)]^S = \mathrm{e}^{-\alpha S} , \tag{3.9}$$

with

$$\alpha = -\ln [\beta_1 f(x)] = \tfrac{1}{4} \ln \frac{2D}{u} , \tag{3.10}$$

using the parametric representation of eq. (3.8). The parameter α increases as D goes to infinity. For large D, nothing special happens to it at the singular point B for $u = \frac{1}{5}$ and α remains strictly positive.

The picture we get of the singularity is that the tree configurations of these tubes of cubes (hydra-like configurations, using the terminology of the percolation problem [15]) become larger and larger, reaching an infinite length at the singularity. This looks very much like the condensation of (branched) polymers. A plaquette can very easily be deformed into a tube; at high dimension, a large surface is therefore very easily deformed locally, but it seems to maintain its global rigidity.

4. Discussion and conclusions

As discussed in sect. 1, there remains the possibility of a first-order transition; this problem cannot be handled by only the high-temperature expansion. A possible tool is provided by some rigorous inequalities on the free energy. A lower bound may be obtained from the mean-field approximation using Peierls' inequality. We recall the argument in appendix E, and in figs. 1 and 2 we draw this lower bound for $D = 8$ in the Z_2 case. As D increases, this curve goes towards the left until it reaches the vertical axis as D goes to infinity. It is thus clear that only the beginning of the arc AB remains physically acceptable. In the x variable, this region furthermore shrinks as D increases. As we did not meet any singularity before the point B, we can only imagine one possibility. Somewhere at a point T on AB, the system undergoes a first-order transition. This does not correspond to a singularity in the free energy.

Indeed cooling the system from high temperature across this postulated transition will leave it momentarily on the same arc AB, which acts now as a metastable region, before jumping into the new phase after some relaxation time. There is evidence of such a behaviour in a recent Monte-Carlo "experiment" on a four-dimensional Z_2 gauge system [16]. In this picture, F appears as a multi-valued function. In our expansion, however, we do not observe the stable branch of F beyond the transition point T, which coincides (in the x variable and at infinite D) with the vertical axis. New methods are needed to find this branch and this is the subject of a forthcoming paper [8].

In any case, our expansion is relevant for the study of the whole strong coupling phase, including the metastability region. As D decreases, the first-order transition point moves towards higher values on the arc AB and may reach B at some critical dimension D_c which quite likely depends on the gauge group. In appendix D, we have computed the first two corrections. There the transition acquires the characteristics of a second-order transition and is correctly described by the behaviour around the singular point B. At this value, the correlation length of two plaquette becomes infinite. It seems, however, that nothing happens to the Wilson loop in high enough dimension. Nevertheless the corrections involved at this critical dimension may force a zero in the coefficient α of the area law (3.9) at the critical temperature. Simple-minded power-counting arguments suggest that logarithmic corrections to the area law are present at the critical point, at least in dimensions $D = 6$ and become stronger as D decreases.

An open problem, which we have not been able to solve, is the computation of the critical dimension for the second-order transition at $x = x_c$, i.e., in which dimension the critical exponent γ defined by $\chi_p \sim 1/[(x_c - x)^\gamma]$ (χ_p is the plaquette susceptibility) is no longer equal to $\frac{1}{2}$. A possible way to attack this problem would consist in finding the connection with polymer physics also for finite dimension and in using the techniques of ref. [9].

We are grateful to C. Itzykson for stimulating discussions. One of us (G.P.) would also like to thank A.A. Migdal and A.M. Polyakov for discussions on the relevance of closed and open surfaces to gauge theories.

Appendix A

Diagrammatic rules and notations

This appendix contains a brief review of some aspects of the high-temperature expansion in lattice gauge theories. More details can be found in ref. [1]. The exponentiated action is decomposed in Fourier series (see eqs. (3.4), (3.5)) on the gauge group G as follows

$$\exp A = \prod_{p} \tilde{\beta}_0 (1 + \sum_{r \neq 0} d_r \beta_r \chi_r(U_p)) , \tag{A.1}$$

where r labels the irreducible representations of the group G. U_p is the product of the gauge fields along the boundary of the plaquette p. d_r and χ_r are the dimension and the character (trace) of the representation r. The trivial representation $r = 0$ contributes a factor $\tilde{\beta}_0$ which has been factorized out. β_r are the coupling constants [related to the Yang-Mills coupling constant g (or the inverse temperature $\beta = 1/2g^2$) and vanishing as g goes to infinity].

