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G. De Franceschi and F. Palumbo: SPONTANEOUS SUPERSYMMETRY BREAKING AND SUPERCONDUCTIVITY IN A NONRELATIVISTIC MODEL.

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ABSTRACT.

The conjecture that supersymmetry breaking implies superconductivity is supported by the analysis of a class of supersymmetric nonrelavistic models involving only fermions. Here the investigation is extended to a nonrelativistic model involving both fermions and bosons, which in a sense is the nonrelativistic version of the Wess-Zumino model. A sufficient condition is established for the validity of the conjecture. This condition can be possibly violated at most in a two-dimensional subspace of the threedimensional subspace of the coupling constants.

1. - INTRODUCTION.

We study the ground state properties of a model of supersymmetry, which in a sense is the nonrelativistic version of the Wess-Zumino model[(1)]. We are motivated by the conjecture[(2)] that, when supersymmetry is spontaneously broken, the fermion density is nonvanishing, and fermions are generally in a coherent state. This conjecture is supported by the analysis of a class of nonrelativistic supersymmetric models involving only fermions, where the ground state is superconductive (coherent), except for a special choice of the coupling constants for which, however, the fermion density is nonvanishing[(2)].

We now extend the investigation to a model involving both fermions and bosons. This model also suggests at first glance that supercondu͟c tivity should be present as a consequence of supersymmetry breaking, due to the appeareance in the Hamiltonian of terms analogous to the Bo͟ goliubov symmetry breaking terms[(3)].

We consider the interaction of two scalar multiplets - each one contai͟n ing two scalar bosons and one Pauli spinor - including quadratic, cubic and quartic terms. The bilinear terms are like the mass terms of the relativistic theory, but unlike the relativistic case the sign of the "mass" is not fixed.

When it is negative, the mass parameter acquires the physical mea͟n ing of a chemical potential, and the quadratic term can be interpreted as a constraint on the average particle density. Breaking of supersymmetry is thus explicitly imposed, and it is not a consequence of the interaction. Hence we will not consider the case of negative mass parameters, the more so because it has no analogy with the relativistic the͟o ry, which we have in mind in studying the present model. Spontaneous breaking is indeed an infrared phenomenon and it is perhaps not unlikely that the results obtained in a nonrelativistic framework have some rel͟e vance to the relativistic formulation.

We are well aware that a comparison between a relativistic and a no͟n relativistic theory may be dangerous , because the former is severely constrained by relativistic invariance. For instance we have three independent couplings while there is only one for a single multiplet in the r͟e lativistic case. A major difference is that in the present case the anticommutator of the spinorial generators gives only a central charge, a possibility which deos not occur in the relativistic case. Another difference is that the present model is not renormalizable, though it can be made finite by introducing an appropriate form factor without destroying supersymmetry[(4)].

The relativistic and nonrelativistic case have, however, also peculiar properties in common. One is that supersymmetry is not spontaneously broken for a scalar multiplet interacting with itself[5]. Another one is that supersymmetry fixes a critical value for the ratio of the fermion-boson to boson-boson coupling constant. For larger values of the ratio the Hamiltonian is not bounded from below, while it is bounded[6] otherwise.

The paper is organized as follows. In sect. 2 the model is presented and the above properties are shown, while the connexion between spontaneous breaking and superconductivity is discussed in sect. 3.

2. - THE MODEL.

2. 1. - Grading of the Galilei group.

We consider the grading of the Galilei group obtained by introducing the spinorial[(+)] generators Q_α with anticommutation relations[7]

$$\begin{aligned} \{ Q_\alpha , Q_\beta \} &= 0 \\ \{ Q_\alpha , Q^{+\beta} \} &= \delta_\alpha^{\ \beta} M, \end{aligned} \qquad (1)$$

where the mass operator M is a central charge. The Q_α commute with all the Galilei generators except angular momentum

$$\left[Q_\alpha , J_K \right] = \frac{1}{2} (\sigma_K Q)_\alpha , \qquad (2)$$

σ_K being the Pauli matrices.

