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ABSTRACT.

Nonrelativistic non-abelian gauge theories are formulated. In these theories the electric potentials are auxiliary fields, while the magnetic potentials satisfy self-coupled equations with admissible localized solutions.

Nonrelativistic versions of the so far well understood relativistic theories provide an accurate description of low-energy phenomena. Moreover they require simple calculations, especially as far as bound states are concerned whose properties, at least for two- and three-body systems can be determined by numerical computations to any desired accuracy. Well known exemples are Coulomb and One-Boson-Exchange potentials in quantum mechanics.

A nonrelativistic version of non-abelian gauge theories is particularly desirable since perturbation theory is not applicable at low energy, and the bound state problem is of fundamental importance. It is not at all obvious, however, that a nonrelativistic version exist, because nonrelativistic massless real fields cannot be quantized, while a characteristic feature of relativistic non-abelian gauge fields is their self interaction.

In this paper we will in fact see that nonrelativistic non-abelian gauge theories can be formulated but have novel features with respect to ordinary quantum mechanics.

In order to keep as close as possible to the relativistic formulation we start from equations of motion for the spinor field which are linear in space-time derivatives[1]

$$-i \partial_t \psi + i \partial_k \sigma_k \chi = 0 \ , \qquad -i \partial_k \sigma_k \psi + 2 m \chi = 0 \ . \tag{1}$$

In the above equations σ_k are Pauli matrices, m the fermion mass, and χ an auxiliary fermion field. Its elimination yields the Schrödinger equation for ψ. Eq.(1) are invariant[1] under Galilei transformations

$$\psi'(x', t') = e^{if(x, t)} \psi(x, t) \ ,$$

$$\chi'(x', t') = e^{if(x, t)} \left[\chi(x, t) - \frac{1}{2} v_k \sigma_k \psi(x, t) \right] \ , \tag{2}$$

where

$$f(x, t) = \frac{1}{2} m v^2 t + m v_k x_k \ . \tag{3}$$

We now introduce gauge fields V^a, A_k^a belonging to the regular representation of a compact Lie group G, whose generators are represented by Hermitian matrices t^a with commutation relations

$$\left[t^a, t^b \right] = i f^{abc} t^c \ . \tag{4}$$

The gauge fields have gauge transformations

$$V^a \rightarrow V^a - \partial_t \omega^a - g f^{abc} \omega^b V^c \ ,$$

$$A_k^a \rightarrow A_k^a + \partial_k \omega^a - g f^{abc} \omega^b A_k^c \ , \tag{5}$$

and Galilei transformations[1]

$$V^a \rightarrow V^a + v_k A_k^a \ , \qquad A_k^a \rightarrow A_k^a \ . \tag{6}$$

Assuming the fermion fields to belong to any representation of G, we couple fermions to gauge fields by the standard replacement of ordinary derivatives by covariant derivatives

$$-iD_t\psi + iD_k\sigma_k\chi = 0 \,, \qquad -iD_k\sigma_k\psi + 2m\chi = 0 \,, \qquad (7)$$

where

$$D_t = \partial_t + igV^a t^a \,, \qquad D_k = \partial_k - igA_k^a t^a \,. \qquad (8)$$

Eqs.(7) are gauge-invariant and Galilei-invariant. Using standard procedures we can show that fermion current

$$\varrho = \psi^* \psi \,,$$

$$J_k = \frac{1}{2mi}(\psi^* \partial_k \psi - \partial_k \psi^* \psi) - \frac{g}{m}A_k^a \psi^* t^a \psi + \qquad (9)$$

$$+ \frac{1}{2m}\varepsilon_{kh\ell}\,\partial_k(\psi^*\sigma_\ell\psi)$$

is conserved

$$\partial_t \varrho + \partial_k J_k = 0 \,. \qquad (10)$$

Moreover, using the matrix elements of the covariant derivative in the adjoint representation

$$D_t^{ab} = \partial_t \delta^{ab} + gf^{abc}V^c \,, \qquad D_k^{ab} = \partial_k \delta^{ab} - gf^{abc}A_k^c \,. \qquad (11)$$

we can define the color current·

$$\varrho^a = \psi^* t^a \psi \,,$$

$$J_k^a = \frac{1}{2mi}(\psi^* t^a D_k \psi - D_k^* \psi^* t^a \psi) + \qquad (12)$$

$$+ \frac{1}{2m}\varepsilon_{kh\ell}\,D_h^{ab}\psi^* t^b \sigma_\ell \psi \,,$$

which satisfies the equation

$$D_t^{ab}\varrho^b + D_k^{ab}J_k^b = 0 \,. \qquad (13) \cdot$$

We see that fermions have a magnetic moment given by

$$\mu_k = \frac{g}{2m}\sigma_k$$

as predicted for a relativistic fermion (in absence of radiative corrections).

This is a consequence of the use of eqs.(1) for the spinor field and of minimal coupling leading to eqs.(7).

It remains to determine the equations for the gauge fields. As a first step we define the tensors

$$F^a_{ok} = \partial_t A^a_k + \partial_k V^a + g f^{abc} A^b_k V^c ,$$

$$F^a_{ij} = \partial_i A^a_j - \partial_j A^a_i + g f^{abc} A^b_i A^c_j , \qquad (14)$$

which have the following Galilei transformations

$$F^a_{ok} \rightarrow F^a_{ok} + v_i F^a_{ik} , \qquad F^a_{ij} \rightarrow F^a_{ij} . \qquad (15)$$

The unique gauge-invariant equation we can write for F^a_{ok} is identical to the relativistic one

$$D^{ab}_k F^b_{ok} = g \varrho^a . \qquad (16)$$

This equation is Galilei-invariant provided

$$D^{ab}_j F^b_{ij} = 0 . \qquad (17)$$

This last equation is also the unique Galilei-invariant equation which can be written for F^a_{ij}. F^a_{ij} in fact cannot be coupled to the current J^a_k because under Galilei transformations

$$J^a_k \rightarrow J^a_k + v_k \varrho^a , \qquad (18)$$

and with the tensors (14) and the covariant derivatives (11) we cannot construct anythink transforming like that.

Eq. (16) shows that the field V^a can be expressed in terms of ϱ^b and A^b_K. It is therefore an auxiliary field whose elimination from eqs.(7) provides the fermion-fermion potential, which turns out to depend on the magnetic potential. This latter, however, is neither an auxiliary field (it cannot be expressed in terms of other quantities), nor a dynamical variable (its equation of motion is time-independent) nor an external field (it is self-coupled).

Due to this peculiarity the theory does not fit automatically into the scheme of ordinary quantum mechanics, but requires a prescription concer

ning the solutions to eq. (17). The analogy with the relativistic case suggests the use of the path-integral method with a sum over all the solutions. A complete settlement of this point however, demands further investigation and a classification of the solutions.

$A_i^a = 0$ is a solution. For such solution eq. (16) reduces to the Poisson equation for V^a. But also nontrivial solutions have been found by T. T. Wu and C. N. Yang[2] for SU2 and by A. C. T. Wu and T. T. Wu[3] for SU3. Let us indeed explicitly note that we do not need to require a finite action for A_i^a, but only that it be such that the fermion energy be bounded from below. Therefore Coleman's argument[4], showing that there are no nontrivial solutions to eq. (17) with finite action is not relevant to the present case.

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