For instance, in the Z_2 case,

$$A = \beta \sum_{p} \sigma_{ij}\sigma_{jk}\sigma_{kl}\sigma_{li} , \tag{A.2}$$

where the fields σ_{ij}, located on the links ij, take the values ± 1. There are only two one-dimensional representations, the trivial one $r = 0$ and the fundamental one $r = 1$ (Z_2 itself). Therefore,

$$\tilde{\beta}_0 = \cosh \beta , \qquad \beta_1 = \tanh \beta . \tag{A.3}$$

Expanding the product in (A.1), we introduce the following definitions:

A *graph* is a function of the plaquettes into the set of representations (i.e., a representation is assigned to each plaquette of the lattice). As the trivial representation plays a trivial role, a graph is also a subset $\mathcal{G}$ of the set of all plaquettes, to each of which a non-trivial representation has been assigned.

A *diagram* $\mathcal{D}$ is defined by the topologic properties of a set of plaquettes. The number of way a diagram can be mapped on the lattice (with N sites and periodic boundary conditions) is called the *configuration number* (cn) of the diagram and is denoted by $\{\mathcal{D}\}$. $\{\mathcal{D}\}$ is a polynomial in N and vanishes for $N = 0$; its degree is the number of connected parts of $\mathcal{D}$. It is also a polynomial in the dimension D of the lattice, which vanishes for $D = 0$, 1, and 2. The linear part in N of $\{\mathcal{D}\}$ is called the *reduced configuration number* (rcn) of $\mathcal{D}$ and will be denoted by $[\mathcal{D}]$. Finally, the degree of D of the rcn $[\mathcal{D}]$ is defined as the *dimensionality* dim $(\mathcal{D})$ of the diagram.

The expansion of the product (A.1) associates a contribution to each graph. The sum of the contributions of all graphs pertaining to the same diagram $\mathcal{D}$ is called the contribution of this diagram. Its contribution to the partition function Z factorizes into the product of $\{\mathcal{D}\}$ and a factor $c(\mathcal{D})$, independent of D and N, homogenous of degree $n(\mathcal{D})$ (where $n(\mathcal{D})$ is the number of plaquettes of the diagram) in the set of variables β_r. After integration over the gauge fields, this factor vanishes, unless the product of the representations assigned to the plaquettes adjacent to each link contains the trivial representation. A diagram fulfilling this condition is said to be *closed*. In particular, it has no boundary (no link belonging to only one plaquette). For the Z_2 gauge group, closed diagrams are those for which each link is shared by an even number of plaquettes. Finally

$$Z = \tilde{\beta}_0^{ND(D-1)/2} \sum_{\text{closed}\,\mathcal{D}} \{\mathcal{D}\}\, c(\mathcal{D}) . \tag{A.4}$$

Furthermore, $c(\mathcal{D})$ factorizes into the product $\Pi_i\, c(\mathcal{D}_i)$, where $\mathcal{D}_i$ are all the connected parts of the diagram

An example of closed diagrams is given by the diagrams homeomorphic to the sphere. For the corresponding graphs, all plaquettes must belong to the same representation r. Furthermore, we have

$$c(\mathcal{D}) = \sum_{r\neq 0} d_r^2 \beta_r^{n(\mathcal{D})} \tag{A.5}$$

for this class of diagrams. Lowest-order diagrams are:

(i) the cube, with n (cube) = 6, {cube} = $\frac{1}{6}ND(D-1)(D-2)$;

(ii) the double-cube (surface of two cubes sharing a face), with n(double-cube) = 10, {double-cube} = $\frac{1}{2}ND(D-1)(D-2)(2D-5)$.

