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ABSTRACT.

The Q^2-dependence of the nucleon structure function F_3 is analyzed using an approach inspired by the Bloch-Nordsieck theorem in the context of QCD. Detailed comparison with the Altarelli-Parisi (AP) equations, which give identical results to the renormalization group, is made. Moments are obtained and compared successfully with the existing data. A regularization scheme is proposed which avoids singular distributions. In the appropriate limit, it reduces to the AP result.

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1. - INTRODUCTION.

In this paper we present an application to QCD of an approach inspired by the classic Bloch-Nordsieck (BN) theorem to obtain the Q^2-dependence of the moments of the non-singlet quark distributions. The underlying physical picture is as follows. The observed x-distribution is supposed to be a folding of the "primitive" (scaling) probability of finding a quark in the hadron with the soft and collinear-hard gluon x-distribution given by the BN theorem. In Section 2 we show that for the moments $\langle x^n \rangle$, the above implies a factorization into 2 terms: one part depends upon the primitive distribution, and it is thus not known (a priori), but is independent of Q^2, and the other one is a calculable QCD distribution which gives the relevant Q^2 dependence. This happy circumstance obtains only for non-singlet quark distributions and offers thus the possibility of a clean test of this approach, through a comparison with the moments of the (nucleon) structure function F_3.

Previously, moments of structure functions have been analyzed in detail by the renormalization group[1, 2](RG) or equivalently through the Altarelli-Parisi (AP) equations[3]. These equations also provide a summation, albeit a different one from the one presented here, of the gluon distribution. The essential physical difference between the two may be summarized in the following way. Standard perturbation theory begins with a free quark at x = 1, i.e. a $\delta(1-x)$ distribution, supplemented (to first order in α) by the soft-gluon spectrum proportional to (α/x) for $x \rightarrow 0$. AP regularize this spectrum by defining a distribution $(1/x)_+$ such that, upto this order, the quark spectrum is correctly prescribed. Their evolution equations then provide the vehicle for computing the (logarithmic) scaling violations in Q^2. Our approach (a là BN) on the other hand has, a priori, no IR divergence[4]. The corresponding soft-gluon distribution, for $x \rightarrow 0$, is proportional to $(\alpha/x^{1-\beta})$, where β itself is of order α, and this automatically provides an integrable spectrum. Thus, if one asks the question, say in QED, as to the probability of finding an electron without any associated photons, the BN approach immediately gives the null answer, whereas for every order in perturbation theory, the corresponding result is infinite (and it is only the final sum which gives the correct answer, zero). The point is that for any finite α, however small but not zero, there is never a "free" electron, i. e. there is strictly speaking, no $\delta(1-x)$ term. In fact, in Section 4, we modify the kernel of the AP equations using the β-method and show that there are no singular distributions whatsoever and that to first order our results match precisely with those given by the AP (or RG) equations, as they should. These may be useful for Monte Carlo simulations of the quark and gluon evolutions, especially in the context of jets, for which clearly a procedure avoiding any singular distributions is required.

Section 2 defines the problem and sets up the algorithm for the separation of the Q^2 dependence of the moments. In Section 3 we develop our approach and obtain the formula for the moments. A comparison with RP (AG) moments is also made. In Section 4 we show the modification of the AP equations using the β-method. Section 5 deals with the phenomenological application of the moments derived in Section 3 to analyze the experimental $\nu(\bar{\nu})$ data on $F_3^{(5)}$. The paper closes with some brief concluding remarks.

2. - SEPARATION OF PRIMITIVE AND RUNNING MOMENTS.

