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Random Magnetic Fields, Supersymmetry, and Negative Dimensions

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We prove the equivalence, near the critical point, of a D -dimensional spin system in a random external magnetic field with a $(D-2)$ -dimensional spin system in the absence of a magnetic field. This is due to the hidden supersymmetry of the associated stochastic differential equation. We identify a space with one anticommuting coordinate with a space having negative dimensions -2 .

The critical behavior of a spin system in a random external magnetic field (i.e., the infrared behavior of a scalar field theory in presence of a random external source) has recently been investigated.¹ By explicit computations it has been found^{1,2} that the values of the most-infrared-divergent diagrams³ in dimensions D are equal to the values of the same diagrams without magnetic fields in dimensions $D-2$. In this Letter we show that this apparently mysterious result has a simple geometrical interpretation, which stems from a hidden supersymmetry⁴ of the associated stochastic equation.

Let us define the free energy F_R averaged over a Gaussian random magnetic field by the functional integrals⁵:

$$\begin{aligned}
 F[h] &= \ln \int \mathcal{D}\varphi \exp\{-\int d^Dx [\mathcal{L}(x) + h(x)\varphi(x)]\}, \\
 F_R &= \int \mathcal{D}h F[h] \exp[-\frac{1}{2}\int d^Dx h^2(x)], \\
 \mathcal{L}(x) &= -\frac{1}{2}\varphi(x)\Delta\varphi(x) + V(\varphi(x)).
 \end{aligned}
 \tag{1}$$

For definiteness we can consider $V(\varphi) = \frac{1}{2}m^2\varphi^2 + g\varphi^4$. The perturbative expansion in g for F_R can be easily constructed either by direct inspection,

or by using the replica trick.⁶ The most-infrared-divergent diagrams contain the maximum number of h^2 insertions, as follows from dimensional analysis. If the other diagrams are neglected we obtain the tree approximation for $F(h)$. Using the correspondence between the tree approximation⁷ and the classical nonlinear differential equation, we find that in this limit the two point Green's function is given by

$$\begin{aligned}
 \langle \varphi(x)\varphi(0) \rangle_R & \\
 & \sim \int \mathcal{D}h \varphi_h(x)\varphi_h(0) \exp[-\frac{1}{2}\int d^Dy h^2(y)],
 \end{aligned}
 \tag{2}$$

where $\varphi_h(x)$ is the solution of the equation

$$-\Delta\varphi + V'(\varphi) + h = 0.
 \tag{3}$$

Equation (3) can also be regarded as a differential stochastic equation, h being a stochastic Gaussian function having autocorrelation $\langle h(x)h(y) \rangle = \delta^D(x-y)$. The results of Refs. 1 and 2 imply that the Green's functions of the stochastic differential equations (3) are the same as those generated by the Lagrangian of Eq. (1) in $D-2$ dimensions. Let us see why.

Using standard manipulations,⁸ we find

$$\begin{aligned}
 \langle \varphi(x)\varphi(0) \rangle & \sim \int \mathcal{D}\varphi \mathcal{D}h \varphi(x)\varphi(0) \delta(-\Delta\varphi + V'(\varphi) + h) \det[-\Delta + V''(\varphi)] \exp[-\frac{1}{2}\int h^2(y)d^Dy] \\
 & \sim \int \mathcal{D}\varphi \mathcal{D}\omega \mathcal{D}\psi \exp[-\int d^Dy \mathcal{L}_R(y)] \varphi(x)\varphi(0),
 \end{aligned}
 \tag{4}$$

$$\mathcal{L}_R = -\frac{1}{2}\omega^2 + \omega[-\Delta\varphi + V'(\varphi)] + \bar{\psi}[-\Delta + V''(\varphi)]\psi,$$

where ψ is an anticommuting scalar field⁹ (a ghost field). The Lagrangian \mathcal{L}_R is invariant under the supersymmetry transformations:

$$\begin{aligned}
 \delta\varphi &= -\bar{a}\epsilon_\mu x_\mu\psi, & \delta\omega &= 2\bar{a}\epsilon_\mu\partial_\mu\psi, \\
 \delta\psi &= 0, & \delta\bar{\psi} &= \bar{a}(\epsilon_\mu x_\mu\omega + 2\epsilon_\mu\partial_\mu\varphi),
 \end{aligned}
 \tag{5}$$