Rule for the "free energy"

The free energy $F = (1/N) \ln Z$ differs, in our definition, from the one of usual thermodynamics by a factor $-\beta$. Due to the existence of the thermodynamic limit $N \to \infty$ for F, one has

$$F = \tfrac{1}{2}D(D-1)\ln \tilde{\beta}_0 + \sum_{\text{closed}\,\mathcal{D}} [\mathcal{D}]\; c(\mathcal{D})\ , \tag{A.6}$$

where the rcn now replaces the cn. Note that the summation must be carried over all diagrams, including disconnected ones. Indeed some effects of excluded volume appear since we do not use cumulants. However, the contribution of disconnected parts may be easily obtained. We have already noticed that $c(\mathcal{D})$ factorizes. Let us now construct the product $\{\mathcal{D}_1\}\{\mathcal{D}_2\}$ by superposition of two realizations of diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ on the lattice; this product is therefore a linear combination of the $\mathcal{D}_i$, where $\mathcal{D}_i$ are all possible diagrams obtained from $\mathcal{D}_1$ and $\mathcal{D}_2$ by identification of some plaquettes. Among these diagrams, the one made of $\mathcal{D}_1$ and $\mathcal{D}_2$ disconnected (i.e., without any identification of plaquettes) has more connected parts than any other one. A recursive use of these identities allows the reduction of the cn of any disconnected diagram to the cn of connected diagrams only. We point out that taking the factor of N in these relation gives identically zero on the left-hand side ($\{\mathcal{D}_1\}\{\mathcal{D}_2\} \sim N^2$); hence there arises a linear relation between the corresponding rcn, which will be used later on in appendix B.

Appendix B

Dominant diagrams at high dimensions

In order to find the dominant diagram as $D \to \infty$, we will use the following theorem.

B.1. Theorem

$$\dim(\mathcal{D}) \leqslant \frac{n(\mathcal{D}) + 6}{4}\ , \qquad \text{if } \mathcal{D} \text{ is closed}\ , \tag{B.1}$$

The equality is reached only for those diagrams bounding a connected tree of three-dimensional cubes (i.e., a set of cubes such that, if one joins the centers of adjacent cubes (cubes sharing a plaquette), the obtained figure is a connected tree). The proof is two-fold.

(a) *Connected diagrams.* dim($\mathcal{D}$) is, in this case, the maximum number of effectively used dimensions for a geometrical realization on the lattice of the diagram $\mathcal{D}$. Let us consider such a maximal realization and perform the following reduction process. Suppress a part of the lattice limited by two consecutive hyperplanes $k < x_j < k + 1$ and containing at least one plaquette of the diagram. Due to the closure condition, at least four plaquettes have disappeared in this process; other plaquettes may be duplicated and must be replaced by a single plaquette or no plaquette at all, in order to obtain a new closed diagram. This operation decreases the number of plaquettes, by at least 4, while the dimensionality decreases at most one. Therefore, if the inequality is valid for the reduced diagram, it is *a fortiori* valid for the original one. This allows a proof by recurrence of the inequality because the lowest-order diagram is the cube, which fulfills eq. (B.1).

The equality is reached only if, in this reduction process, exactly four plaquettes disappear at each step and if we finally obtain the cube, which is the only 3-dimensional diagram saturating the inequality. These four plaquettes are arranged as the lateral surface of a three-dimensional cube. So we obtain that the connected tree of cubes saturates the inequality (B.1).

(b) Disconnected diagrams. At the end of appendix A, a recursive process for computing the rcn of disconnected diagrams has been discussed. In these relations, the diagram with the most important number of connected parts also have strictly more plaquettes than any other diagram for which some identifications have been done. Hence the strict inequality (B.1) results by recurrence.

We conclude that the connected trees of three-dimensional cubes are the only diagrams saturating (B.1).

B.2. Summation of the dominant diagrams

We recall that the contribution of a diagram to the free energy F is given by eq. (A.6). As D goes to infinity, only the diagrams which saturate the inequality (B.1) (that is only trees of cubes) will contribute.