Let us assume that the probability of finding a quark inside a hadron is given by the (primitive) distribution (dP_o/dX) where X is the fraction of hadronic energy (or longitudinal momentum) carried by the said quark. Now, this quarks can emit and absorb gluons and will find itself at the end with the fractional energy x. The probability of losing a certain fraction of energy, call it $d\bar{P}/(dx/X)$, is to be calculated using the BN approach in QCD. Since the two steps are assumed independent, it is easy to see that the observed x-distribution is given by the equation

$$\left(\frac{dP}{dx}\right) = \int_x^1 \left(\frac{dX}{X}\right)\left(\frac{dP_o}{dX}\right)\left(\frac{d\bar{P}}{dx/X}\right) . \tag{1}$$

Defining,the moments

$$\langle x^n \rangle \equiv \int_0^1 \frac{dx}{x}\, x^n \left(\frac{dP}{dx}\right) , \tag{2}$$

eq. (2) lets us obtain the factorized form

$$\langle x^n \rangle = \langle X^n \rangle \langle \bar{x}^n \rangle , \tag{3}$$

where, in an obvious notation, $\langle X^n \rangle$ denotes the moment of the primitive distribution P_o, and $\langle \bar{x}^n \rangle$ the corresponding moment of $\bar{P}$. $(d\bar{P}/dy)$ is properly normalized to a Dirac δ-fn. $\delta(1-y)$ when the QCD coupling constant $\alpha = 0$, to obtain the primitive scaling distribution (dP_o/dy) in this limit. Since $\langle X^n \rangle$, by definition, have no Q^2--dependence, all the observed Q^2 variation is contained in the running moments $\langle \bar{x}^n \rangle$ for which we obtain explicit expressions in the next Section. From now on, we shall drop the bars over $\langle \bar{x}^n \rangle$ and call them simply $\langle x^n \rangle$.

From what has been said earlier it should be clear that the above is true only for non-singlet quark distributions. For the general case, both quark and gluon distributions enter and we have a system of matrix equations from which the primitive moments do not drop out. It is for this reason that the analysis, say of F_2, is more model dependent since the primitive distributions need to be specified. In this work, we shall not deal with this problem.

3. - BN MOMENTS.

In QED, the Bloch-Nordsieck theorem and the overall energy--momentum conservation leads to the following soft-photon formula for the energy distribution[6]

$$\frac{dP}{d\omega} = \int_{-\infty}^{\infty} \left(\frac{dt}{2\pi}\right) e^{i\omega t - \beta h(E,t)} , \tag{4}$$

where

$$h(E,t) = \int_0^E \frac{dk}{k} (1 - e^{-ikt}) , \tag{5}$$

and

$$\beta = - \int (d^2 \underset{\sim}{n})\, j_\mu(n)\, j^{\mu +}(n) , \tag{6}$$

is the single-photon spectrum written in terms of the classical current

$$j_\mu(k) = \frac{ie}{(2\pi)^{3/2}} \sum_i \eta_i \left(\frac{p_{i\mu}}{p_i \cdot k} \right) . \tag{7}$$

Defining, the fraction $x = \omega/E$, a standard calculation yields[6]

$$\frac{dP_\gamma}{dx} = \frac{x^{\beta-1}}{\gamma^\beta \Gamma(\beta)} , \qquad (x \lesssim 1) \tag{8}$$

where $\gamma = e^C = 1.781$ is Euler's constant.

If we include the hard-photon correction, i. e. replace $1/k$ by $\frac{1 + (1 - k/E)^2}{2k}$ as given by the Weizäcker-Williams approximation, then for large E, eq. (5) may be replaced by

$$\bar{h}(E,t) \simeq h(E,t) + \int_0^E \frac{dk}{k} \left(- \frac{k}{E} + \frac{k^2}{2E^2} \right) = h(E,t) - \frac{3}{4} .$$

Thus, eq. (8) is modified to read

$$\frac{d\bar{P}_\gamma}{dx} = \frac{e^{\frac{3}{4}\beta}\, x^{\beta-1}}{\gamma^\beta \Gamma(\beta)} . \tag{9}$$

The corresponding electron spectrum would be given by

$$\frac{d\bar{P}_e}{dx} = \frac{e^{\frac{3}{4}\beta}(1-x)^{\beta-1}}{\gamma^{\beta}\,\Gamma(\beta)} \quad . \tag{10}$$

The nth-moment of this distribution is

$$\langle x^n \rangle \equiv \int_0^1 dx\, x^{n-1} \left(\frac{d\bar{P}_e}{dx}\right) = \frac{e^{\frac{3}{4}\beta}}{\gamma^{\beta}} \frac{\Gamma(n)}{\Gamma(n+\beta)} \quad . \tag{11}$$

The above argument ignores the production of e^+e^- pairs and is thus particularly suitable for $(e-\bar{e})$ or "non-singlet"-distributions in general for which it (i. e. pair production) cancels out.