(+) Our conventions are as follows: Quantities like Q_α, ψ_α, with a lower index transform according to $Q_\alpha \rightarrow u_\alpha^{\ \beta} Q_\beta$, quantities like Q^*_α, $\psi^{+\alpha}$, with an upper index, like $Q^{*\alpha} \rightarrow u^{*\alpha}{}_\beta Q^{*\beta}$, where u is a SU_2 matrix and u^* its complex conjugate. Indices are raised as follows: $\psi^\alpha = (i\,\sigma_2)^{\alpha\beta} \psi_\beta$.

The resulting graded algebra is a contraction (velocity of light $\to \infty$) of the Wess-Zumino algebra.

The Q_α's and M have the following representation on superfields[8]

$$
\begin{aligned}
Q_\alpha &= i\left(\frac{\partial}{\partial \theta^{*\alpha}} + \frac{1}{2} M \theta_\alpha\right) \\
Q^{+\alpha} &= -i\left(\frac{\partial}{\partial \theta_\alpha} + \frac{1}{2} M \theta^{*\alpha}\right)
\end{aligned}
\tag{3}
$$

$$M = i \frac{\partial}{\partial s}$$

with covariant derivatives[9]

$$
\begin{aligned}
D_\alpha &= \frac{\partial}{\partial \theta^{*\alpha}} - \frac{1}{2} M \theta_\alpha \\
D^{+\alpha} &= -\frac{\partial}{\partial \theta_\alpha} + \frac{1}{2} M \theta^{*\alpha} .
\end{aligned}
\tag{4}
$$

2. 2. - The scalar multiplet.

The scalar superfield

$$
\begin{aligned}
S = {} & A - \sqrt{m}\, \psi^\alpha \theta_\alpha + \frac{1}{2} m B \theta^\alpha \theta_\alpha + \frac{i}{2} \frac{\partial}{\partial s} A \theta^{*\alpha} \theta_\alpha + \\
& - \sqrt{m}\, \frac{i}{4} \frac{\partial}{\partial s} \psi_\alpha \theta^{*\alpha} \theta^\beta \theta_\beta + \frac{1}{16} \left(\frac{\partial}{\partial s}\right)^2 A \theta^\alpha \theta_\alpha \theta^{*\beta} \theta^*_\beta
\end{aligned}
\tag{5}
$$

is determined by the condition

$$D_\alpha S = 0 \tag{6}$$

The constant m having the dimension of a mass has been introduced in order that the fields have canonical dimensions. The two scalars A and B and the Pauli spinor ψ transform under infinitesimal supertransformations with parameters η, η^* according to

$$
\begin{aligned}
\delta_\eta A &= -\psi^\alpha \eta_\alpha \\
\delta_\eta B &= \frac{i}{2} \eta^{*\alpha} \frac{\partial}{\partial s} \psi_\alpha \\
\delta_\eta \psi_\alpha &= -2 B \eta_\alpha - i \frac{\partial}{\partial s} A (i \sigma_2)_{\alpha\beta} \eta^{*\beta} .
\end{aligned}
\tag{7}
$$

We assume all the fields A, B and ψ to be eigenstates of M with eigenvalue m, and to obey canonical commutation relations. The spinorial generators have then the following representation

$$Q_\alpha = i\sqrt{m} \int d^3 x \left[A(x)(i\sigma_2)_{\alpha\beta} \psi^{+\beta}(x) - B^+(x)\psi_\alpha(x) \right] . \tag{8}$$