Let us map such a dominant diagram on the lattice. In a step-by-step process, we add a cube adjacent to an already placed cube. After the choice of the common plaquette, we are left with $2D - 5 \sim 2D$ possibilities. In fact, some of those may be forbidden (if they touch already placed cubes) but they are finite in number and the choice is always of order $2D$, where D is large. Apart from this factor $2D$ for each added cube (and also the choice of the first placed cube), this construction process is exactly the same as the drawing of a tree on a Bethe lattice of coordination number 6 (since a cube has 6 faces). The counting is performed in appendix C; we have only to replace the counting variable t by $2D\beta_r^4$ [$2D$ is the previously discussed factor; each added cube brings four plaquettes; the final graph is homeomorphic to the sphere and thus all plaquettes must be assigned (see appendix A) to the same representation r; hence the term $\beta_r^4]$. We have furthermore to adjust the multiplicative constant for the first placed cube. Hence the final result at large D:

$$F = \tfrac{1}{2}D(D-1)\ln\tilde{\beta}_0 + \sum_{r\neq 0} \frac{d_r^2\beta_r^2 D^2}{12} g(2D\beta_r^4)\,, \tag{B.2}$$

with the function $g(t)$ of appendix C for $q = 6$. In the paper, we used the fact that setting $\tilde{\beta}_0$ to unity changes F by a trivial constant and that only β_1 contributes as D goes to infinity (when the action involves only the fundamental trace representation) in order to write the simplified formula (3.7). Using the variables

$$x_r = 2D\beta_r^4\,, \tag{B.3}$$

we finally recast the result as

$$F - \tfrac{1}{2}D(D-1)\ln\tilde{\beta}_0 = \frac{D^{3/2}}{12\sqrt{2}} \sum_{r\neq 0} d_r^2 x_r^{1/2} g(x_r) + O(D^{5/4})\,. \tag{B.4}$$

The first two corrections to this formula are computed in appendix D.

Appendix C

Graph counting on a q-coordinated Bethe lattice

Let us recall that, in a q-coordinated Bethe lattice (Cayley tree), each vertex is linked to q other vertices and that no closed paths exist. We consider a lattice with $\mathcal{N}$ nodes and neglect the surface effects due to the finite size of this lattice. Let p_n be the number of connected trees. We shall evaluate the generating function

$$g(t) = \frac{1}{\mathcal{N}} \sum_n p_n t^n\,. \tag{C.1}$$

Let us count first the number q_n of rooted trees containing a given node of the lattice and which are connected to the other n vertices only through one given bond (selected among the q bonds originating from this node). The corresponding generating function

$$f(t) = \sum_n q_n t^n \tag{C.2}$$

obeys the relation

$$f(t) = 1 + t f^{q-1}(t) \tag{C.3}$$

(the 1 corresponds to the case $n = 0$). This equation completely defines the function $f(t)$.

There are two possible procedures for constructing a graph.

(a) Choose a node among the $\mathcal{N}$ nodes of the Bethe lattice, then dress the q bonds starting from this node with a rooted tree. This method will count n times each graph with n vertices.

(b) Choose a link among the $\frac{1}{2}q\mathcal{N}$ bonds of the Bethe lattice, then dress the two extremities of this link with rooted trees. This method will count n_L times each graph with n_L bonds.

Since $n_L = n - 1$, the exact result is obtained by subtracting the two preceding biased results and then

$$g(t) = tf^q(t) - \tfrac{1}{2}q(f(t) - 1)^2 , \tag{C.4}$$

or, using eq. (C.3) in order to eliminate t,

$$g(t) = \tfrac{1}{2}(f(t) - 1)(q - (q - 2)f(t)) . \tag{C.5}$$

A useful parametric representation of these equations is given by

$$\begin{aligned} t &= u(1 - u)^{q-2} , \\ f(t) &= \frac{1}{1 - u} , \\ g(t) &= \frac{u(1 - \frac{1}{2}qu)}{(1 - u)^2} . \end{aligned} \tag{C.6}$$

Appendix D

Corrections

We derive here the first two corrections for any gauge group. As a complete and rigorous proof is rather lengthy and tedious, we only explain its mechanism; the interested reader may, indeed, easily restore the omitted points.