In QCD, in the leading logarithm approximation, IR divergences have been shown to cancel for color singlets(7, 8). In this approximation, the non-abelian nature of QCD manifest itself essentially in replacing the coupling constant by the running coupling constant - otherwise, the results are practically unchanged, apart from obvious group theory factors. Thus, eq. (11) should remain valid for non-singlet quark distributions as well. We shall specify β for this case shortly.

Let us compare the above expression (eq. (11)) to those obtained through the renormalization group or equivalently, the Altarelli-Parisi equations for the moments of the non-singlet quark structure functions, for large t $(= \ln Q^2/\Lambda^2)$:

$$\frac{M(n,t)}{M(n,0)} \simeq e^{\frac{A_n^{NS}}{2\pi b}\ln t} \quad , \tag{12}$$

where

$$A_n^{NS} = \int_0^1 dx\, x^{n-1} P_{qq}(x) = C_2(R)\left[-\frac{1}{2} + \frac{1}{n(n+1)} - 2\sum_{j=2}^{n}\frac{1}{j}\right] , \tag{13}$$

$C_2(R) = \dfrac{N_c^2 - 1}{2N_c}$ and $b = \left(\dfrac{11N_c - 2N_f}{12\pi}\right)$ for $SU(N_c)$ color and N_f = number of flavors.

For large n, eqs. (11) and (12) lead to the same n dependence, provided we identify

$$\beta \simeq \frac{C_2(R)}{\pi b} \ln t \quad . \tag{14}$$

This allows us to interpret β as

$$\beta = \frac{1}{\pi} C_2(R) \int dt \ \alpha(t) \quad . \tag{15}$$

For QED, $C_2(R) \to 1$ and we get the correct expression (for large Q^2)

$$\beta_{QED} \simeq \frac{\alpha}{\pi} (\ln \frac{Q^2}{m^2}) \quad . \tag{16}$$

For QCD, if we use the asymptotic freedom formula $\alpha(t) = \frac{1}{bt}$, eq. (15) gives back

$$\beta_{QCD}(t) \underset{(t\ \text{large})}{\simeq} \frac{C_2(R)}{\pi b} \ln t \quad ,$$

in agreement with eq. (14).

Also, from eq. (11) we can compute the variation of $\langle x^n \rangle$ with β (and thus Q^2):

$$\frac{d\langle x^n \rangle}{d\beta} = \langle x^n \rangle \left[\frac{3}{4} - \ln \gamma - \psi(n+\beta) \right] \quad , \tag{17}$$

where $\psi(z) = \frac{d}{dz} \ln \Gamma(z)$. We may contrast it with the RG result, eq. (12), for the same quantity

$$A_n(0) \equiv \frac{A_n^{NS}}{2C_2(R)} = \frac{3}{4} - \ln \gamma - \psi(n) - \frac{(2n+1)}{2n(n+1)} \tag{18}$$

To first order in the coupling constant, we may set $\beta \doteq 0$ on the right hand side of eq. (17) and compare it to eq. (18). The difference is then seen to be of order $1/n$ for large n.

At first sight it seems strange that the two results differ (by non-leading terms in n) to first order in α (or β). This discrepancy can be traced to the particular normalization of $P_{qq}(x)$ and hence the consequent A_n^{NS}. For example, it leads to $A_1^{NS} = 0$. In the next Section, when we modify the AP equations using our β-regularization but keeping their normalization, to first order, our result is identical to theirs.

4. - MODIFIED AP EQUATIONS.

We now set up equations of the AP type with the β-correction in the kernel so as not to encounter any singular distributions. To facilitate comparison, we shall follow as closely as possible their development[3].