Invariants must be independent of s and must therefore contain an even number of superfields. The only quadratic invariant is

$$\mathcal{N} = -\frac{4}{m^2}(S^+S)_{LAST} = A^+A + B^+B + \psi^{+\alpha}\psi_\alpha , \tag{9}$$

which is the total density of particles. From this we can construct the kinetic energy density

$$\mathcal{T} = -\frac{2\hbar^2}{m^3}\left(S^+ \frac{\overleftarrow{\partial}}{\partial x_k} \frac{\partial}{\partial x_k} S \right)_{LAST} = \tag{10}$$
$$= \frac{1}{2m}\left(A^+ \frac{\overleftarrow{\partial}}{\partial x_k}\frac{\partial}{\partial x_k} A + B^+ \frac{\overleftarrow{\partial}}{\partial x_k}\frac{\partial}{\partial x_k} B + \psi^{+\alpha} \frac{\overleftarrow{\partial}}{\partial x_k}\frac{\partial}{\partial x_k}\psi_\alpha \right).$$

There are four quartic invariants

$$\mathcal{V}_+ = -\frac{1}{m^2}(S^+S^+SS)_{LAST} = A^+A^+AA + A^+B^+AB + 2A^+A\psi^{+\alpha}\psi_\alpha + \tag{11}$$
$$-\frac{1}{2}A^+B^+\psi^\alpha\psi_\alpha + \frac{1}{2}\psi^{+\alpha}\psi^+_\alpha AB - \frac{1}{4}\psi^{+\alpha}\psi^+_\alpha\psi^\beta\psi_\beta ,$$

$$\mathcal{V}_- = -\frac{1}{m^2}(R^+R^+RR)_{LAST} = B^+B^+BB + A^+B^+AB + 2B^+B\psi^{+\alpha}\psi_\alpha + \tag{12}$$
$$+\frac{1}{2}A^+B^+\psi^\alpha\psi_\alpha - \frac{1}{2}\psi^{+\alpha}\psi^+_\alpha AB - \frac{1}{4}\psi^{+\alpha}\psi^+_\alpha\psi^\beta\psi_\beta ,$$

$$\mathcal{V}_{\prime\prime} = -\frac{2}{m^2}(R^+S^+SS + S^+S^+SR)_{LAST} ,$$

$$\mathcal{V}_{\prime\prime}' = -\frac{2i}{m^2}(R^+S^+SS - S^+S^+SR)_{LAST} , \tag{13}$$

where

$$R = -\frac{1}{2m}\bar{D}^\alpha \bar{D}_\alpha S. \tag{14}$$

For the sake of simplicity we will not consider $\mathcal{V}'$ and $\mathcal{V}''$ so that finally we have the following Hamiltonian density

$$\mathcal{H} = \mathcal{T} + g_+ \mathcal{V}_+ + g_- \mathcal{V}_- + \mu \mathcal{N} \tag{15}$$

2. 3. - Ground state properties.(*)

Unlike the relativistic case the Hamiltonian is not determined by the Q_α's, it is not positive definite and in general not bounded from below. Rearranging terms, $\mathcal{V}_+$ can be rewritten as

$$\mathcal{V}_+ = (AB - \frac{1}{2}\psi^\alpha \psi_\alpha)^+ (AB - \frac{1}{2}\psi^\alpha \psi_\alpha) + A^+ A^+ AA + 2A^+ A \psi^{+\alpha} \psi_\alpha , \tag{16}$$

which is a sum of positive semidefinitive operators. Since $\mathcal{V}_-$ also can be put into this form, we conclude that a necessary and sufficient condition for lower boundedness is

$$g_+ , \; g_- \geq 0. \tag{17}$$

This condition obviously ensures also the absence of symmetry breaking (although if, e. g. $\mu = g_+ = 0$, the supersymmetric vacuum is degenerate with any coherent state of A-bosons only). The absence of symmetry breaking can be shown even if the terms $\mathcal{V}'$ and $\mathcal{V}''$ are taken into account.

It is convenient to introduce two new coupling constants

$$G = \frac{1}{2}(g_+ + g_-) \geq 0 \qquad ; \qquad g = \frac{1}{2}(g_+ - g_-) \tag{18}$$

and rewrite $\mathcal{H}$ as

$$\mathcal{H} = \mathcal{T} + G(\mathcal{V}_+ + \mathcal{V}_-) + g(\mathcal{V}_+ - \mathcal{V}_-) + \mu \mathcal{N}. \tag{19}$$

(x) The results of this section have been presented in ref. (4).