A first remark is that the contribution of dominant graphs has been computed in appendix B up to corrections in $1/D$. These corrections will not be evaluated here, because we are only interested in the first two corrections in $D^{-1/4}$ and $D^{-1/2}$. We use here the parameters

$$x_r = 2D\beta_r^4 , \tag{D.1}$$

which are kept constant as we expand in $D^{-1/4}$.

D.1. First correction

Surfaces homeomorphic to the sphere have an even number of plaquettes and thus do not contribute here. The only possibility is a singular closed curve limiting three open surfaces made of plaquettes. The suppression of one of these surfaces will again give a closed surface homeomorphic to the sphere of lowest order; that is, a dominant diagram. Therefore the only possibility is that the singular closed curve is an elementary square; the open surfaces are thus either a single plaquette or a dominant diagram with one omitted plaquette (rooted tree of cubes in the language of appendix C). Neglecting the excluded volume effects (which are of order $1/D$) we are finally led to a generating functional of the form

$$\tfrac{1}{2}D^2\left[\frac{1}{3!}(f(t)-1)^3+\frac{1}{2!}(f(t)-1)^2\right] . \tag{D.2}$$

Now we turn to the group factors. By integration over the links belonging to two plaquettes, each open surface must have the same representation assigned to its plaquettes and yields a factor

$$d_r\beta_r^n\chi_r(U_{\rm p}) .$$

The integration over the singular line is now achieved, using

$$\int \chi_r(U)\,\chi_s(U)\,\chi_t(U)\mathcal{D}\,U = N_{rst} , \tag{D.3}$$

where N_{rst} is the number of trivial representations in the decomposition of the product $r \times s \times t$. Gathering the pieces, we finally obtain the correction to the free energy.

$$\frac{D^{5/4}}{12\cdot 2^{3/4}}\sum_{rst} N_{rst}d_rd_sd_t(x_rx_sx_t)^{1/4}[f(x_r)\,f(x_s)\,f(x_t)-f(x_r)-f(x_r)$$
$$-f(x_t)+2] . \tag{D.4}$$

This correction vanishes for the Z_2 gauge group ($N_{111}=0$). Indeed, closed diagrams have an even number of plaquettes and the series become in this case an expansion in $D^{-1/2}$.

D.2. Second corrections

There are three contributions.

(a) *Non-connected diagrams.* To this order, only diagrams made of two-dominant contribute. Their rcn is computed according to the method described at the end of appendix A. We finally get

$$-\tfrac{1}{32}D\,[\sum_r d_r^2x_r^{1/2}(f^2(x_r)-1)]^2 . \tag{D.5}$$

Note the minus sign which appears in non-connected contributions.

(b) Connected diagrams homeomorphic to the sphere. Let us first remark that replacing a plaquette by the "skin" of a rooted cube-tree does not change the order of a diagram in our expansion. All diagrams thus can be obtained by dressing the

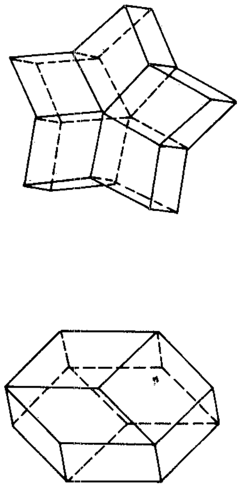

Fig. 3. Some diagrams contributing to the second corrections.

plaquettes of some fundamental ones (the cube for the dominant diagrams). Up to the considered order, these fundamental diagrams are drawn in fig. 3. Apart from a particular one (fig. 3a), they are composed of a "ring" of k cubes ($k = 3, 4, \ldots$; $k = 5$ in fig. 3b). Their evaluation is a straightforward (although lengthy) exercise. Finally, the result is

$$\tfrac{1}{2}D\sum_r d_r^2\left[\frac{x_r^3 f^{12}(x_r)}{12} - \frac{x_r^2 f^8(x_r)}{12} - x_r f^4(x_r) - \ln(1 - x_r f^4(x_r))\right]. \tag{D.6}$$

The logarithm proceeds from the summations over all possible fundamental diagrams. Its effect is to add a singularity when $x_r f^4(x_r)$ is equal to unity, that is exactly at point C of fig. 1. This correction erases the arc CD of the curve.

