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THE MASS MATRIX OF $N = 8$ SUPERGRAVITY

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We analyze the group-theoretic basis of the mass formulae which hold in $N = 8$ spontaneously broken supergravity, using the supersymmetry and the internal symmetries of this theory. The proportionality of gauge couplings and masses can be related to properties of central charges.

It has been noted recently [1] that, in a wide class of spontaneously broken supersymmetric field theories, certain mass relations are satisfied with the consequence that severe limitations in the resulting particle spectrum of the theory emerge. Interestingly enough these relations in turn imply that the cosmological term induced by one-loop quantum corrections is at most logarithmically divergent, in contrast with non-supersymmetric theories where it is usually quartically divergent [2]. Most surprisingly, in the recently constructed spontaneously broken version of $N = 8$ supergravity [3], the symmetry is so stringent that the particle spectrum of the broken theory satisfies several mass relations, stating that the graded trace of the first three powers of the square-mass operator vanishes [4]. As a consequence the one-loop induced cosmological term is finite [2]. It is the aim of the present paper to give a simple group theoretic explanation of these mass relations in $N = 8$ supergravity. As a byproduct we are able to discuss certain supersymmetric subtheories of $N = 8$ spontaneously broken supergravity in which massive multiplets with central charges [5] appear.

According to ref. [4] spontaneously broken $N = 8$ supergravity in four dimensions can be obtained from $N = 1$ eleven-dimensional supergravity [6] by applying dimensional reduction in two steps. First one uses standard dimensional reduction from $D = 11$ to $D = 5$ space–time dimensions. Then one uses a generalized dimensional reduction [3] from $D = 5$ to $D = 4$ to obtain the final version of the theory. In the intermediate reduction procedure the unbroken five-dimensional theory has an Sp(8) local invariance in much the same way as the four-dimensional unbroken theory has an SU(8) local invariance [7]. The five-dimensional supergravity theory is a massless theory in which (free) particle states belong to Sp(8) multiplets. We indicate by J the five-dimensional spin. Then the $J = 2$ state is an Sp(8) singlet, the $J = 3/2$ state is an Sp(8) octet and the $J = 1, \frac{1}{2}, 0$ states are in the 27, 48, 42 dimensional representations of Sp(8), respectively. Note that these states have degeneracy $2J + 1$ because the little group of massless states in five-dimensions is SO(3). On the other hand, if one groups together states having the same J_3 namely if one decomposes SO(3) representations into SO(2) representations (appropriate to massless particles in four-dimensions), then one gets the usual SO(8) classification, i.e., a singlet with $|J_3| = 2$, an octet with $|J_3| = 3/2$, 28 states with $|J_3| = 1$, 56 states with $|J_3| = \frac{1}{2}$ and 70 states with $J_3 = 0$.

Using the method of generalized dimensional reduction from $D = 5$ to $D = 4$ space–time dimensions one gets masses by maintaining a dependence of the fields upon the 5th coordinate associated to a representation of a group element of the Cartan subalgebra of Sp(8). The mass spectrum of the four-dimensional theory is given by the eigenvalues of an arbitrary element of the Sp(8) Cartan subalgebra in the representation to which the field describing the particle of spin J belongs. Since the Cartan subalgebra of Sp(8) is four-dimensional the mass spectrum of the broken $N = 8$ theory in four dimensions is parametrized by four arbitrary real parameters. If the four dimensional mass of a state of a given spin is non-zero, then the degeneracy is the same as that of the corresponding massless state in five dimensions. When the four-dimensional mass vanishes, one must take into account that the five dimensional massless state corresponds to several four dimensional massless states, according to the appropriate reduction of SO(3) representations into SO(2) representations. Following ref. [4], we write an element of the Cartan subalgebra of a compact (unitary) Sp($2n$) as follows

$$\mathcal{M}(m_k) = \sum_{k=1}^{n} m_k \lambda_k , \tag{1}$$

where

$$\lambda_k = \epsilon \otimes e_{kk} , \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2}$$

and e_{ij} are n by n matrices with elements 1 in the intersection of the ith row with the jth column and 0 otherwise. The eigenvalues of $\mathrm{i}\mathcal{M}(m_k)$ in the relevant representations of Sp(8) are given in table 1, where the numbers in front of the parentheses are the multiplicities. The moduli of these eigenvalues are the four-dimensional masses of the corresponding particle states. The representative of the matrix $\mathrm{i}\mathcal{M}$ plays the role of the mass matrix in the dimensionally reduced four dimensional theory.