We will now give a proof of the property mentioned at the beginning that supersymmetry fixes a critical value for the ratio of the fermion-boson to boson-boson coupling constants.

Let us replace in $\mathcal{V}_+$ the coefficient of ($A^+B^+ \psi^\alpha\psi_\alpha$+ h. c.) by $-\frac{1}{2}(1+\lambda)$ so that $\mathcal{V}_+$ becomes

$$\left[AB - \frac{1}{2}(1+\lambda)\psi^\alpha\psi_\alpha\right]^+ \left[AB - \frac{1}{2}(1+\lambda)\psi^\alpha\psi_\alpha\right]$$
$$+ A^+A^+AA + 2A^+A\psi^{+\alpha}\psi_\alpha + \frac{1}{4}\lambda(\lambda+2)\psi^{+\alpha}\psi^+_\alpha\psi^\alpha\psi_\alpha . \qquad (20)$$

This expression is supersymmetric only for $\lambda = 0$.

If $-2 < \lambda < 0$, (20) is the sum of positive semidefinite terms, whose ground state is the vacuum. That in the other cases (20) is not bounded from below can be seen by considering its expectation value in a class of trial states, which we will also use in Sect. 3 to study the connection between symmetry breaking and superconductivity. They are the product of a coherent state of A-bosons times a coherent state of B-bosons times a coherent (superconductive) state of fermions.

A tipical state in this class is characterized by the following expectation values for the operators contained in $\mathcal{H}$

$$\langle \int d^3x \ \psi^\alpha(x)\psi_\alpha(x) \rangle = \Omega D$$
$$\langle \int d^3x \ \psi^{+\alpha}(x)\psi^+_\alpha(x)\psi^\beta(x)\psi_\beta(x) \rangle = -\Omega(|D|^2 + \varrho^2), \qquad (21)$$

where Ω is the volume of quantization,

$$D = 2 \sum_k u(k)\, v(k) \ ,$$
$$\varrho = 2 \sum_k |v(k)|^2 . \qquad (22)$$

Here u(k) and v(k) are the BCS functions, obeying the normalization condition

$$|u(k)|^2 + |v(k)|^2 = 1 . \qquad (23)$$

For the bosons

$$\langle A^{+n} A^m \rangle = a^{*n} a^m , \qquad \langle B^{+n} B^m \rangle = b^{*n} b^m . \qquad (24)$$

If we take

$$|v| = \begin{cases} \sim \varepsilon , & k < K \\ 0 , & k > K \end{cases}$$
$$|u| = \begin{cases} \sim 1 , & k < K \\ 1 , & k > K \end{cases} \qquad (25)$$

then

$$|D| \sim \varepsilon K^3 , \qquad \varrho \sim \varepsilon^2 K^3 ,$$
$$\langle \int d x^3 \mathcal{T} \rangle \sim \varepsilon^2 K^5 . \qquad (26)$$

If, in addition, we take $a = \tau b$, $\tau b^2 = \frac{1}{2}(1 + \lambda)D$, we can choose ε and τ so small that $\langle \mathscr{H} \rangle$ is negative and divergent like K^6.

2. 4. - Coupled Multiplets.

We consider now the coupling of two multiplets. If they have the same mass, again only quadratic and quartic invariants are available. One can show then that lower boundedness of the Hamiltonian forbids spontaneous breaking.

We therefore assume the mass of one of the multiplets to be twice the mass of the other. The quantities referring to the heavier one will be denoted by the label 2.