(c) Diagrams with singular lines. The singular line may be either a single square (limiting four surfaces) or two elementary squares (limiting three surfaces, and which may eventually share a corner or a link). Again the computation is straightforward, and we get

$$\tfrac{1}{96}D\sum_{rstu}(x_r x_s x_t x_u)^{1/4} N_{rstu} d_r d_s d_t d_u [f(x_r)\,f(x_s)\,f(x_t)\,f(x_u) - (f(x_r)\,f(x_s) + \mathrm{sym})$$
$$+ 2(f(x_r) + f(x_s) + f(x_r) + f(x_u)) - 3]$$
$$+ \tfrac{5}{32}D \sum_{rstuv} d_r d_s d_u d_v N_{rst} N_{tuv} (x_r x_s x_u x_v)^{1/4} (f(x_r)\,f(x_s) - 1)$$
$$\times (f(x_u)\,f(x_v) - 1)\,\frac{f(x_t) - 1}{5 - 4f(x_t)}\,. \tag{D.7}$$

D.3. Discussion

A possible use of the previous formulae is the computation of the corrections to the critical temperature. As D goes to infinity. F behaves as $A(x_c - x)^\alpha$ plus regular terms. We assume that, as in the spin systems, A and x_c have a $D^{-1/4}$ expansion while the power α is constant. More precisely,

$$A = A_0 D^{3/2} + A_1 D^{5/4} + A_2 D + \ldots, \tag{D.8}$$

$$x_c = x_0 + x_1 D^{-1/4} + x_2 D^{-1/2} + \ldots . \tag{D.9}$$

Consequently, the singular part of F reads

$$F_{\text{sing}} = D^{3/2}(x_0 - x)^{3/2}\left[A_0 + \left(A_1 + \frac{3A_0 x_1}{x_0 - x}\right)D^{-1/4} + \left(A_2 + \frac{3(A_1 x_1 + A_0 x_2)}{2(x_0 - x)} + \frac{3A_0 x_1^2}{8(x_0 - x)^2}\right)D^{-1/2} + \ldots\right]. \tag{D.10}$$

Comparison with the expansion of F allows the extraction of the series (D.8) and (D.9). Our hypothesis is supported by two facts. First, the absence of logarithmic singularities in the corrections at the critical point. Secondly, the most singular parts in the corrections are related and this provides a consistency check, which works. In the Z_2 case, we find

$$x_c = \frac{256}{3125} - \frac{45696\sqrt{10}}{3515625} D^{-1/2} + O(D^{-1}) . \tag{D.11}$$

We recall that $x = 2D\beta_1^4$ and, therefore, this corresponds to four singularities in the β plane. They are at equal distance from the origin at infinite D, but the corrections remove this degeneracy. Of course, only the singularity on the real positive β axis is physical. This pattern is characteristic of pure gauge theory on hypercubical lattices at large dimension. It is seen, even for $D = 4$, in a numerical analysis of available high-temperature expansions.