The "equivalent number of gluons" at x in a quark is taken from the bremsstrahlung formula to be

$$\mathcal{N}_g \equiv \hat{\mathcal{P}}_g(x, \beta) = C(\beta)\, x^{\beta-1}\left[1+x^2\right] \quad , \tag{19}$$

where $C(\beta)$ is a normalization constant to be specified shortly. The corresponding number of quarks inside a quark will be

$$\hat{\mathcal{P}}_q(x, \beta) = \hat{\mathcal{P}}_g(1-x, \beta) \quad . \tag{20}$$

The above expression can be read: the probability for a certain missing energy is equal to the probability of finding a quark with that energy missing.

The transition probability for quark bremmstrahlung is thus

$$\frac{d\,\hat{\mathcal{P}}_q(x, \beta)}{d\beta} \equiv p_{qq}(x, \beta) \quad . \tag{21}$$

In analogy with AP then the following master equation is proposed:

$$\frac{d\,\mathscr{P}_q(x,\beta)}{d\beta} = \int_x^1 \frac{dy}{y}\, p_{qq}(y,\beta)\, \mathscr{P}_q(\frac{x}{y},\beta) \quad . \tag{22}$$

The moments of the quark densities $\mathscr{P}_q(x,\beta)$,

$$M^n(x,\beta) = \int_0^1 dx\, x^{n-1}\, \mathscr{P}_q(x,\beta) \quad , \tag{23}$$

will thus satisfy the equation

$$\frac{dM^n(\beta)}{d\beta} = A_n(\beta)\, M^n(\beta) \; , \tag{24}$$

where

$$A_n(\beta) = \int_0^1 \frac{dy}{y}\, y^n\, p_{qq}(y,\beta) \quad . \tag{25}$$

$C(\beta)$ can be fixed by requiring that

$$\int_0^1 dx\, \mathscr{N}_q(x) = 1 \quad . \tag{26}$$

Eqs. (19) and (26) then give

$$C(\beta) = \frac{\beta}{1 + \frac{1}{(1+\beta)(1+\beta/2)}} \quad . \tag{27}$$

Now eqs. (19-27) allow us to obtain

$$A_n(\beta) = \frac{d}{d\beta}\left[\frac{a_n(\beta)}{a_1(\beta)}\right] , \tag{28}$$

where

$$a_n(\beta) = \frac{\Gamma(n)\,\Gamma(\beta)}{\Gamma(n+\beta)}\left[1 + \frac{(n+1)n}{(n+\beta)(n+\beta+1)}\right]. \tag{29}$$

A straightforward calculation then gives for $\beta = 0$,

$$A_n(0) = \frac{3}{4} - \ln \gamma - \psi(n) - \frac{(2n+1)}{2n(n+1)} \quad ,$$

which is exactly the AP (RG) result given in eq. (18).

From the normalization condition, eq. (26), it follows that

$$\int_0^1 p_{qq}(x, \beta)\, dx = 0 \; , \tag{30}$$

which is equivalent to $A_1(\beta) = 0$ as given by eq. (28).

Also,the momentum conservation condition

$$p_{qq}(x, \beta) = p_{Gq}(1-x, \beta) \tag{31}$$

is valid here for all x - in contrast to AP - due to our regularization procedure. It easily follows that

$$\int_0^1 dx\, x \left[p_{qq}(x, \beta) + p_{Gq}(x, \beta) \right] dx = 0 \; . \tag{32}$$

Relation (30) allows us to obtain the sum rule

$$A_n(\beta) = \sum_{k=0}^{n-1} (-)^k \binom{n-1}{k} A^{gq}_{k+1}(\beta) \; , \tag{33}$$

where

$$A^{gq}_k = \int_0^1 dx\, x^{k-1}\, p_{gq}(x) \; . \tag{34}$$

Thus, we have all the AP results without ever encountering singular distributions. As stated in the introduction, if one wishes to use the basic quark and gluon densities p_{qq} and p_{gq} etc. for numeri̲cal computational purposes (say of Mote Carlo type), it is important that the input probabilities always remain finite and bounded without recourse to sigular limits. For such applications, the above method appears eminently suitable.