Interaction terms - no more than quartic - in the fields can only be cubic and there are four of them

$$W_1 = \frac{1}{m^2} (S_2^+ S^2 + S^{+2} S_2)_{LAST} = - A_2^+ A^2$$
$$- A_2 A^{+2} + B_2^+ \left(\frac{1}{2} \psi^\alpha \psi_\alpha - A B \right) + \left(- \frac{1}{2} \psi^{+\alpha} \psi_\alpha^+ - A^+ B^+ \right) B_2 \qquad (27)$$
$$- \sqrt{2}\, \psi_2^{+\alpha} \psi_\alpha A - \sqrt{2}\, A^+ \psi^{+\alpha} \psi_{2\alpha} ,$$

$$W_2 = \frac{1}{m^2}(S_2^+ S R + S^+ R^+ S_2)_{LAST} = -B_2^+ B^2 - B_2 B^{+2}$$
$$- A_2^+ (\frac{1}{2}\psi^\alpha \psi_\alpha + A B) - (-\frac{1}{2}\psi^{+\alpha} \psi_\alpha^+ + A^+ B^+) A_2 + \qquad (28)$$
$$-\sqrt{2}\, \psi_2^{+\alpha} \psi_\alpha B - \sqrt{2}\, B^+ \psi^{+\alpha} \psi_{2\alpha} ,$$

$$W' = \frac{i}{m^2}(S_2^+ S^2 - S_2 S^{+2}),$$
$$\qquad (29)$$
$$W'' = \frac{i}{m^2}(S_2^+ S R - S^+ R^+ S_2).$$

For the sake of simplicity we will not consider W' and W", so that we are left with the following Hamiltonian density

$$\mathcal{H} = \mathcal{T}_1 + \mathcal{T}_2 + G(\mathcal{V}_+ + \mathcal{V}_-) + G_2(\mathcal{V}_{2+} + \mathcal{V}_{2-}) + g(\mathcal{V}_+ - \mathcal{V}_-)$$
$$+ g_2(\mathcal{V}_{2+} - \mathcal{V}_{2-}) + f_1 W_1 + f_2 W_2 + \mu \mathcal{N} + \mu_2 \mathcal{N}_2. \qquad (30)$$

The above Hamiltonian density has a global U(1) invariance related to mass conservation. In the presence of supersymmetry breaking the expectation values of A_2, B_2, cannot vanish simultaneously and the terms

$$-\frac{1}{2}\langle A_2^+ \rangle \psi^\alpha \psi_\alpha + \frac{1}{2}\psi^{+\alpha}\psi_\alpha^+ \langle A_2 \rangle , \quad \frac{1}{2}\langle B_2^+ \rangle \psi^\alpha \psi_\alpha - \frac{1}{2}\psi^{+\alpha}\psi_\alpha^+ \langle B_2 \rangle$$

are analogous to the symmetry breaking terms introduced by Bogoliubov[3]. Unless these terms cancel out with each other, the ground state must be superconductive, as we shall see in the next section.

3. - SUPERSYMMETRY BREAKING AND SUPERCONDUCTIVITY.

The expectation value of the Hamiltonian H, whose density is given in eq. (30), in the trial state specified in sect. 2. 3 is

$$\langle H \rangle = \Omega\,(|a|^2+|b|^2+2\varrho)\left[G(|a|^2+|b|^2)+g(|a|^2-|b|^2)\right]$$

$$+\mu(|a|^2+|b|^2+\varrho)+\frac{1}{2}G(|D|^2+\varrho^2)+(|a_2|^2+|b_2|^2+2\varrho_2)\cdot$$

$$\cdot\left[G_2(|a_2|^2+|b_2|^2)+g_2(|a_2|^2-|b_2|^2)\right]+\mu_2(|a_2|^2+|b_2|^2+\varrho_2)+ \qquad (31)$$

$$+\frac{1}{2}G_2(|D_2|^2+\varrho_2^2)-f_1\left[a^2a_2^*+a\,b\,b_2^*+\text{c. c.}\right]-f_2\left[b^2b_2^*+\right.$$

$$\left.+\,a\,b\,a_2^*+\text{c. c.}\right]-(C\,D+C_2D_2+\text{c. c.})\,,$$

where

$$C = \frac{1}{2}g\,a^*b^*-\frac{1}{2}f_1\,b_2^*+\frac{1}{2}\,f_2\,a_2^*$$

$$C_2 = \frac{1}{2}g_2\,a_2^*\,b_2^*, \qquad (32)$$

and where - see later discussion - the kinetic energy terms have been omitted.