In order to prove the mass relations which have been noted in ref. [4] we give some mathematical preliminaries. Consider the Cartan subalgebra of Sp($2n$) whose arbitrary element is given by eq. (1). It is

$$\mathrm{Tr}(\mathcal{M}_\pi)^q = 0, \tag{3}$$

if q is odd and

$$\mathrm{Tr}(\mathcal{M}_\pi)^{2q} = \mathcal{P}^q_\pi(\alpha_k), \tag{4}$$

where $\mathcal{M}_\pi$ denotes a representation of $\mathcal{M}$ and $\mathcal{P}$ is a homogeneous polynomial of degree q, symmetric in the variables $\alpha_k = (m_k)^2$.

In order to prove eqs. (3) and (4), it is sufficient to find two group elements A and B of Sp($2n$) such that

$$A\lambda_k A^{-1} = \lambda_h , \quad h \neq k, \quad A\lambda_h A^{-1} = \lambda_k , \qquad A\lambda_i A^{-1} = \lambda_i , \quad \text{for all } i = 1, \dots , n;\ i \neq h, k \tag{5}$$

and

$$B\lambda_r B^{-1} = -\lambda_r , \qquad B\lambda_s B^{-1} = \lambda_s , \quad \text{for all } s = 1, \dots , n;\ s \neq r. \tag{6}$$

Let us recall that $A, B \in \mathrm{Sp}(2n)$ if they are unitary and $A^{\mathrm{T}}\Omega A = \Omega$, $B^{\mathrm{T}}\Omega B = \Omega$ where Ω is the symplectic metric $\Omega = \epsilon \otimes 1$ (1 unit matrix). It is easy to verify that, in the $2n$ dimensional representation of Sp($2n$), these group elements are given by

$$A = 1 \otimes (1 - e_{hh} - e_{kk} + e_{hk} + e_{kh}), \qquad B = \mathrm{i}\,\sigma_1 \otimes e_{rr} + 1 \otimes (1 - e_{rr}). \tag{7}$$

Eq. (6) implies that $\mathcal{P}^q_\pi$ is symmetric in m_k and eq. (7) that it depends only upon $(m_k)^2$.

We now consider the graded trace

$$\mathrm{GrTr}\,(\mathcal{M})^{2q} = \sum_{J=0}^{2} (-)^{2J}(2J+1)\,\mathrm{Tr}(\mathcal{M}_J)^{2q} = \mathcal{P}^q(\alpha_k), \tag{8}$$

Table 1

Five-dimensional spin	Sp(8) representation	Eigenvalues of $\mathrm{i}\mathcal{M}(m_k)$
$J = 2$	1	0
$J = 3/2$	8	$\pm m_k$
$J = 1$	27	$3(0), \pm(m_i \pm m_j); \quad i < j$
$J = 1/2$	48	$2(\pm m_k), \pm(m_i \pm m_j \pm m_k); i < j < k$
$J = 0$	42	$2(0), \pm(m_i \pm m_j), \pm(m_1 \pm m_2 \pm m_3 \pm m_4); i < j$

where $\mathcal{M}_J$ is the representative of $\mathcal{M}$ in the representation of spin J. We can easily show that

$$\mathrm{GrTr}(\mathcal{M})^{2q} = \mathcal{P}^q(\alpha_k) = 0, \tag{9}$$

for all q, if one of the α_k vanishes. This is due to the fact that, if a subset of h of the α_k vanishes, then the theory possesses an unbroken $N = 2h$ extended supersymmetry. Indeed the 8 spin 3/2 fields belong to the defining representation of Sp(8) and their square mass is just given by α_k. The presence of this unbroken supersymmetry insures that eq. (9) is valid for each separate supermultiplet, since the masses of a supermultiplet are equal and the number of boson states equals the number of fermion states. From eq. (9), because of the homogeneity and symmetry of $\mathcal{P}^q(\alpha_n)$ it follows as a corollary that

$$\mathcal{P}^q(\alpha_k) = 0, \tag{10}$$

identically for $q = 0, \ldots, n-1$. In the Sp(8) theory $\mathcal{P}^0 = \mathcal{P}^1 = \mathcal{P}^2 = \mathcal{P}^3 = 0$. However, $\mathcal{P}^4 \neq 0$ because the term proportional to $\alpha_1\alpha_2\alpha_3\alpha_4$ in $\mathcal{P}^4$ comes only from the $(J = 0)$ 42-dimensional representation of Sp(8). This proves the mass relations.