5. - PHENOMENOLOGY OF F_3 MOMENTS

We now present a fit to the moments of the nucleon structure function F_3 based on expression (11). We shall assume, for lack of a better knowledge, the asymptotic form, eq. (12),

$$\beta \simeq \frac{C_2(R)}{\pi b} \ln t$$

valid for large $t = \ln Q^2/\Lambda^2$, throughout for all Q^2. Clearly, this is a gross approximation and reflects our ignorance of the non-perturbative confinement region.

The data are taken from ref. (5). For this Q^2 region, we shall assume 3 flavors. Thus, for $N_c = 3$, we have

$$\beta \simeq \left(\frac{16}{27}\right) \ln \ln Q^2/\Lambda^2 \quad . \tag{35}$$

We have also chosen $\Lambda = 0.75$ GeV, the same value which gives the best fit to first-order QCD result. Figure 1 shows the comparison between data and eq. (11) for the moments of F_3 for $n = 2, 3, 4$ and 5. The agreément seems quite satisfactory down to $Q^2 \gtrsim 0.8\ \mathrm{GeV}^2$. It is interesting to note that the theoretical expression also has a maximum even though its position is not right. Thus, the qualitative features of the low energy data are reflect in this simple formula.

Fig. 2 shows the $n = 1$ moment. The GLS sum rule gives a constant value (3) asymptotically. To first order in QCD there is no variation in Q^2. Since in ref. (5) several second order QCD estimates are shown for $\Lambda = 0.2$, 0.4 and 0.75 GeV, we also plot our expressions for the same Λ. The experimental errors are large for this case so no firm conclusions can be drawn. A smaller value of Λ (than those needed for $n = 2, 3, 4$ and 5) does seem to work better.