Due to the large number of variables it is rather complicate to obtain an explicit solution of this variational problem.

(Note that the number of boson variables is twice that of the relativistic case, because a nonrelativistic scalar boson is described by a complex field.)

We will confine ourselves to give a sufficient condition for superconductivity to occur if there is spontaneous supersymmetry breaking in the boson sector only.

The condition is simply that C and C_2 in eqs. (31), (32) be not simultaneously vanishing at the minimum of the purely bosonic potential. In fact, if this is the case, the contribution of the term $(CD + C_2D_2 + c.c.)$ in eq. (31) can always be made negative and larger in modulus than the remaining contributions involving ϱ, ϱ_2, $|D|$, $|D_2|$ and the fermionic kinetic energy. This can be seen e. g. by chosing ε small enough in eq. (25). Let us now show that the above condition is fulfilled provided none of the coupling constants g_2, f_1, f_2 is vanishing . Optmization with respect to the phases of a, b, a_2, b_2 yields for the bosonic potential

$$\begin{aligned} \langle H_B \rangle / \Omega = & (|a|^2+|b|^2)\left[G(|a|^2+|b|^2)+g(|a|^2-|b|^2+\mu\right] \\ & +(|a_2|^2+|b_2|^2\left[G_2(|a_2|^2+|b_2|^2)+g_2(|a_2|^2-|b_2|^2)\right. \\ & \left.+\mu_2\right] - 2(|a_2||a|+|b_2||b|)(|f_1||a|+|f_2||b|) \end{aligned} \tag{33}$$

From the hypothesis that there is supersymmetry breaking in the boson sector, it follows that the minimum occurs at

$$|a|^2+|b|^2 \neq 0 \,, \quad |a_2|^2+|b_2|^2 \neq 0 \,, \tag{34}$$

because otherwise the only attractive term in eq. (33) would vanish(*).

We now distinguish the case i) that either $|a|$ or $|b|$ vanish, and ii) that both $|a|, |b| \neq 0$.

In the first case C and C_2 cannot both vanish, because then, from eq. (32), it would follow that also $|a_2| = |b_2| = 0$.

In case ii) we can show that both $|a_2|$ and $|b_2| \neq 0$ so that $C_2 \neq 0$, thus completing the proof.

(*) On the other hand it is easy to see that breaking actually occurs for μ and μ_2 not too large.

For this purpose we introduce the parametrization

$$|a_2| = \sigma_2 \cos\varphi_2 \;,\; |b_2| = \sigma_2 \sin\varphi_2 \qquad 0 \leq \varphi_2 \leq \pi/2 \tag{35}$$

We get

$$\left.\frac{\partial}{\partial\varphi_2}\frac{\langle H_B\rangle}{\Omega}\right|_{\varphi_2=0} = -2\sigma_2 |b|(|f_1||a|+|f_2||b|) < 0$$

$$\left.\frac{\partial}{\partial\varphi_2}\frac{\langle H_B\rangle}{\Omega}\right|_{\varphi_2=\frac{\pi}{2}} = 2\sigma_2 |a|(|f_1||a|+|f_2||b|) > 0, \tag{36}$$

which shows that the minimum occurs at $\varphi_2 \neq 0, \pi/2$.

In conclusion, although we have neither proved nor disproved the conjecture that supersymmetry breaking neccesarily implies superconductivity, our results show that the presence of this phenomenon cannot be excluded a priori, as it is usually done. On the contrary breaking of supersymmetry without superconductivity can take place, in the present model, at most in a subspace of dimensions 2 of the three-dimensional space of the coupling contants g_2, f_1, f_2.

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