\bar{a} being an infinitesimal anticommuting number and ϵ_μ an arbitrary vector. The invariance under

these supersymmetry transformations [Eq. (5)] is quite unexpected.¹⁰ It is useful to introduce the superspace⁴ characterized by a D -dimensional commuting coordinate x and by an anticommuting coordinate θ ($\theta^2 = \bar{\theta}^2 = \theta\bar{\theta} + \bar{\theta}\theta = 0$) and the superfield

$$\Phi(x, \theta) = \varphi(x) + \bar{\theta}\psi(x) + \bar{\psi}(x)\theta + \theta\bar{\theta}\omega(x).
 \tag{6}$$

Higher orders in θ are identically zero as a re-

It of the anticommuting properties of θ . The action [Eq. (4)] can be written as $\int d^D x d\theta \mathcal{L}_{ss}(\Phi)$, with

$$\mathcal{L}_{ss}(\Phi) = -\frac{1}{2} \Phi \Delta_{ss} \Phi + V(\Phi), \quad (7)$$

where $\Delta_{ss} = \Delta + \partial^2 / \partial \bar{\theta} \partial \theta$ is the Laplacian in the superspace and the integration in θ selects the term proportional to $\theta \bar{\theta}$ [e.g., $\int d\theta \Phi(x, \theta) = -\pi^{-1} \omega(x)$].¹¹

The supersymmetry transformations [Eq. (5)] are simply rotations in superspace leaving invariant the metric $x^2 + \theta \bar{\theta}$. We argue that the superspace (x, θ) is equivalent to an ordinary $(D-2)$ -dimensional space. Indeed a space with only one anticommuting coordinate θ is equivalent to an ordinary space with negative dimensions -2 . (Space with negative dimensions are defined by analytic continuations¹² from positive dimensions.) This can be seen from the relation

$$\int d\theta f(\bar{\theta}\theta) = -\frac{1}{\pi} \frac{d}{dz} f(z) \Big|_{z=0} = \lim_{D \rightarrow -2} \int d^D r f(r^2) = \lim_{D \rightarrow -2} S_D \int r^{D-1} dr f(z^2), \quad S_D = 2\pi^{D/2} / \Gamma(\frac{1}{2}D). \quad (8)$$

S_D is the surface of the unit sphere in D dimensions.

Let us consider a space of dimension $D-2$ and formally decompose it as the sum of a space of dimension D and of another space of dimension -2 . The previous argument implies that an ordinary space of dimension $D-2$ is equivalent to the D -dimensional superspace. The precise meaning of the equivalence is the following:

$$\int d^{D-2} x F(Y_i, x, x^2) = \int d^D x d\theta F(Y_i, x, x^2 + \bar{\theta}\theta), \quad (9)$$

where Y_i are some $(D-2)$ -dimensional vectors. For example,

$$\int d^{D-2} x f(x^2) = \int d^D x d\theta f(x^2 + \bar{\theta}\theta) = -\pi^{-1} \int d^D x f'(x^2) = \int d^{D-2} x f'(x^2). \quad (10)$$

Equation (9) is sufficient to prove, at all orders in perturbation theory, that the Green's functions computed in the $D-2$ space are the same as those computed in the D -dimensional superspace. Indeed, the perturbative expansion for the Lagrangian (7) [which is equivalent to the stochastic Eq. (3)] can be written directly from Feynmann's rule in configuration superspace using the technique of superpropagator.¹³ The final integrals have the form of Eq. (9) and the equivalence of the stochastic Eq. (3) with the $(D-2)$ -dimensional field theory is therefore proved in the perturbative expansion. It has its root in the hidden supersymmetry of the system and in the geometrical equivalence of an anticommuting-variable space with a negative-dimensional space.

It may be useful to establish this equivalence rigorously beyond perturbation theory. The stochastic differential Eq. (3) may provide us with a different framework to study the properties of a field theory. It would be quite interesting to see if and how this formalism can be extended to gauge theories.

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¹¹The factor $-1/\pi$ has been introduced for later convenience.

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