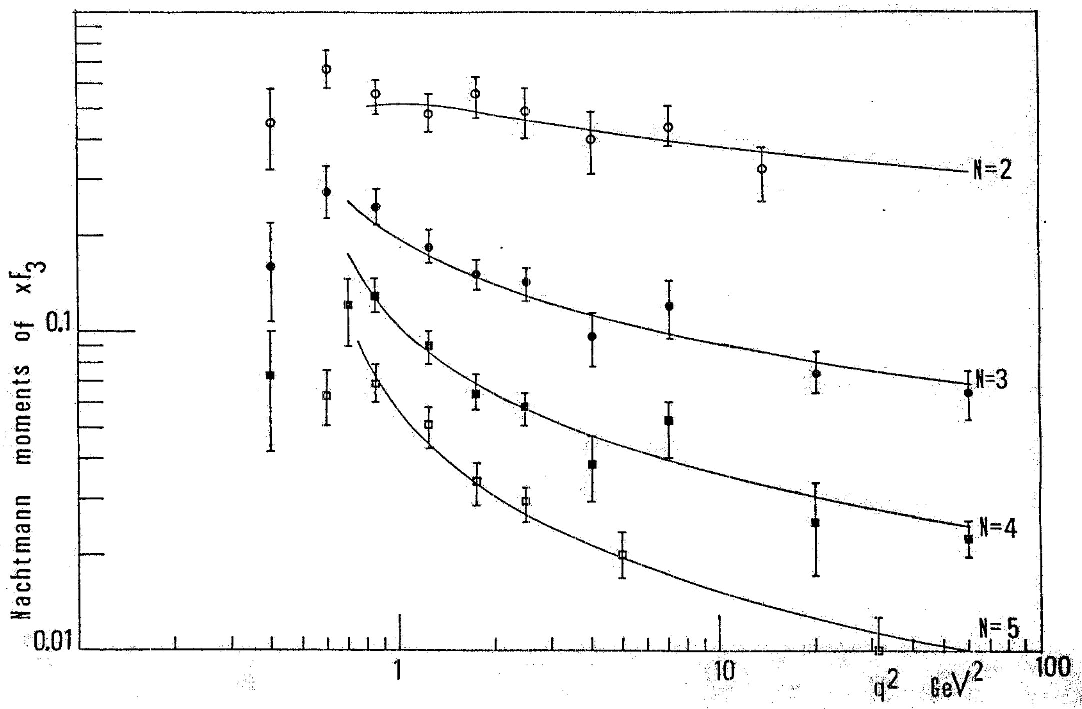

FIG. 1 - Moments of F_3 for n = 2, 3, 4 and 5 vs. Q^2, from eqs. (11) and (35) with Λ = 0.75 GeV. Data are from ref. (5).

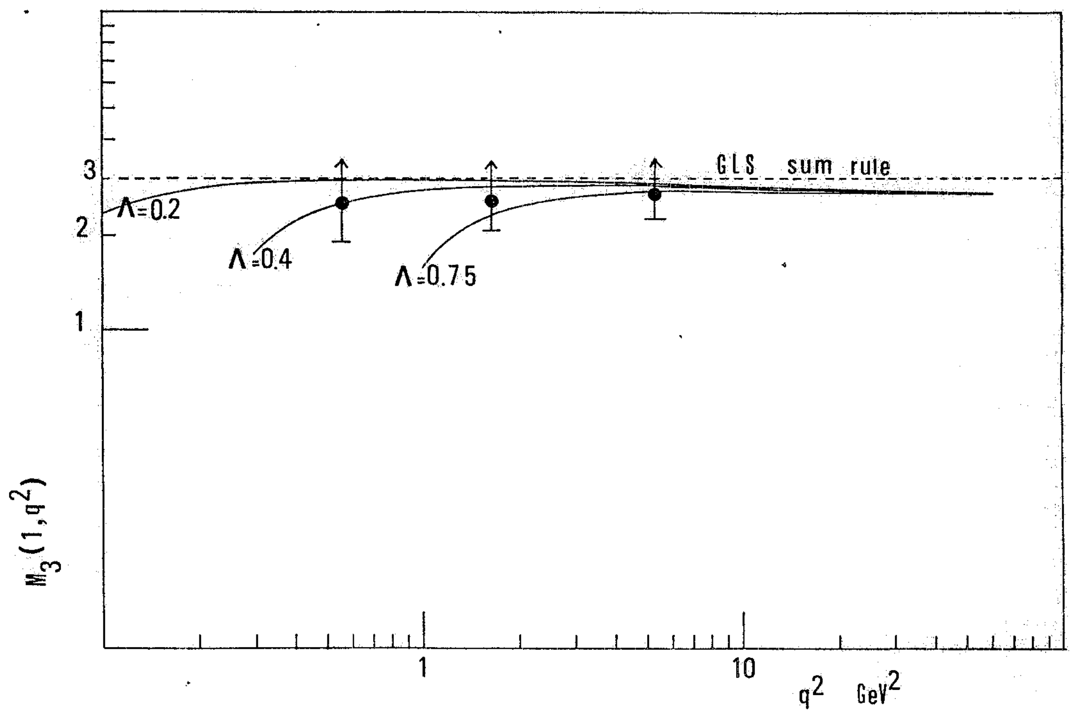

FIG. 2 - n = 1 moment for F_3 vs. Q^2 from eqs. (11) and (35) with Λ = 0.2, 0.4 and 0.75 GeV. Data are from ref. (5).

6. - CONCLUSIONS.

The analysis presented here seems quite well supported by the data even for rather low values of $Q^2 \gtrsim \Lambda^2$ and for the lower moments the fits are somewhat superior to the RG result. It apperas remarkable that we are able to do as well as we do with the asymptotic formula for β (valid for large Q^2) all the way. It is an interesting open problem to derive theoretically a reasonable low energy expression for β. This is likely to be difficult since it inevitably involves entering the region of confinement. We conjecture that for sufficiently low Q^2, β becomes a constant and probably vanishes. Such is the behavior in QED and may turn out to be so in QCD as well.

FOOTNOTES AND REFERENCES.

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