One can check (see table 2) that, when h of the α_n's vanish, both massless and massive states group themselves in supermultiplets of $N = 2h$ extended supersymmetry. All massive supermultiplets have non-vanishing central charges and are equivalent to a pair of massive multiplets of $N = h$ extended supersymmetry with vanishing central charges [8]. Particles of the same four-dimensional spin group together, inside an irreducible massive supermultiplet, in antisymmetric representations of Sp($2h$). On the other hand massless particles of the same helicity group together, inside an irreducible supermultiplet, in antisymmetric representations of SO($2h$). In ref. [3] some of the generators of the unbroken "flat group" correspond to vector fields with minimal coupling. In the symmetric limit, when h of the α_k vanish, one sees that these generators are the central charges. This explains why the minimal couplings of ref. [3] are proportional to the masses [9].

From eq. (10) it is clear that the number of mass relations depends on the rank of Sp($2n$). This means that if we consider a subtheory of the Sp(8) theory with a residual symmetry Sp($2h$) there will be in general $h - 1$ residual mass relations. We can explicitly construct, as far as the particle spectrum is concerned, completely broken subtheories of the $N = 8$ theory by first constructing subtheories of the five-dimensional unbroken theory and by performing then the generalized dimensional reduction of it to obtain a completely broken four-dimensional theory.

Example: Sp(6) supergravity in five-dimensions. We decompose the Sp(8) supermultiplet into Sp(6) supermultiplets according to the following chain:

$$\begin{aligned}&[1(2),8(3/2),27(1),48(1/2),42(0)]\\ &\quad = [1(2),6(3/2),15(1),20(1/2),14(0)]\\ &\quad \oplus 2\,[1(3/2),6(1),14(1/2),14(0)]\,.\end{aligned} \tag{11}$$

Table 2

Masses	Multiplets
	$m_4 = 0$
0	[1(2),2(3/2),1(1)] 3[1(1),2(1/2),2(0)]
$\lvert m_i\rvert$	2[1(3/2),2(1),1(1/2)] 2[1(1/2),2(0)]
$\lvert m_i \pm m_j\rvert$	2[1(1),2(1/2),1(0)]
$\lvert m_i \pm m_j \pm m_k\rvert$	2[1(1/2),2(0)]
	$m_3 = m_4 = 0$
0	[1(2),4(3/2),6(1),4(1/2),2(0)], 2[1(1),4(1/2),6(0)]
$\lvert m_i\rvert$	2[1(3/2),4(1),6(1/2),4(0)]
$\lvert m_i \pm m_j\rvert$	2[1(1),4(1/2),5(0)]
	$m_2 = m_3 = m_4 = 0$
0	[1(2),6(3/2),16(1),26(1/2),30(0)]
$\lvert m\rvert$	2[1(3/2),6(1),14(1/2),14(0)]
	$m_1 = m_2 = m_3 = m_4 = 0$
0	[1(2),8(3/2),28(1),56(1/2),70(0)]

Table 3

Spin (in 5 dimensions)	Masses	Sp(6) representation
2	0	1
3/2	$\pm m_i$	6
1	$\pm (m_i \pm m_j)$, 3(0)	15
1/2	$2(\pm m_i)$, $\pm (m_i \pm m_j \pm m_k)$	20
0	2(0), $\pm (m_i \pm m_j)$	14
Spin		
3/2	0	1
1	$\pm m_i$	6
1/2	$\pm (m_i \pm m_j)$, 2(0)	14
0	$\pm (m_i \pm m_j \pm m_k)$, $\pm m_i$	14

We take the Cartan subalgebra of Sp(6) contained in the Cartan subalgebra of Sp(8). The reduced mass matrix of Sp(6) in the Sp(8) multiplet is given by table 1 with $m_4 = 0$. This gives rise to table 3. Five-dimensional irreducible Sp(6) supergravity is given by the first supermultiplet. If all $m_i \neq 0$ ($i = 1,2,3$), supersymmetry is completely broken. If $m_1 = 0$, there is a residual $N = 2$ supersymmetry; if $m_1 = m_2 = 0$, there is a residual $N = 4$ supersymmetry. In this theory $\mathrm{GrTr}(\mathcal{M})^{2q} = 0$ for $q = 0, 1, 2$. We can proceed further if desired by reducing the Sp(6) theory down to the Sp(4) and the Sp(2) broken symmetry theories with 2 and 1 mass parameters, respectively. Note that there is only one gauged central charge in all these theories.

It is clear that in the class of theories so far considered the Sp(2) subtheory ($N = 2$) does not fulfill any mass relation in contrast to the class of models of ref. [1] in which the quadratic mass formula is satisfied even in $N = 1$ supersymmetry. It would be interesting to derive that formula in a group-theoretical way along lines similar to those of the present investigation.

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