Appendix E

Inequalities on the free energy

The starting point is the Peierls' inequality based on the convexity of the exponential function

$$\int d\mu\, e^X \geqslant \left(\int d\mu\right) \exp\left(\frac{\int d\mu X}{\int d\mu}\right), \tag{E.1}$$

where $d\mu$ is any positive measure. If, for instance, we choose, for the measure, the sum over all configurations with a weight $\exp(-\beta_0\mathcal{H})$ ($\mathcal{H}$ being any statistical hamiltonian) and, for X, $(\beta_0 - \beta)\mathcal{H}$, we obtain on the left-hand side the partition function $Z(\beta)$ at temperature β while the integration of the measure gives it at temperature β_0. Taking now the logarithm, we obtain

$$F(\beta) \geqslant F(\beta_0) + (\beta - \beta_0) \left.\frac{\partial F}{\partial \beta}\right|_{\beta=\beta_0} , \tag{E.2}$$

with $F(\beta) = (1/N) \ln Z(\beta)$. This proves the convexity of $F(\beta)$ as a function of the inverse temperature, a well-known result of thermodynamics.

Turning now to the lattice gauge theory (3.3), we choose for the measure

$$\mathrm{d}\mu = \prod_{\langle ij \rangle} \mathcal{D}R_{ij}\, \mathrm{e}^{\chi(KR_{ij})} ,$$

and for X

$$X = A - \chi(K \sum_{\langle ij \rangle} R_{ij}) .$$

We again get, on the left-hand side, the partition function, which does not depend on K. Therefore, one can maximize on K and obtain the mean-field result [2]:

$$F(\beta) \geqslant \operatorname*{Max}_{K} \left\{ D\left[u(k) - \chi\left(K\frac{\mathrm{d}u}{\mathrm{d}K}\right)\right] + \tfrac{1}{2}\beta D(D-1)\,\chi\left(\left(\frac{\mathrm{d}u}{\mathrm{d}K}\right)^4\right) \right\} , \tag{E.3}$$

where the function $u(K)$ is defined as

$$u(K) = \ln\left(\int \mathcal{D}R \exp \chi(KR)\right) . \tag{E.4}$$

The parameter K is a matrix in the representation whose character is χ, and derivatives are of course to be understood in the matrix sense. We emphasize that, although the mean-field approximation is highly suspicious in the lattice gauge theories (it breaks local gauge invariance), the inequality is rigorous.

References

[1] K.G. Wilson, Phys. Rev. D10 (1974) 2445; Erice Lectures notes.
[2] R. Balian, J.M. Drouffe and C. Itzykson, Phys. Rev. D10 (1974) 3376; D11 (1975) 2098; 2104; D19 (1979) 2514;
J.M. Drouffe and C. Itzykson, Phys. Reports 38 (1978) 133.
[3] J. Kogut and L. Susskind, Phys. Rev. D11 (1975) 395.
[4] J.M. Drouffe, Phys. Rev. D18 (1978) 1174.
[5] T. Banks, R. Myerson and J. Kogut, Nucl. Phys. B129 (1977) 493.
[6] S. Elitzur, Phys. Rev. D12 (1975) 3978.
[7] F. Englert, Phys. Rev. 129 (1963) 567;
M.E. Fisher and D.S. Gaunt, Phys. Rev. 133 (1964) A224;
R. Abe, Prog. Theor. Phys. 47 (1972) 62.

[8] J.M. Drouffe, G. Parisi and N. Sourlas, in preparation.
[9] L.C. Lubensky and J. Isaacson, Phys. Rev. Lett. 41 (1978) 829; and references therein.
[10] A. Polykov, unpublished.
[11] M. Wortis, *in* Phase transitions and critical phenomena, ed. C. Domb and S. Green (Academic Press, 1974) vol. 3, and references therein.
[12] J.M. Kosterlitz, D.J. Thouless and R.C. Jones, Phys. Rev. Lett. 36 (1976) 1217;
D.J. Thouless, P.W. Anderson and R.G. Palmer, Phil. Mag. 35 (1977) 593;
B. Derrida, private communication.
[13] G. Parisi, Phys. Lett. 81B (1979) 357.
[14] J. Kogut, D.K. Sinclair and L. Susskind, Nucl. Phys. B114 (1976) 199.
[15] D. Stauffer, Phys. Rev. Lett. 41 (1978) 1333.
[16] M. Creutz, L. Jacobs and C. Rebbi, Phys. Rev. Lett. 42 (1979) 1390.